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Construction of Improved Runge-Kutta Nystrom Method for Solving Second-Order Ordinary Differential Equations

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Abstract: Improved Runge-Kutta Nystrom (IRKN) method for the numerical solution of second-order ordinary differential equations is constructed. The scheme arises from the classical Runge-Kutta Nystrom method also can be considered as two step method. IRKN methods require less number of stages which lead to less number of function evaluations per step, compared with the existing Runge-Kutta Nystrom (RKN) methods. Therefore, the methods are computationally more efficient at achieving the higher order of local accuracy. The algebraic order conditions of the method using the Taylor's series expansion are obtained and the methods of order 3, 4 and 5 are derived. The stability properties of method are discussed and numerical examples are given to show the efficiency of the proposed methods compared to the existing RKN methods.

Key words: Improved Runge-Kutta Nystrom method • Runge-Kutta Nystrom method • second-order ordinary differential equations • algebraic order conditions

INTRODUCTION

The special second-order ordinary differential equations are given by:

$$y'' = f(x,y),\ \ y(x_0) = y_0,\ y'(x_0) = y'_0 \tag{1}$$

The second-order ordinary differential equations (1), can be reduced into system of first order ordinary differential equations (ODEs) or can be solved by using Runge-Kutta Nystrom (RKN) methods or multistep methods, directly. Udwadia and Farahani [1] proposed the Accelerated Runge-Kutta (ARK) methods for solving autonomous first order ODEs. Rabiei *et al*. [2] developed the Accelerated Runge-Kutta Nystrom (ARKN) methods for solving special second-order autonomous ordinary differential equations in form of $y'' = f(y)$. Rabiei and Ismail [3, 4] by improving the ARK methods for solving general form of ordinary differential equations, constructed the Improved Runge-Kutta method for solving first order ODEs. In this paper, we developed the Improved Runge-Kutta Nystrom (IRKN) method for solving second-order ODEs in form of $y'' = f(x,y)$.

The third-order Improved Runge-Kutta Nystrom method used only 2 stages while there is not any existing Runge-Kutta Nystrom method with two stages. Also the fourth and fifth order IRKN methods used 3 and 4 stages, respectively.

In section 2, the general form of IRKN methods is constructed and the order conditions of method using Taylor's series expansion are obtained in section 3. In section 4, the derivation of the method is given followed by the stability of the method in section 5. The number of tested problems to show the efficiency of the methods compared with the existing RKN methods, are given in the last section.

CONSTRUCTION OF METHOD

Consider the IRK method with s-stages from [3, 4] as follows

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$$y_{n+1} = y_n + h(b_1 k_1 - b_{-1} k_{-1} + \sum_{i=2}^{s} b_i (k_i - k_{-i})), \quad 1 \le n \le N-1 \tag{2}$$

$$k_1 = f(x_n, y_n)$$

$$k_{-1} = f(x_{n-1}, y_{n-1})$$

$$k_i = f(x_n + c_i h, y_n + h\sum_{j=1}^{i-1} a_{ij} k_j)$$

$$k_{-i} = f(x_{n-1} + c_i h, y_{n-1} + h\sum_{j=1}^{i-1} a_{ij} k_{-j})$$

for $2 \le i \le s$, where h is the fixed step size. Following the approach discussed by Dormand on the derivation of RKN method [5] and based on IRK method [3, 4] we developed the IRKN method for solving the second-order ODEs directly. Consider the IRKN method as follows:

$$y_{n+1} = y_n + h(b_1 f_1 - b_{-1} f_{-1} + \sum_{i=2}^{s} b_i (f_i - f_{-i})) \tag{3}$$

$$y'_{n+1} = y'_n + h(b_1 g_1 - b_{-1} g_{-1} + \sum_{i=2}^{s} b_i (g_i - g_{-i})) \tag{4}$$

where

$$g_1 = g(x_n, y_n), \; g_{-1} = g(x_{n-1}, y_{n-1}), \quad f_1 = y'_n, \; f_{-1} = y'_{n-1}$$

for $2 \le i \le s$. We have:

$$g_i = g(x_n + c_i h, y_n + h\sum_{k=1}^{i-1} a_{ik} f_k), g_{-i} = g(x_{n-1} + c_i h, y_{n-1} + h\sum_{k=1}^{i-1} a_{ik} f_{-k}) \tag{5}$$

$$f_i = y'_n + h\sum_{j=1}^{i-1} a_{ij} g_j, \; f_{-i} = y'_{n-1} + h\sum_{j=1}^{i-1} a_{ij} g_{-j} \tag{6}$$

Substituting (6) into (5) give us

$$g_i = g(x_n + c_i h, y_n + h\sum_{k=1}^{i-1} a_{ik} (y'_n + h\sum_{j=1}^{k-1} a_{kj} g_j)) \tag{7}$$

$$g_{-i} = g(x_{n-1} + c_i h, y_{n-1} + h\sum_{k=1}^{i-1} a_{ik} (y'_{n-1} + h\sum_{j=1}^{k-1} a_{kj} g_{-j})) \tag{8}$$

Using the row sum condition for RK method $c_i = \sum_{j=1}^{i-1} a_{ij}$ and doing some simplifications we have:

$$g_i = g(x_n + c_i h, y_n + h c_i y'_n + h^2 \sum_{j=1}^{i-1} \bar{a}_{ij} g_j)) \tag{9}$$

$$g_{-i} = g(x_{n-1} + c_i h, y_{n-1} + h c_i y'_{n-1} + h^2 \sum_{j=1}^{i-1} \bar{a}_{ij} g_{-j})) \tag{10}$$

Where $\bar{a}_{ij} = \sum_{k=1}^{s} a_{ik} a_{kj}$, for j = 1,…,i-1 and i = 2,…,s. By substituting (6) into (3) we obtained

$$y_{n+1} = y_n + h(b_1 y'_n - b_{-1} y'_{n-1} + \sum_{i=2}^{s} b_i \{ (y'_n + h\sum_{j=1}^{i-1} a_{ij} g_j) - (y'_{n-1} + h\sum_{j=1}^{i-1} a_{ij} g_{-j}) \})$$

$$= y_n + h(b_1 y'_n - b_{-1} y'_{n-1} + \sum_{i=2}^{s} b_i y'_n - \sum_{i=2}^{s} b_i y'_{n-1} + h\sum_{i=2}^{s} b_i (\sum_{j=1}^{i-1} a_{ij} g_j - \sum_{j=1}^{i-1} a_{ij} g_{-j}))$$

$$y_{n+1} = y_n + h((b_1 + \sum_{i=2}^{s} b_i) y'_n - (b_{-1} + \sum_{i=2}^{s} b_i) y'_{n-1} + h\sum_{i=2}^{s} b_i \sum_{j=1}^{i-1} a_{ij} (g_j - g_{-j}))$$

Using the order conditions for first and second order IRK methods [3, 4] we obtained:

$$y_{n+1} = y_n + \frac{3h}{2} y'_n - \frac{h}{2} y'_{n-1} + h^2 \sum_{i=2}^{s} \overline{b}_i (g_i - g_{-i}) \tag{11}$$

where

$$\overline{b}_{i+1} = \sum_{j=2}^{s} b_j a_{ji}, \ i = 1,...,s$$

Because the special second order ODEs does not include the y' therefore for some simplification we denoted the $\overline{a}_{ij}$ by a_{ij} in formulas (9) and (10). Also we replaced g_i with k_i in formulas (11). Therefore the general form of explicit IRKN method with s-stages is given by:

$$y_{n+1} = y_n + \frac{3h}{2} y'_n - \frac{h}{2} y'_{n-1} + h^2 \sum_{i=2}^{s} \overline{b}_i (k_i - k_{-i})$$

$$y'_{n+1} = y'_n + h(b_1 k_1 - b_{-1} k_{-1} + \sum_{i=2}^{s} b_i (k_i - k_{-i})) \tag{12}$$

$$k_1 = f(x_n, y_n), \qquad k_{-1} = f(x_{n-1}, y_{n-1})$$

$$k_i = f(x_n + c_i h, y_n + h c_i y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} k_j)$$

$$k_{-i} = f(x_{n-1} + c_i h, y_{n-1} + h c_i y'_{n-1} + h^2 \sum_{j=1}^{i-1} a_{ij} k_{-j}))$$

for i = 2,…,s. It is convenient to represent the coefficients of explicit IRKN method from equation (12) by Table 1.

However, the method of transformation forces $a_{i,i-1} = 0, \overline{b}_s = 0$, conditions which need not be imposed on the formulas (9), (10) and (11). The formulae for a_{ij} may be replaced by the simpler relations as Nystrom row sum condition which is defined by

Table 1: Table of coefficients for explicit IRKN

0					
c_2	a_{21}				
c_3	a_{31}	a_{32}			
.	.	.	.		
.	.	.	.		
.	.	.	.		
c_s	a_{s1}	a_{s2}	…	a_{ss-1}	
b_{-1}	b_1	b_2	…	b_{s-1}	b_s
		$\overline{b}_2$	…	$\overline{b}_{s-1}$	$\overline{b}_s$

$$\frac{1}{2}c_i^2 = \sum_{j=1}^{i-1} a_{ij}, \quad i = 1,\ldots,s \tag{13}$$

and by using these values of c_i we can obtain the values of $\bar{b}_i$ and other parameters.

ORDER CONDITIONS

To find the order conditions for IRKN method we applied the Taylor's series expansion to equations (12) and compare with the Taylor series expansion of y_{n+1} for the method order p, which is given by [3, 4].

$$y_{n+1} = y(x_n + h) = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \frac{h^3}{3!}y'''(x_n) + \ldots + O(h^{p+1})$$

By using the Taylor's series expansion the order conditions of method for y_n' and y_n up to order five are presented in Table 2.

DERIVATION OF METHOD

Consider the third order IRKN method with two-stages (s = 2) from formulas (12), method is called IRKN3. To find the coefficients of IRKN3, order conditions up to order three for y_n and y_n' from Table 2 must be satisfied. Here, we choose the value of $b_{-1} \in [-1,1]$ as a free parameter, by substituting $b_{-1} = -\frac{1}{3}$ into order conditions we obtained all coefficients of IRKN3 [2]. Fourth order method with three stages (IRKN4) is derived by substituting the $c_2 = \frac{1}{4}$ and $c_3 = \frac{3}{4}$ as free parameters into order conditions of fourth-order method and Nystrom row sum conditions from equation (13). For fifth order method with four stages (IRKN5), we choose the $c_2 = \frac{1}{4}, c_3 = \frac{1}{4}$ and $c_4 = \frac{3}{4}$ as free parameters in solving order conditions of fifth-order method. Also in Nystrom row sum conditions from equation (13), we choose $a_{41} = \frac{4}{32}, a_{42} = 0$ and obtained the remaining parameters. All obtained sets of coefficients of IRKN3, IRKN4 and IRKN5 are given in Table 3-5.

Table 2: Order conditions for IRKN method

Order of method	Order condition for y_n'	Order condition for y_n
First order	$b_1 - b_{-1} = 1$	
Second order	$b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2}$	
Third order	$\sum_{i=2}^{s} b_i c_i = \frac{5}{12}$	$\sum_{i=2}^{s} \bar{b}_i = \frac{5}{12}$
Fourth order	$\sum_{i=2}^{s} b_i c_i^2 = \frac{1}{3}$	$\sum_{i=2}^{s} \bar{b}_i c_i = \frac{1}{6}$
Fifth order	$\sum_{i=2}^{s} b_i c_i^3 = \frac{31}{120}$	$\sum_{i=2}^{s} \bar{b}_i c_i^2 = \frac{31}{360}$
	$\sum_{i=2}^{s} b_i a_{ij} c_j = \frac{31}{720}$	

Table 3: Table of coefficients for IRKN 3

0		
1/2	1/8	
-1/3	2/3	5/6
		5/12

Table 4: Table of coefficients for IRKN 4

0			
1/4	1/32		
3/4	0	9/32	
1/18	19/18	1/6	11/6
		7/24	1/8

Table 5: Table of coefficients for IRKN 5

0				
1/4	1/32			
½	-1/6	7/24		
¾	4/32	0	5/32	
1/45	46/45	-1/15	-1/10	29/45
		49/180	7/180	19/180

STABILITY ANALYSIS

Consider the following form of IRKN method [6, 7].

$$y_{n+1} = y_n + \frac{3h}{2} y'_n - \frac{h}{2} y'_{n-1} + h^2 \sum_{i=2}^{s} \bar{b}_i \left(f(x_n + c_i h, Y_i) - f(x_{n-1} + c_i h, Y_i) \right)$$

$$y'_{n+1} = y'_n + h\left(b_1 f(x_n, Y_1) - b_{-1} f(x_{n-1}, Y_{-1}) + \sum_{i=2}^{s} b_i \left(f(x_n + c_i h, Y_i) - f(x_{n-1} + c_i h, Y_{-i}) \right)\right) \quad (14)$$

$$Y_1 = y_n, \quad Y_{-1} = y_{n-1}$$

$$Y_i = y_n + h c_i y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} f(x_n + c_j h, Y_j), \quad i = 2,..,s$$

$$Y_{-i} = y_{n-1} + h c_i y'_{n-1} + h^2 \sum_{j=1}^{i-1} a_{ij} f(x_{n-1} + c_j h, Y_{i-1}), \quad i = 2,...,s$$

Apply test equation $y'' = -\lambda^2 y$ and by replacing $f(x,y) = -\lambda^2 y$ into equation (14) we have:

$$y_{n+1} = y_n + \frac{3h}{2} y'_n - \frac{h}{2} y'_{n-1} + (-\lambda^2 h^2) \sum_{i=2}^{s} \bar{b}_i (Y_i - Y_{-i}) \quad (15)$$

$$y'_{n+1} = y'_n + (-\lambda^2 h)\left(b_1 Y_1 - b_{-1} Y_{-1} + \sum_{i=2}^{s} b_i (Y_i - Y_{-i})\right) \quad (16)$$

For i = 2,…,s also we have

$$Y_i = y_n + h c_i y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} Y_j \quad (17)$$

$$Y_{-i} = y_{n-1} + h c_i y'_{n-1} + h^2 \sum_{j=1}^{i-1} a_{ij} Y_{-j} \quad (18)$$

Multiplying equation (16) by h we have:

$$h y'_{n+1} = h y'_n + (-\lambda^2 h^2)\left(b_1 Y_1 - b_{-1} Y_{-1} + \sum_{i=2}^{s} b_i (Y_i - Y_{-i})\right) \quad (11)$$

Now we rewrite the equations (15) and (18) in the following matrix form:

$$\begin{bmatrix} y_{n+1} \\ y_n \\ hy'_{n+1} \\ hy'_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{-1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_n \\ y_{n-1} \\ hy'_n \\ hy'_{n-1} \end{bmatrix} + H \begin{bmatrix} 0 & 0 & \overline{b}_2 & \dots & \overline{b}_s \\ & & 0 & & \\ b_{-1} & b_1 & b_2 & \dots & b_s \\ & & 0 & & \end{bmatrix} \left(\begin{bmatrix} 0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_s \end{bmatrix} - \begin{bmatrix} Y_{-1} \\ 0 \\ Y_{-2} \\ \vdots \\ Y_{-s} \end{bmatrix} \right) \tag{19}$$

Where $H = -(\lambda^2 h^2)$, $\lambda \in R$. We can define matrix form of equations (17) and (18) as follows:

$$\begin{bmatrix} 0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & c_2 & 0 \\ 1 & 0 & c_3 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & c_s & 0 \end{bmatrix} \begin{bmatrix} y_n \\ y_{n-1} \\ hy'_n \\ hy'_{n-1} \end{bmatrix} + H \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ a_{21} & a_{21} & 0 & \dots & 0 & 0 \\ a_{31} & a_{31} & a_{32} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ a_{s1} & a_{s1} & a_{s2} & & a_{ss-1} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_s \end{bmatrix} \tag{20}$$

and

$$\begin{bmatrix} Y_{-1} \\ 0 \\ Y_{-2} \\ \vdots \\ Y_{-s} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & c_2 \\ 0 & 1 & 0 & c_3 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & c_s \end{bmatrix} \begin{bmatrix} y_n \\ y_{n-1} \\ hy'_n \\ hy'_{n-1} \end{bmatrix} + H \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ a_{21} & a_{21} & 0 & \dots & 0 & 0 \\ a_{31} & a_{31} & a_{32} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ a_{s1} & a_{s1} & a_{s2} & & a_{ss-1} & 0 \end{bmatrix} \begin{bmatrix} Y_{-1} \\ 0 \\ Y_{-2} \\ \vdots \\ Y_{-s} \end{bmatrix} \tag{21}$$

Subtracting (21) from (20) we have

$$\begin{bmatrix} 0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_s \end{bmatrix} - \begin{bmatrix} Y_{-1} \\ 0 \\ Y_{-2} \\ \vdots \\ Y_{-s} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & c_2 & -c_2 \\ 1 & -1 & c_3 & -c_3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & -1 & c_s & -c_s \end{bmatrix} \begin{bmatrix} y_n \\ y_{n-1} \\ hy'_n \\ hy'_{n-1} \end{bmatrix} + H \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ a_{21} & a_{21} & 0 & \dots & 0 & 0 \\ a_{31} & a_{31} & a_{32} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ a_{s1} & a_{s1} & a_{s2} & & a_{ss-1} & 0 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_s \end{bmatrix} - \begin{bmatrix} Y_{-1} \\ 0 \\ Y_{-2} \\ \vdots \\ Y_{-s} \end{bmatrix} \right) \tag{22}$$

Define:

$$Y = \begin{bmatrix} 0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_s \end{bmatrix} - \begin{bmatrix} Y_{-1} \\ 0 \\ Y_{-2} \\ \vdots \\ Y_{-s} \end{bmatrix}, \quad Z_n = \begin{bmatrix} y_n \\ y_{n-1} \\ hy'_n \\ hy'_{n-1} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ a_{21} & a_{21} & 0 & \dots & 0 & 0 \\ a_{31} & a_{31} & a_{32} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ a_{s1} & a_{s1} & a_{s2} & & a_{ss-1} & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & c_2 & -c_2 \\ 1 & -1 & c_3 & -c_3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & -1 & c_s & -c_s \end{bmatrix}$$

Simplifying equation (22) we obtain:

$$Y = \left(I - HA\right)^{-1} E\, Z_n \tag{23}$$

where I is the identity matrix. Substituting (23) into (19) we have

$$Z_{n+1} = \left(M + HB\left((I - HA)^{-1} E\right)\right) Z_n \tag{24}$$

Where

$$M = \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{-1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ & & & \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & \bar{b}_2 & \dots & \bar{b}_s \\ & & 0 & & \\ b_{-1} & b_1 & b_2 & \dots & b_s \\ & & 0 & & \\ & & & & \end{bmatrix}$$

We can rewrite equation (24) as follows:

$$Z_{n+1} = D(H)Z_n \tag{25}$$

Where

$$D(H) = \begin{bmatrix} 1 + H\bar{b}^T N^{-1} e_1 & H\bar{b}^T N^{-1} e_2 & \frac{3}{2} + H\bar{b}^T N^{-1} e_3 & \frac{-1}{2} + H\bar{b}^T N^{-1} e_4 \\ 1 & 0 & 0 & 0 \\ Hb^T N^{-1} e_1 & Hb^T N^{-1} e_2 & 1 + Hb^T N^{-1} e_3 & Hb^T N^{-1} e_4 \\ 0 & 0 & 1 & 0 \\ & & & \end{bmatrix}$$

$$N = (I - HA), \quad \bar{b} = [0 \quad 0 \quad \bar{b}_2 \quad \dots \quad \bar{b}_s\,]^T$$
$$b = [b_{-1} \quad b_1 \quad b_2 \quad \dots \quad b_s]^T$$

$$e_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ c_2 \\ \vdots \\ c_s \end{bmatrix}, \quad e_4 = -e_3$$

D(H) is called the Stability matrix and the stability polynomial related with this method is given by:

$$\rho(\xi, H) = \det(\xi I - D(H)) \tag{26}$$

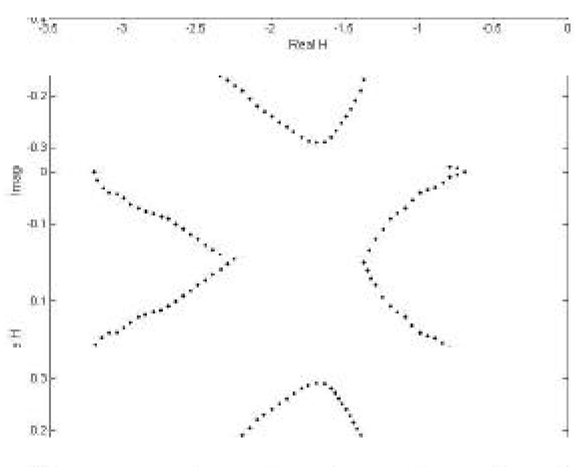

Fig. 1: Stability region of IRKN3 for $H = -(\lambda^2 h^2)$

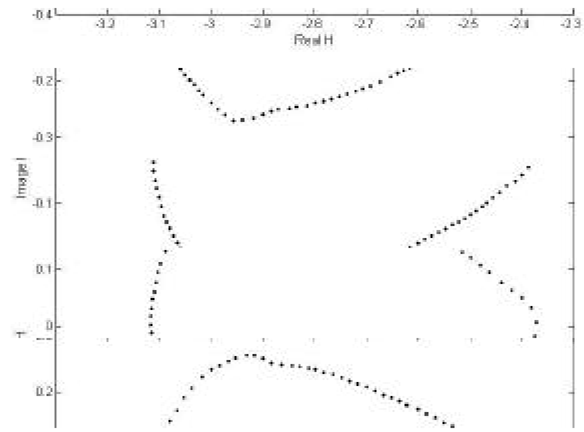

Fig. 2: Stability region of IRKN4 for $H = -(\lambda^2 h^2)$

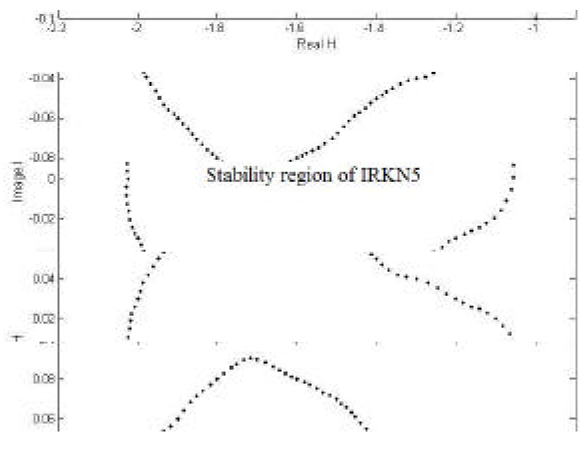

Fig. 3: Stability region of IRKN5 for $H = -(\lambda^2 h^2)$

Stability region of the methods is the set of values of $H = -(\lambda^2 h^2)$, $\lambda \in R$, such that all the roots of stability polynomial are inside the unit circle. Here the stability regions of IRKN3, IRKN4 and IRKN5 are plotted in Fig. 1-3.

NUMERICAL EXAMPLES

In this section, we tested a standard set of second-order initial value problems to show the efficiency and accuracy of the proposed method. The exact solutions of $y(x)$ and $y'(x)$ are used to estimate the global error and approximate the starting values of y_1 and y'_1 at the first step $[x_0\ x_1]$. The following problems are solved for $x \in [0\ 10]$.

Problem 1 (The undamped Duffing's equations [8])

$$y'' + y + y^3 = 0.002\cos(1.01x)$$
$$y(0) = 0.200426728067, \quad y'(0) = 0$$

Exact solution computed by Galerkin method and given by:

$$y(x) = \sum_{i=0}^{4} a_{2+1} \cos[1.01(2i+1)x]$$

with

$$a_1 = 0.200179477536,\ a_3 = 0.24\times10^{-3}$$
$$a_5 = 0.304014\times10^{-6},\ a_7 = 0.374\times10^{-9}\ \text{and}\ a_9<10^{-12}$$

Problem 2 (An almost periodic Orbit problem studied by Stiefel and Bettis [9])

$$y'' + y = 0.001e^{ix}, y(0) = 1, y'(0) = 0.9995i$$

Exact solution: $y(x) = (1-0.0005ix)e^{ix}$. we write in equivalent form

$$y_1'' + y_1 = 0.001\cos(x),\quad y_1(0) = 1,\quad y_1'(0) = 0$$

$$y_2'' + y_2 = 0.001\sin(x),\quad y_2(0) = 0,\quad y_2'(0) = 0.9995$$

Exact solution:

$$y_1(x) = \cos(x) + 0.0005x\sin(x),\quad y_2(x) = \sin(x) - 0.0005x\cos(x)$$

The numerical results of new methods are compared with existing RKN methods to show the efficiency of method in addition the following abbreviations are used in to present the results in this paper.

IRKN3, IRKN4 and IRKN5: The Improved Runge-Kutta Nystrom method of order three, four and five with 2, 3 and 4 stages respectively, derived in this paper.

RKNV3: The third order Runge-Kutta Nystrom method with zero dissipation three stages given in van der Houwen and Sommeijer [10]

RKND3: The third order three stages Runge-Kutta Nystrom method given by Dormand.[5]

RKNC4: The fourth order three stages classical Rnge-Kutta Nystrom method given by Garcia *et al*.[11]

RKNV4: The fourth order Runge-Kutta Nystrom method with ten order dispersion, fifth order dissipation four stages given by Van der Huwen and Sommeijer [10]

The numerical results for tested problems are given by Table 6 and 7 which show the maximum global error against the different values of step size h = 0.1, 0.01, 0.005.

From Table 6 and 7, it is clear that the third order new method with only 2 stages (IRKN3) obtained the better accuracy compared with the existing third order methods with 3 stages (RKNV3 and RKND3). Therefore IRKN3 with less number of function evaluations and high accuracy is more efficient than the same order existing RKN methods. IRKN4 with three stages compared with the fourth order, three stages existing method (RKNC4), is more accurate. Also IRKN5 with four stages compared with same stages existing method of order four (RKNV4) gives the higher accuracy.

Table 6: Maximum global error versus step size h for IRKN and RKN methods in solving test problem 1

Method	h = 0.1	h = 0.01	h = 0.005
IRKN3	2.47 E-5	2.32 E-8	2.90 E-9
IRKN4	1.05 E-7	3.43 E-11	1.01 E-11
IRKN5	6.07 E-8	1.01 E-11	1.07 E-11
RKNV3	2.37 E-3	2.38 E-5	5.09 E-6
RKND3	3.01 E-5	3.01 E-8	3.77 E-9
RKNC4	6.68 E-7	8.41 E-11	3.48 E-11
RKNV4	3.36 E-6	8.11 E-10	5.13 E-10

Table 7: Maximum global error versus step size h for IRKN and RKN methods in solving test problem 2

Method	h = 0.1	h = 0.01	h = 0.005
IRKN3	1.13 E-4	1.07 E-7	1.43E-8
IRKN4	5.52 E-7	5.47 E-11	3.84 E-12
IRKN5	2.86 E-7	2.89 E-12	8.98 E-14
RKNV3	1.07 E-2	1.07 E-4	2.69 E-5
RKND3	1.49 E-4	1.18 E-7	1.85E-8
RKNC4	1.40 E-5	3.47 E-9	2.47 E-9
RKNV4	3.09 E-6	3.05 E-10	1.09 E-11

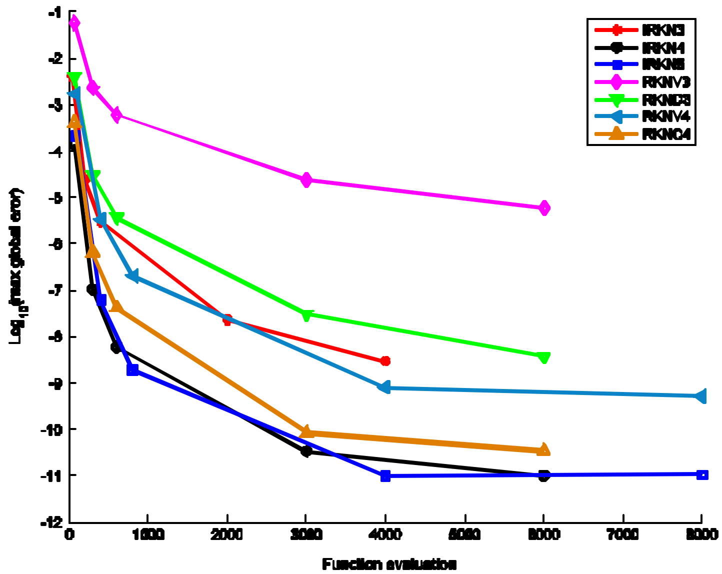

Fig. 4: Logarithm of maximum global error versus number of function evaluations for third order IRKN and RKN methods for problem 1

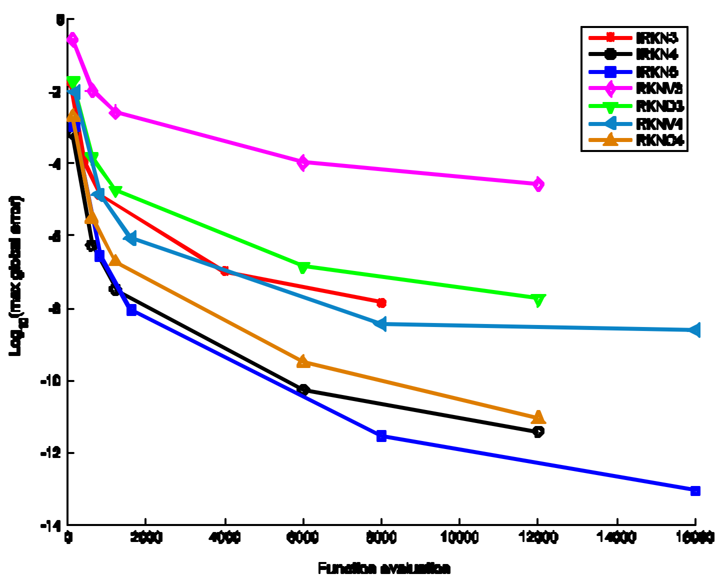

Fig. 5: Logarithm of maximum global error versus number of function evaluations for fourth order IRKN and RKN methods for problem2

Figure 4 and 5, show the efficiency of new methods by giving number of function evaluations versus the logarithm of maximum global error with h=0.1 for tested problems.

From Fig. 4 and 5, it is observed that the curves of number of function evolutions against the relative error accuracy for IRKN3 is lower than two existing methods (RKNV3 and RKND3). That shows the efficiency of third order IRKN method compared with the existing ones. Also the curves of accuracy for IRKN4 and IRKN5, are lower than RKNC4 and RKNV4, respectively. Therefore, IRKN method of order 3, 4 and 5 with better error accuracy and less number of function evaluations are more efficient than the existing RKN methods.

CONCLUSION

In this paper we constructed the explicit Improved Runge-Kutta Nystrom method for solving second order ODEs. The scheme was based on Runge-kutta Nystrom method also can be considered as two step methods. The IRKN methods derived by using less number of stages which lead to less number of function evaluations at less time, that is, the efficiency of new methods.

In this paper, the method of order three derived with only two stages while there is not any existing Runge-kutta Nystrom method with two stages. Third and fourth order methods also derived with three and four stages, respectively. The efficiency of method by solving number of second order standard test problems was shown.

Therefore, we can conclude that, IRKN methods with less number of function evaluations and better accuracy are computationally more efficient than the existing RKN methods.

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