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On Coefficient Bounds within Linear Operators

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Abstract: Wthin the differential operator $\nabla_w^k f(z)=f^{(k-1)}(z)+\beta(z-w)f^{(k)}(z)$ a more generalization of subclasses of analytic functions *f* in the open disc $\mathcal{U}_w=\{z:|z-w|<1\}$ is introduced. Coefficient bounds of w-p-valent subclasses within the differential operator ∇_w^k are calculated. Some properties of the growth, distortion and quasi-Hadmard product are obtained.

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INTRODUCTION

Many subclasses of p-valent univalent analytic functions have been studied since the last century by many authors and there are more than one thousand of research papers written on the subject some of the latest of them are [1, 2, 4, 5, 7, 9-15]. The study of p-valent univalent mappings is a fairly recent area of research. So, it is natural to consider the properties of analytic univalent functions as a starting point for our study of p-valent univalent mappings. A general theme will be "What are the properties of p-valent analytic functions that are still powerful within differential operators to solve some physical problems?". It is known that the functions within differential operators that arise in physical problems are generally nonlinear, therefore the geometry functions theory and conformal mappings provides a powerful tool for obtaining the solutions of these problems which were difficult to solve by any other methods.
Let A(p) be the class of functions

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{n+p}z^{n+p},\quad a_{n+p}\geq 0 \text{ and } p,n\in\mathbb{N}=\{1,2,3,4,...\}$$

which are analytic in $\mathcal{U}=\{z:|z|<1\}$ and the subclass of all p-valent starlike functions of order α in A(p) is defined to be

$$T(p,\alpha)=\{f(z)\in A(p):\mathrm{Re}\{\frac{zf'(z)}{f(z)}\}>\alpha,\ \alpha\in[0,\frac{1}{p}),\text{and } z\in\mathcal{U}\}$$

Among the last twelve years many authors have studied many valuable and interesting results of univalent functions, even with various generalizations as appeared in many literatures. For a fixed point w in the unit disc $\mathcal{U}$, Kanas and Ronning [4] introduced a more generalization form of analytic functions in the unit disc $\mathcal{U}$ of the form

$$f(z)=(z-w)+\sum_{k=2}^{\infty}a_k(z-w)^k,\ a_k\in\mathcal{C}$$

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which are denoted by $\Gamma(w)$, ST(w) and CV(w) and they obtained some results related to the other univalent functions. Acu and Owa [5] introduced bounds for classes of ω-close-to-convex functions, ω-α-convex functions and other further studies of these classes. Al-Kasasbeh and Darus [6, 8] introduced classes of analytic univalent functions that are defined in the open disc $\mathcal{U}_w=\{z:|z-w|<1\}$ and they proved corresponding results to these classes.

In this paper, for any complex number w in the open disc $\mathcal{U}_w=\{z:|z-w|<1\}$, we define the classes $\mathcal{A}_w(p)$ to be the w-p-valent analytic functions

$$f(z)=(z-w)^p-\sum_{n=1}^{\infty}a_{n+p}(z-w)^{n+p},\quad a_{n+p}\geq 0 \text{ and } p,n\in\mathbb{N}=\{1,2,3,4,...\}$$

and in the open disc $\mathcal{U}_w$, we define the subclass $ST^w(p,\lambda)$ of $\mathcal{A}_w(p)$ which consists of w-p-valent starlike functions of order λ as follows:

$$ST^w(p,\lambda)=\{f(z)\in A_w(p):\operatorname{Re}\{\frac{(z-w)f'(z)}{f(z)}\}>\lambda,\ 0\leq\lambda<\frac{1}{p}, \text{and } z\in\mathcal{U}_w\}$$

For a nonnegative parameter β and $k\in\mathbb{N}$, Al-Kasasbeh and Darus [6, 8] defined the linear differential operator ∇_w^k for an analytic function *f* in the open disc $\mathcal{U}_w=\{z:|z-w|<1\}$ to be

$$\nabla_w^k f(z)=f^{(k-1)}(z)+\beta(z-w)f^{(k)}(z)$$

where

$$f^{(0)}(z)=f(z),\quad \nabla_w^1 f(z)=f(z)+\beta(z-w)f'(z)$$

$$\nabla_w^2 f(z)=f'(z)+(z-w)\beta f''(z)$$

and so on

and proved the follows investigated properties.

Remark 1.1: [6]. Let *f* be analytic in $\mathcal{U}_w=\{z:|z-w|<1\}$ and $f\in A_w(p)$ then,

(1) If $\operatorname{Re}\{\nabla_w^1 f(z)\}>0$ then $\operatorname{Re}\{f(z)\}>0$ and $\operatorname{Re}\{f'(z)\}>0$

(2) If $\operatorname{Re}\{\nabla_w^k f(z)\}>0$ then $\operatorname{Re}\{f^{(k-1)}(z)\}>0$ and $\operatorname{Re}\{f^{(k)}(z)\}>0$

(3) If $\operatorname{Re}\{\nabla_w^k f(z)\}>\alpha$ then $\operatorname{Re}\{f^{(k)}(z)\}>\alpha$, and $\operatorname{Re}\{f^{(k-1)}(z)\}>\alpha$

In this article, the linear differential operator ∇_w^k within the class of w-p-valent analytic functions is introduced as follows.

Definition 1.2: Let $f(z)\in\mathcal{A}_w(p)$ for $z\in\mathcal{U}_w$ and $0\leq\lambda<\frac{1}{p}$. Then

$$f(z)\in ST\nabla_w^k(p,\beta,\lambda) \text{ if and only if } \operatorname{Re}\left\{\frac{\nabla_w^k f(z)}{f^{(k-1)}(z)}-1\right\}>\lambda$$

The class $ST\nabla_w^k(p,\beta,\lambda)$ is a generalization of various subclasses of univalent functions, it is easy to note that if $k=1$, $w=0$, $\lambda=\alpha$, $\beta=1$ and $p=1$, then $ST\nabla_0^1(1,0,\lambda)\equiv\mathcal{T}^*(\lambda)$ due to Silverman [1] in the unit disc $\mathcal{U}_0$. Also, if $k=1$, $w=0$, $\lambda=\alpha$, $\beta=1$ and $0\leq\lambda<\frac{1}{p}$, then $ST\nabla_0^1(p,0,\lambda)\equiv\mathcal{T}_k^*(p,n)$ due to Owa [2, 3] in the unit disc $\mathcal{U}_0$.

THE MAIN RESULTS

The coefficient bounds for a function f in the class $ST\nabla_w^k(p,\beta,\lambda)$ which is analytic in the open disc $\mathcal{U}_w$ are estimated and calculated.

Theorem 2.1: A function $f(z) \in \mathcal{A}_w(p)$ is in the class $ST\nabla_w^k(p,\beta,\lambda)$ if and only if

$$a_{n+p} \le \frac{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]}{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]} \tag{2.1}$$

Proof: Since $f(z) \in ST\nabla_w^k(p,\beta,\lambda)$, for $0 \le \lambda < \frac{1}{p}$ and $z \in \mathcal{U}_w$, then

$$\operatorname{Re}\left(\frac{\nabla_w^k f(z)}{f^{(k-1)}(z)} - 1\right) = \operatorname{Re}\left(\frac{\beta(z-w)f^{(k)}(z))}{f^{(k-1)}(z)}\right) > \lambda$$

Hence

$$\beta\left(\prod_{i=1}^{k}(p-i)(z-w)^{p-k+1} - \sum_{n=1}^{\infty} a_{n+p} \prod_{i=1}^{k}(p+n+1-i)(z-w)^{p+n-k}\right)$$
$$> \lambda\left(\prod_{i=1}^{k-1}(p+1-i)(z-w)^{p-k+2} - \sum_{n=1}^{\infty} a_{n+p} \prod_{i=1}^{k-1}(p+n+1-i)(z-w)^{p+n-k+2}\right)$$

Assume that (z-w) approaches to 1, since it is observed that |z-w|<1 in the disc $\mathcal{U}_w$, so that

$$\prod_{i=1}^{k}(p+1-i)[(p-k+1)\beta-\lambda] \ge \sum_{n=1}^{\infty} a_{n+p}\{\prod_{i=1}^{k}(p+n+1-i)[(p+n-k+1)\beta-\lambda]\}$$

For $0 \le \lambda < \frac{1}{p}$, the result is obtained and

$$a_{n+p} \le \frac{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]}{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]} \tag{2.2}$$

To prove the necessity (the converse direction), assume by contradiction that f isn't belong to the class $ST\nabla_w^k(p,\beta,\lambda)$, then by the definition, for $0 \le \lambda < \frac{1}{p}$,

$$\operatorname{Re}\left\{\frac{(z-w)\nabla_w^k f(z)}{f^{(k-1)}(z)} - 1\right\} < \frac{1}{p}$$

Hence

$$\left(\prod_{i=1}^{k-1}(p+1-i)(z-w)^{p-k+2} - \sum_{n=1}^{\infty} a_{n+p} \prod_{i=1}^{k-1}(p+n+1-i)(z-w)^{p+n-k+2}\right)$$
$$> p\beta\left(\prod_{i=1}^{k}(p-i+1)(z-w)^{p-k+1} - \sum_{n=1}^{\infty} a_{n+p} \prod_{i=1}^{k}(p+n+1-i)(z-w)^{p+n-k}\right)$$

Suppose that r = |z-w| approaches to 1, then

$$\prod_{i=1}^{k}(p+1-i)[(p-k+1)\beta-\frac{1}{p}] > \sum_{n=1}^{\infty} a_{n+p}\prod_{i=1}^{k}(p+n+1-i)[(p+n-k+1)\beta-\frac{1}{p}]$$

for $0 \le \lambda < \frac{1}{p}$, which implies that

$$a_{n+p} > \frac{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-1]p}{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-1]p} \quad (2.3)$$

and this is impossible and contradiction to our assumption for $0 \le \lambda < \frac{1}{p}$

$$a_{n+p} \le \frac{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]}{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]} \quad (2.4)$$

Thus $f(z) \in ST\nabla_{w}^{k}(p,\beta,\lambda)$ and the result is sharp for

$$f(z)=(z-w)^{p} - \frac{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]}{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]}(z-w)^{p+n}$$

Next, the distortion and growth properties are discussed.

Theorem 2.2: Let $f(z) \in ST\nabla_{w}^{k}(p,\beta,\lambda)$, for $z \in \mathcal{U}_{w} = \{z : r = |z-a| < 1\}$, then

$$r^{p} - (\frac{\beta-\lambda}{2(2\beta-\lambda)})r^{p+1} \le |f(z)| \le r^{p} + (\frac{\beta-\lambda}{2(2\beta-\lambda)})r^{p+1}$$

and

$$p\,r^{p-1} - (\frac{\beta-\lambda}{2(2\beta-\lambda)})(p+1)r^{p} \le |f'(z)| \le p\,r^{p-1} + (\frac{\beta-\lambda}{2(2\beta-\lambda)})(p+1)r^{p}$$

with quality for

$$f(z)=(z-w)^{p} - (\frac{\beta-\lambda}{2(2\beta-\lambda)})(z-w)^{p+1}$$

Proof: For p = n = 1 in (2.1), we have

$$\sum_{n=1}^{\infty} a_{n+p} \le \frac{\beta-\lambda}{2(2\beta-\lambda)} \quad (5)$$

Thus

$$|f(z)| \le r^{p} + \sum_{n=1}^{\infty} a_{n+p} r^{p+1} \le r^{p} + r^{p+1}\sum_{n=1}^{\infty} a_{n+p} \le r^{p} + (\frac{\beta-\lambda}{2(2\beta-\lambda)})r^{p+1}$$

and

$$|f(z)| \ge r^{p} - \sum_{n=1}^{\infty} a_{n+p} r^{p+1} \ge r^{p} - r^{p+1}\sum_{n=1}^{\infty} a_{n+p} \ge r^{p} - (\frac{\beta-\lambda}{2(2\beta-\lambda)})r^{p+1}$$

Also, from (2.2) and Theorem 2.1, it follows that

$$\sum_{n=1}^{\infty}(p+n)a_{n+p} \le \frac{\beta-\lambda}{2(2\beta-\lambda)}$$

For r = |z-w|<1, we have

$$|f'(z)| \le p|z-w|^{p-1}+\sum_{n=1}^{\infty}(p+1)a_{n+p}|z-w|^{p} \le pr^{p-1}+(p+1)r^{p+1}\sum_{n=1}^{\infty}a_{n+p} \le pr^{p-1}+(\frac{\beta-\lambda}{2(2\beta-\lambda)})(p+1)r^{p}$$

and

$$|f'(z)| \ge p|z-w|^{p-1}-\sum_{n=1}^{\infty}(p+1)a_{n+p}|z-w|^{p} \ge pr^{p-1}-(p+1)r^{p}\sum_{n=1}^{\infty}a_{n+p} \ge pr^{p-1}-(\frac{\beta-\lambda}{2(2\beta-\lambda)})(p+1)r^{p}$$

This complete the proof of the theorem.

Lemma 2.3: Let *f* and g be two functions in the class $ST\nabla_{w}^{k}(p,\beta,\lambda)$, such that

$$f(z)=(z-w)^{p}-\sum_{n=1}^{\infty}a_{n+p}(z-w)^{n+p}, \quad a_{n+p} \ge 0 \text{ and } g(z)=(z-w)^{p}-\sum_{n=1}^{\infty}b_{n+p}(z-w)^{n+p}, b_{n+p} \ge 0$$

Then

$$G(z)=(1-\eta)g(z)+\eta f(z)=(z-w)^{p}-\sum_{n=1}^{\infty}C_{n+p}(z-w)^{n+p}$$

is belong to the class $ST\nabla_{w}^{k}(p,\beta,\lambda)$, for $0\le\eta\le1$.

Proof: Since *f* and $g \in ST\nabla_{w}^{k}(p,\beta,\lambda)$, for $0\le\lambda\le1$ then

$$a_{n+p} \le \frac{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]}{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]} \text{ and } b_{n+p} \le \frac{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]}{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]}$$

So that

$$(1-\eta)a_{n+p}+\eta b_{n+p} \le \eta(\frac{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]}{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]})+(1-\eta)(\frac{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]}{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]})$$

$$=\frac{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]}{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]}$$

Assume that

$$C_{n+p}=\eta a_{n+p}+(1-\eta)b_{n+p} \le \frac{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]}{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]}$$

Then

$$G(z)=(z-w)^{p}-\sum_{n=1}^{\infty}C_{n+p}(z-w)^{n+p}$$

is in the class $ST\nabla_{w}^{k}(p,\beta,\lambda)$.

Lemma 2.4: If λ_1 and $\lambda_2 \in [0,1]$ such that $\lambda_1<\lambda_2$, then

$$\{ST\nabla_w^k(p,\beta,\lambda_1)\} \subseteq \{ST\nabla_w^k(p,\beta,\lambda_2)\}$$

Proof: Assume that $f(z) \in ST\nabla_w^k(p,\beta\ \lambda_1)$ then

$$\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda_1]}{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda_1]} a_{n+p} \leq 1$$

And since $\lambda_1 < \lambda_2$, then

$$\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda_2]}{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda_2]} a_{n+p} \leq \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda_1]}{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda_1]} a_{n+p} \leq 1$$

Thus $f(z) \in ST\nabla_w^k(p,\beta,\lambda_2)$.

Theorem 2.5: Suppose that $f_1(z)=(z-w)^p$ and for each positive integer $n \geq 2$,

$$f_n(z)=(z-w)^p - \frac{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]}{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]}(z-w)^{n+p}, \quad \text{for } z \in \mathcal{U}_w$$

Then $f \in ST\nabla_w^k(p,\beta,\lambda)$ if and only if *f* can be expressed in the form $f(z)=\sum_{n=0}^{\infty}\delta_{n+1} f_{n+1}(z)$ where $\delta_n \geq 0$ and $\sum_{n=0}^{\infty}\delta_{n+1}=1$.

Proof: Assume that

$$f(z)=\sum_{n=0}^{\infty}\delta_{n+1} f_{n+1}(z)=(z-w)^p - \sum_{n=1}^{\infty}\delta_{n+1} a_{n+p}(z-w)^{n+p}$$

Since

$$\sum_{n=1}^{\infty}\delta_{n+1}\left(\frac{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]}{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]}\right)\left(\frac{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]}{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]}\right) \leq \sum_{n=1}^{\infty}\delta_{n+1} \leq 1-\delta_1 \leq 1$$

By Theorem 2.1 $f \in ST\nabla_w^k(p,\beta,\lambda)$.

Conversely, let $f \in ST\nabla_w^k(p,\beta,\lambda)$. Then

$$a_{n+p} \leq \frac{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]}{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]}, \quad \text{for } n \geq 1$$

Without any loss of generality, assume that

$$\delta_{n+1} = \frac{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]}{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]} a_{n+p}, \quad \text{for } n \geq 1$$

and $\delta_1 = 1 - \sum_{n=1}^{\infty}\delta_{n+1}$. Then

$$f(z) = (z-w)^p - \sum_{n=1}^{\infty}\delta_{n+1}a_{n+p}(z-w)^{n+p}$$

$$= (z-w)^p - \sum_{n=1}^{\infty}\delta_{n+1}\frac{\prod_{i=1}^{k}(p+1-i)[\beta(p-k+1)-\lambda]}{\prod_{i=1}^{k}(p+n+1-i)[\beta(p+n-k+1)-\lambda]}(z-w)^{n+p}$$

$$= (z-w)^p - \sum_{n=1}^{\infty}\delta_{n+1}[(z-w)^{n+p} - f_{n+1}(z)] = (1-\sum_{n=1}^{\infty}\delta_{n+1})(z-w)^{n+p} + \sum_{n=1}^{\infty}\delta_{n+1}f_{n+1}(z)$$

$$= \delta_1(z-w)^{n+p} + \sum_{n=1}^{\infty}\delta_{n+1}f_{n+1}(z) = \delta_1 f_1(z) + \sum_{n=1}^{\infty}\delta_{n+1}f_{n+1}(z) = \sum_{n=1}^{\infty}\delta_n f_n(z)$$

Let

$$f(z) = (z-w)^p - \sum_{n=1}^{\infty}a_{n+p}(z-w)^{n+p} \text{ and } g(z) = (z-w)^p - \sum_{n=1}^{\infty}b_{n+p}(z-w)^{n+p}$$

for $z \in \mathcal{U}_w$. Then the quasi-Hadamard product of f and g is denoted by $f*g$ and is defined to be

$$(f*g)(z) = (z-w)^p - \sum_{n=1}^{\infty}a_{n+p}b_{n+p}(z-w)^{n+p}$$

The quasi-Hadmard product results are presented as follows.

Theorem 2.6: Let $\hat{\beta} = \min\{\beta_1, \beta_2\} > 0$ and suppose that $f(z) \in ST\nabla_w^k(p,\beta_1,\lambda)$ and $g(z) \in ST\nabla_w^k(p,\beta_2,\lambda)$ for $z \in \mathcal{U}_w$. Then $(f*g)(z) \in ST\nabla_w^k(p,\hat{\beta},\lambda)$.

Proof: Since $f(z) \in ST\nabla_w^k(p,\beta_1,\lambda)$, we get

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)[\beta_1(p+n-k+1)-\lambda]}{\prod_{i=1}^{k}(p+1-i)[\beta_1(p-k+1)-\lambda]}a_{n+p} \le 1$$

and since $g(z) \in ST\nabla_w^k(p,\beta_2,\lambda)$, then

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)[\beta_2(p+n-k+1)-\lambda]}{\prod_{i=1}^{k}(p+1-i)[\beta_2(p-k+1)-\lambda]}b_{n+p} \le 1$$

So that

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)\left((p+n-k+1)\hat{\beta}-\lambda\right)}{\prod_{i=1}^{k}(p+1-i)\left((p-k+1)\hat{\beta}-\lambda\right)}a_{n+p}b_{n+p} \le \max\{\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)\left((p+n-k+1)\beta_1-\lambda\right)}{\prod_{i=1}^{k}(p+1-i)\left((p-k+1)\beta_1-\lambda\right)}a_{n+p}$$

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)\left((p+n-k+1)\beta_2-\lambda\right)}{\prod_{i=1}^{k}(p+1-i)\left((p-k+1)\beta_2-\lambda\right)}b_{n+p}\} \le 1$$

Hence $(f*g)(z) \in ST\nabla_w^k(p,\hat{\beta},\lambda)$.

Theorem 2.7: Let $\hat{\beta}=\min\{\beta_1,\beta_2...,\beta_m\}>0$ and suppose that $f_i(z)\in ST\nabla_w^k(p,\beta_i,\lambda)$, for i = 1,2,3,…,m. Then $(f_1 * f_2 * \ldots * f_m)(z)\in ST\nabla_w^k(p,\hat{\beta},\lambda)$.

Proof: It is clear the result is true for m = 2 by Theorem 2.6. Let $\hat{\beta}=\min\{\beta_1,\beta_2...,\beta_m\}$ and $\beta^*=\min\{\hat{\beta},\beta_{m+1}\}$. Suppose that the result is true for any positive integer m and $f_i(z)\in ST\nabla_w^k(p,\beta_i,\lambda)$, for each i = 1,2,3,…,m+1, then

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)\big((p+n-k+1)\beta_i-\lambda\big)}{\prod_{i=1}^{k}(p+1-i)\big((p-k+1)\beta_i-\lambda\big)}b_{n+p,i}\le 1$$

So that

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)\big((p+n-k+1)\beta^*-\lambda\big)}{\prod_{i=1}^{k}(p+1-i)\big((p-k+1)\beta^*-\lambda\big)}(b_{n+p,1})(b_{n+p,2})(b_{n+p,3}).....(b_{n+p,m})(b_{n+p,m+1})\le$$

$$\max\{\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)\big((p+n-k+1)\hat{\beta}-\lambda\big)}{\prod_{i=1}^{k}(p+1-i)\big((p-k+1)\hat{\beta}-\lambda\big)}(b_{n+p,1})(b_{n+p,2})(b_{n+p,3}).....(b_{n+p,m})$$

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)\big((p+n-k+1)\beta_{m+1}-\lambda\big)}{\prod_{i=1}^{k}(p+1-i)\big((p-k+1)\beta_{m+1}-\lambda\big)}b_{n+p,m+1}\}\le \max\{\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)\big((p+n-k+1)\beta_1-\lambda\big)}{\prod_{i=1}^{k}(p+1-i)\big((p-k+1)\beta_1-\lambda\big)}b_{n+p,1},$$

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)\big((p+n-k+1)\beta_2-\lambda\big)}{\prod_{i=1}^{k}(p+1-i)\big((p-k+1)\beta_2-\lambda\big)}b_{n+p,2}\ ,....,\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)\big((p+n-k+1)\beta_m-\lambda\big)}{\prod_{i=1}^{k}(p+1-i)\big((p-k+1)\beta_m-\lambda\big)}b_{n+p,m}$$

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)\big((p+n-k+1)\beta_{m+1}-\lambda\big)}{\prod_{i=1}^{k}(p+1-i)\big((p-k+1)\beta_{m+1}-\lambda\big)}b_{n+p,m+1}\}\le 1$$

This shows that the result is true for m+1. Therefore, by mathematical induction, the result is true for any positive integer m. Hence $(f_1 * f_2 ....* f_m)(z)\in ST\nabla_w^k(p,\hat{\beta},\lambda)$.

Theorem 2.8: Suppose that $f(z)$ and $g(z)\in ST\nabla_w^k(p,\beta,\lambda)$, for $z\in\mathcal{U}_w$. Then

$$(f*g)(z)\in ST\nabla_w^k(p,\beta,\lambda^*),\text{for } \lambda^*=1-\lambda\left(\frac{\lambda-\beta}{\lambda-2\beta}\right)$$

Proof: From Theorem 2.1 we observe that

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)\big((p+n-k+1)\beta-\lambda\big)}{\prod_{i=1}^{k}(p+1-i)\big((p-k+1)\beta-\lambda\big)}a_{n+p}\le 1 \qquad (2.3)$$

and

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)\big((p+n-k+1)\beta-\lambda\big)}{\prod_{i=1}^{k}(p+1-i)\big((p-k+1)\beta-\lambda\big)}b_{n+p}\le 1 \qquad (2.4)$$

Assume that $\lambda^* \in [0,1]$ such that

$$\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{k}(p+n+1-i)\left((p+n-k+1)\beta-\lambda^*\right)}{\prod_{i=1}^{k}(p+1-i)\left((p-k+1)\beta-\lambda^*\right)} a_{n+p}\, b_{n+p} \le 1$$

Apply Cauchy-Schwarz inequality on (2.3)and (2.4)to have

$$\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{k}(p+n+1-i)\left((p+n-k+1)\beta-\lambda\right)}{\prod_{i=1}^{k}(p+1-i)\left((p-k+1)\beta-\lambda\right)} \sqrt{a_{n+p}\, b_{n+p}} \le 1$$

Now, $\sqrt{a_{n+p} b_{n+p}} \le 1-\lambda^*$ if and only if

$$\frac{\prod_{i=1}^{k}(p+n+1-i)\left((p+n-k+1)\beta-\lambda\right)}{\prod_{i=1}^{k}(p+1-i)\left((p-k+1)\beta-\lambda\right)} \le 1-\lambda^*$$

It is observed that

$$\varphi(n) = \frac{\prod_{i=1}^{k}(p+1-i)\left((p-k+1)\beta-\lambda\right)}{\prod_{i=1}^{k}(p+n+1-i)\left((p+n-k+1)\beta-\lambda\right)}$$

is a decreasing quantity, since

$$\varphi(n) = \frac{\prod_{i=1}^{k}(p+1-i)\left((p-k+1)\beta-\lambda\right)}{\prod_{i=1}^{k}(p+n+1-i)\left((p+n-k+1)\beta-\lambda\right)} \le \varphi(2)$$

Which implies that the bound is sharp and

$$\lambda^* = 1-\left(\frac{3(\lambda-\beta)}{2(\lambda-2\beta)}\right)$$

Theorem 2.9: Suppose that $f_i(z) \in ST\nabla_w^k(p,\beta,\lambda)$, for each i = 1,2,…,m. Then

$$(f_1 * f_2 * \ldots * f_m)(z) \in ST\nabla_w^k(p,\beta,\lambda^*), \quad \text{for} \quad \lambda^* = 1-\left(\frac{3(\lambda-\beta)}{2(\lambda-2\beta)}\right)$$

Proof: From Theorem 2.6 it is clear the result is true for m = 2. Suppose that the result is true for any positive integer m and $f_i(z) \in ST\nabla_w^k(p,\beta,\lambda)$, for each i = 1,2,3,…,m+1, then

$$\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{k}(p+n+1-i)\left((p+n-k+1)\beta-\lambda\right)}{\prod_{i=1}^{k}(p+1-i)\left((p-k+1)\beta-\lambda\right)} b_{n+p,i} \le 1 \qquad (2.5)$$

Assume that $\lambda^* \in [0,1]$ such that $\lambda^* = 1-\left(\frac{3(\lambda-\beta)}{2(\lambda-2\beta)}\right)$. Then

$$\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{k}(p+n+1-i)\left((p+n-k+1)\beta-\lambda^*\right)}{\prod_{i=1}^{k}(p+1-i)\left((p-k+1)\beta-\lambda^*\right)}(b_{n+p,1})(b_{n+p,2})\dots(b_{n+p,m})(b_{n+p,m+1}) \le 1$$

Apply Cauchy-Schwarz inequality on (2.5)to have

$$\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{k}(p+n+1-i)\left((p+n-k+1)\beta-\lambda\right)}{\prod_{i=1}^{k}(p+1-i)\left((p-k+1)\beta-\lambda\right)}\sqrt{(b_{n+p,1})(b_{n+p,2})\dots(b_{n+p,m})(b_{n+p,m+1})} \le 1$$

Now, $\sqrt{(b_{n+p,1})(b_{n+p,2})\dots\dots(b_{n+p,m})(b_{n+p,m+1})} \le 1-\lambda^*$ if and only if

$$\frac{\prod_{i=1}^{k}(p+1-i)\left((p-k+1)\beta-\lambda\right)}{\prod_{i=1}^{k}(p+n+1-i)\left((p+n-k+1)\beta-\lambda\right)} \le 1-\lambda^*$$

It is observed that

$$\varphi(n) = \frac{\prod_{i=1}^{k}(p+1-i)\left((p-k+1)\beta-\lambda\right)}{\prod_{i=1}^{k}(p+n+1-i)\left((p+n-k+1)\beta-\lambda\right)}$$

is a decreasing quantity, since

$$\varphi(n) = \frac{\prod_{i=1}^{k}(p+1-i)\left((p-k+1)\beta-\lambda\right)}{\prod_{i=1}^{k}(p+n+1-i)\left((p+n-k+1)\beta-\lambda\right)} \le \varphi(2)$$

Therefore, by mathematical induction, the result is true for any positive integer m. Hence

$$(f_1 * f_2 \dots * f_m)(z) \in ST\nabla_w^k(p,\beta,\lambda^*) \text{ for } \lambda^* = 1-\left(\frac{3(\lambda-\beta)}{2(\lambda-2\beta)}\right)$$

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