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Solving Zhou Chaotic System Using Fourth-Order Runge-Kutta Method

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Abstract: Most of scientific problems and natural phenomena can be modeled by chaotic systems of ordinary differential equations. These problems can be solved by using various methods. In this paper, we are interested to test the Runge-Kutta method of order four on the Zhou chaotic system. This system is a new three-dimensional autonomous chaotic system. Numerical comparisons are made between the Runge-Kutta of order four and the Euler's method. Comparisons were also done between the RK4 methods but with different time steps. It has been observed that the accuracy of RK4 solutions can be increased by decreasing the time step. Our work shows that RK4 method successfully can solve the Zhou system and figures are given for different number of iterations with corresponding range of time, *t*.

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Key words: Zhou chaotic system · Fourth-order Runge-Kutta method · Euler's method

INTRODUCTION

Most of scientific problems and natural phenomena can be modeled by chaotic systems of ordinary differential equations (ODEs). Not all chaotic systems have analytical solutions. This is due to their complexities. Therefore, the numerical methods can be used to obtain the approximation of solutions of the problems. Some numerical methods that can be used to solve the systems are; Euler's method, midpoint method, Heun's method and Runge-Kutta method of different orders.

Recently, Roslan *et al.* [1] have used the Euler's method to solve the chaotic system. They used this method because it is one of the simplest approaches to obtain the numerical solution of a differential equation. An algorithm for Euler's method is used to obtain an approximation for the initial-value problem and was employed to Zhou's chaotic system [2]. They used the C++ software to solve this system and MATLAB to plot the solutions and the results are given for different number of iterations. Although the results obtained is the same butterfly-shaped, but however, this method is not an efficient method and seldom used because of its less accuracy [3].

In this paper, we are interested to test the Runge-Kutta method of order four (RK4) on the Zhou system [2]. We choose the RK4 because it can obtain greater accuracy and does not need the calculation of higher derivatives [3]. Moreover, RK4 has been widely and commonly used for simulating the solution of chaotic systems [4, 5, 6, 7] and was taken as the comparison method [8, 9, 10, 11, 12]. For example, in 2009, Nazri and Rokiah [13] have considered the RK4 method as their benchmark solution.

Previously, some researchers have shown that RK4 successfully can solve the chaotic systems such as Lorenz system [8, 13], Rössler system [11] and Chen system [5, 9, 10, 12]. As such, the objective of this paper is to use the RK4 method in solving the Zhou chaotic system. We want to prove whether this method successfully can solve the Zhou system or not. The organization of this paper is in the following manner. In Section 2 we give some introduction for a new chaotic system that is Zhou system. The definition of fourth-order Runge-Kutta method (RK4) will be defined in Section 3 while in Section 4 we show the algorithm to calculate the RK4. Section 5 is the numerical results and discussion.

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The Zhou Chaotic System: In 2008, Zhou *et al*. [2] have proposed a new system and we called it as Zhou system. They used the controlled Lorenz system [14] to obtain their new system. Zhou system is one of the new chaotic systems in dynamical systems that exhibit chaos. This system is a three-dimensional autonomous system according to the numerical simulation as well as the theoretical analysis. The chaotic attractor of this Zhou system also has the butterfly-shaped which is same as the Lorenz attractor, but both of them are not topologically equivalent.

The Zhou chaotic dynamical system is defined as follows:

$$\frac{dx}{dt} = a(y-x)$$
$$\frac{dy}{dt} = bx - xz \qquad (1)$$
$$\frac{dz}{dt} = xy + cz$$

where x, y z are the state variables and a, b, c are constants. This system is chaotic when $a = 10$, $b = 16$ and $c = -1$. To determine the equilibrium points for this new system (1), we must find its equilibria. Let:

$$a(y-x) = 0$$
$$bx - xz = 0 \qquad (2)$$
$$xy + cz = 0$$

From (2), there exist three equilibria,

$O(0, 0, 0)$, $E^{+}(x_1, y_1, z_1)$, $E^{-}(x_2, y_2, z_2)$

where,

$$x_1 = y_1 = \sqrt{-bc}, \qquad z_1 = b,$$

$$x_2 = y_2 = -\sqrt{-bc}, \qquad z_2 = b.$$

Figure 1 to Figure 4 shows the Zhou system from different views. This system is found to be unstable for all three equilibrium points. The equilibrium points are at the origin, E^{+} and E^{-}.

In addition, some basic dynamical properties of Zhou system are; nonlinearity, symmetry and invariance, dissipativity, has positive Lyapunov exponents as indication of the existence of chaos phenomena etc. The Poincaré mapping, fractal dimension, bifurcation diagram and continuous spectrum of this system also have been studied by Zhou *et al*. [2].

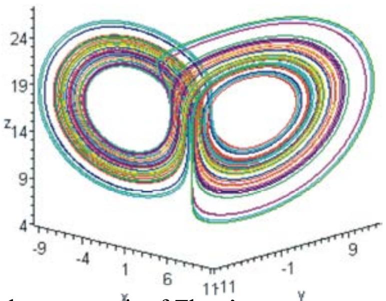

Fig. 1: *xyz* phase portrait of Zhou's attractor

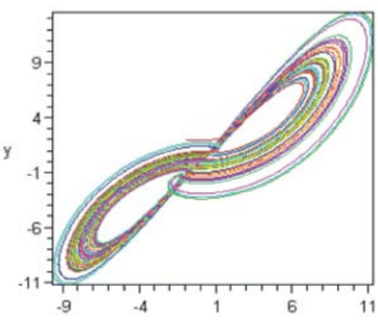

Fig. 2: *x-y* phase portrait of Zhou's attractor

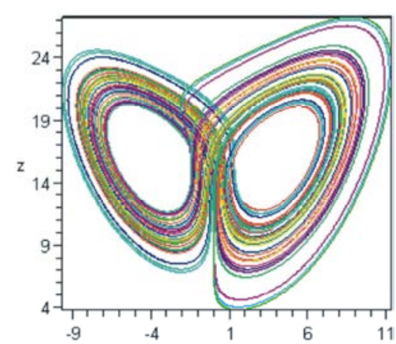

Fig. 3: *x-z* phase portrait of Zhou's attractor

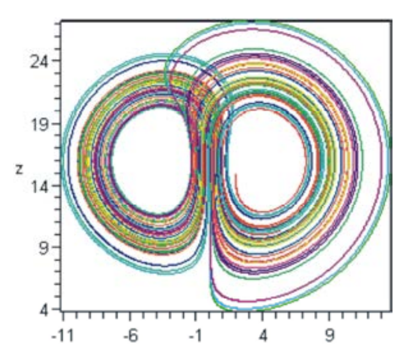

Fig. 4: *y-z* phase portrait of Zhou's attractor

Fourth-Order Runge-kutta Method (RK4): There are some different orders of Runge-Kutta methods, but all of them can be cast in the following general form.

$$y_{i+1} = y_i + \varphi(t_i, y_i, h)h \quad (3)$$

where $\varphi(t_i, y_i, h)$ is called an increment function, which is interpreted as the representative slope over interval. The estimate slope φ is used to extrapolate from an old value y_i to a new value y_{i+1} over a distance h. This is called an explicit method. The general form of this increment function is:

$$\varphi = a_1k_1 + a_2k_2 + \ldots + a_nk_n \quad (4)$$

where the a's are constants and the k's are:

$$\begin{aligned} k_1 &= f(t_i, y_i) \\ k_2 &= f(t_i + p_1h, y_i + q_{11}k_1h) \\ k_3 &= f(t_i + p_2h, y_i + q_{21}k_1h + q_{22}k_2h) \\ &\vdots \\ k_n &= f(t_i + p_{n-1}h, y_i + q_{n-1,1}k_1h + q_{n-1,2}k_2h + \cdots + q_{n-1,n-1}k_{n-1}) \end{aligned}$$

where p's and q's are constants [15].

To solve ODEs problem, we consider an initial value problem (IVP) of the first order differential equation:

$$y' = f(t, y),\ a \le t \le b,\ y(a) = \alpha. \quad (5)$$

The solution of this IVP by using the classical RK4 is given by:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (6)$$

where,

$$\begin{aligned} k_1 &= hf(t_i, y_i), \\ k_2 &= hf(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_1), \\ k_3 &= hf(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_2), \\ k_4 &= hf(t_i + h, y_i + k_3). \end{aligned}$$

This explicit Runge-Kutta method of order four (RK4) requires four evaluations of function [16]. We will use this classical RK4 method to solve the Zhou chaotic system which will be explained in the next section.

The Algorithm: Below is the algorithm to calculate the RK4 as stated in Burden and Faires [17]. We will apply this algorithm to solve the Zhou chaotic system in order to find the values of x, y and z subject to the initial conditions (–1, 2, 15).

To approximate the solution of the IVP in (5) at $(N + 1)$ equally spaced numbers in the interval $[a, b]$:

INPUT endpoints a,b; integer N; initial condition α
OUTPUT approximation w to y at the $(N + 1)$ values of t

Step 1 Set	$\Delta t = (b - a)/N$;
	$t = a$;
	$w = a$;
OUTPUT	(t, w)
Step 2	For $i = 1, 2, .., N$ do Steps 3-5.
Step 3 Set	$K_1 = hf(t, w)$;
	$K_2 = hf(t + h/2, w + K_1 / 2)$;
	$K_3 = hf(t_i + h / 2, w + K_2 / 2)$;
	$K_4 = hf(t + h, w + K_3)$.
Step 4 Set	$w = w + (K_1 + 2K_2 + 2K_3 + K_4) / 6$;
	(compute w_i)
	$t = a + ih$ (compute t_i)
Step 5	OUTPUT (t, w)
Step 6	STOP.

We will apply the algorithm above to solve the Zhou system by using Maple and MATLAB program to plot the solutions.

RESULTS AND DISCUSSIONS

Previously, Roslan *et al*. [1] have used the Euler's method for solving this Zhou system with the aid of C++ software to solve the system and MATLAB to plot the solutions of x, y and z. In this paper, we first make the comparison between the Euler's methods with different time steps. We determine the accuracy of Euler for the solution of (1) for different time steps. We set 7 decimal points for the obtained solutions. From the results presented in Table 1 we can see that the maximum error between the Euler solutions on time steps $\Delta t = 10^{-7}$ and $\Delta t = 10^{-8}$ is 0.4827420 whereas between the time steps of $\Delta t = 10^{-8}$ and $\Delta t = 10^{-9}$ the difference is 0.0053939. The difference for $\Delta t = 10^{-8}$ and $\Delta t = 10^{-9}$ is smaller than $\Delta t = 10^{-7}$ and $\Delta t = 10^{-8}$ therefore the Euler solution on the time step $\Delta t = 10^{-8}$ is sufficiently can be used as our comparison purpose.

Second, we compare the accuracy of the RK4 method with the Euler's method on the chosen time step $\Delta t = 10^{-8}$ The absolute values were used to determine the performance of RK4 against the Euler's method. In Table 2, we first find the error between the RK4 method ($\Delta t = 0.01$) and Euler ($\Delta t = 10^{-8}$). We could see clearly that

Table 1: Differences between Euler solutions for $t \in [0, 20]$

	$\Delta = \lvert EULER_{10^{-7}} - EULER_{10^{-8}}$			$\Delta = \lvert EULER_{10^{-8}} - EULER_{10^{-9}}$		
Time	*x*	*y*	*z*	*x*	*y*	*z*
2	0.0533644	0.0001797	0.0030585	0.0005399	1.7997E-05	0.0003060
4	0.1055537	0.0003590	0.0061138	0.0010798	3.599E-05	0.0006119
6	0.1565663	0.0005377	0.0091661	0.0016194	5.3977E-05	0.0009179
8	0.2064269	0.0007158	0.0122154	0.0021590	7.1958E-05	0.0012238
10	0.2551600	0.0008934	0.0152615	0.0026985	8.9936E-05	0.0015296
12	0.3027896	0.0010705	0.0183046	0.0032378	0.0001079	0.0018354
14	0.3493395	0.0012470	0.0213447	0.0037770	0.0001259	0.0021412
16	0.3948330	0.0014230	0.0243817	0.0043161	0.0001438	0.0024470
18	0.4392929	0.0015984	0.0274156	0.0048550	0.0001618	0.0027528
20	0.4827420	0.0017732	0.0304466	0.0053939	0.0001797	0.0030585

Table 2: Differences between RK4 and Euler solutions for $t \in [0, 20]$

	$\Delta = \lvert RK4_{0.01} - EULER_{10^{-8}}$			$\Delta = \lvert RK4_{0.01} - EULER_{10^{-8}}$			$\Delta = \lvert RK4_{0.01} - EULER_{10^{-8}}$		
Time	*x*	*y*	*z*	*x*	*y*	*z*	*x*	*y*	*z*
2	7.9006407	6.1585734	0.9041884	7.9006415	6.1585744	0.9041891	7.9006415	6.1585744	0.9041891
4	1.1990076	3.0650627	4.0795822	1.1990068	3.0650651	4.0795874	1.1990068	3.0650651	4.0795873
6	0.3388785	2.9060230	4.8138815	0.3388731	2.9060303	4.8138816	0.3388731	2.9060303	4.8138816
8	8.2831329	6.1423708	2.1561032	8.2831488	6.1423711	2.1561505	8.2831488	6.1423711	2.1561505
10	1.2997943	3.2923150	4.7520798	1.2998451	3.2923377	4.7522011	1.2998451	3.2923377	4.7522011
12	0.1252842	3.5261763	4.5436376	0.1253142	3.5262133	4.5435830	0.1253142	3.5262133	4.5435830
14	8.9610576	7.2377190	2.1478526	8.9610183	7.2379223	2.1473481	8.9610183	7.2379224	2.1473481
16	1.0658062	1.5678702	1.2870140	1.0657097	1.5679954	1.2867754	1.0693050	1.5680154	1.2705307
18	0.4529035	3.9653424	4.4082413	0.4531548	3.9656263	4.4075923	0.4531548	3.9656263	4.4075923
20	7.7113938	7.4295621	2.8642130	7.7039812	7.4234912	2.8810156	7.7039810	7.4234910	2.8810160

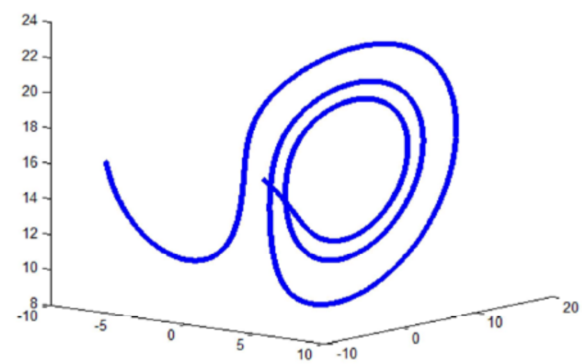

Fig. 5: Zhou's attractor when $0 \le t \le 0$

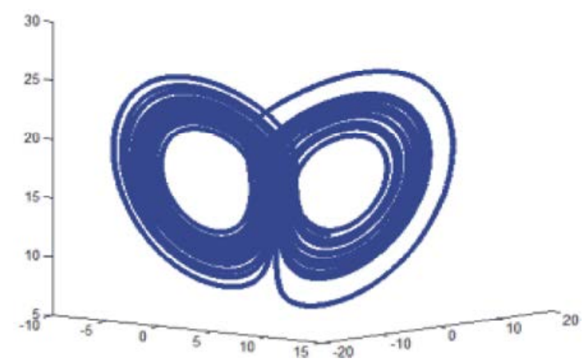

Fig. 7: Zhou's attractor when $0 \le t \le 50$

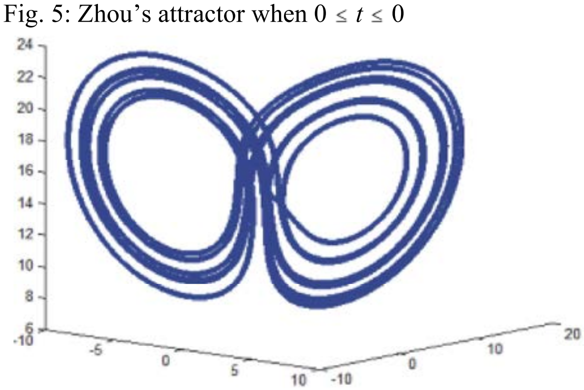

Fig. 6: Zhou's attractor when $0 \le t \le 20$

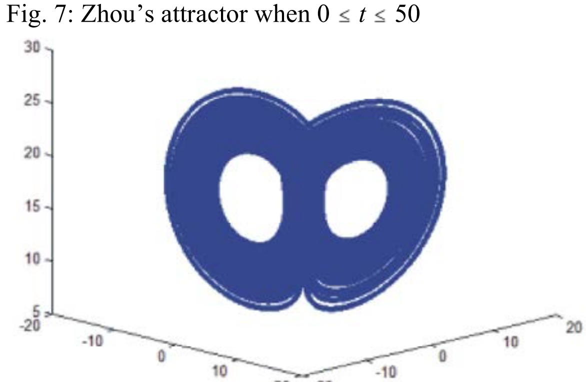

Fig. 8: Zhou's attractor when $0 \le t \le 100$

the maximum error is 7.7113938. Second, the maximum error between RK4 (Δt = 0.001) and Euler ($\Delta t = 10^{-8}$) is 7.7039812 and lastly, for RK4 (Δt = 0.0001) and Euler ($\Delta t = 10^{-8}$ the maximum difference is 7.7039810. The error is smaller for RK4 (Δt = 0.0001) and Euler ($\Delta t = 10^{-8}$). Thus, we can conclude that the accuracy of RK4 solutions can be increased by decreasing the time step.

To solve the three-dimensional system of Zhou chaotic system, we use the Maple program to run the RK4 in order to produce the values of x, y and z when the value of time, t increased. Then these values will be linked to MATLAB program to plot the solutions. The result is shown below in Figure 5 when $0 \leq t \leq 5$ Here, we choose Δt = 0.001 to solve the Zhou system. With the time steps of 0.001, this means that there are 5000 values of x, y and z. Notice that this figure has only one part of butterfly wings. This is due to the lower numbers of iterations used which are 5000.

Next, we show the effect of different ranges of t to Zhou's attractor with the same time steps; 0.001. The more the iterations used, the more the attractor become complete. By using different number of iterations, we can see how the attractor is designed and moves.

CONCLUSIONS

This paper shows that the RK4 method successfully can solve the three-dimensional Zhou chaotic system [2]. This method is used because it RK4 can obtain greater accuracy and does not need the calculation of higher derivatives. From the previous research by Roslan et al. [1], the use of Euler's method can also solve the chaotic system, but however, it is less accuracy compared to RK4 method.

Numerical comparisons have been made between the Runge-Kutta of order four (RK4) and the Euler's method for different time steps. This is done to determine the accuracy of this method with corresponding to time steps. It has been observed that the accuracy of RK4 solutions can be increased by decreasing the time step.

An algorithm for RK4 method is used to solve the initial-value problem for ordinary differential equation of the Zhou chaotic system. This method produces the values of x, y and z. We use the Maple program to solve this system and MATLAB to plot the solutions. The results are given for different ranges of t. By using different number of iterations, we can see how the attractor is designed and moves.

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