World Applied Sciences Journal 19 (12): 1781-1788, 2012
ISSN 1818? 4952

DOI: 10.5829/idosi.wasj.2012.19.12.2093

The Exp-function Method for Solving the Nonlinear Heat Conduction Equation

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Abstract: The Exp-function Method (EFM) with the aid of symbolic computational system can be used to obtain the generalized solitary solutions for the nonlinear evolution equations arising in mechanics. In this paper we study for the analytic treatment of the nonlinear heat conduction equation. Exact solutions with solitons are obtained. The nonlinear heat conduction equation is chosen to illustrate the effectiveness of this method. This method is straightforward and concise and its applications are promising. This method is developed for searching exact travelling wave solutions of nonlinear partial differential equations.

Key words: The exp-function method . the nonlinear heat equation . solitary and soliton solutions

INTRODUCTION

In this article, we use an effective method for constructing a range of exact solutions for the following nonlinear partial differential equations which was first presented by He [1]. A new method called the Exp-function Method (EFM) is presented to look for traveling wave solutions of nonlinear evolution equations (NLEEs). The Exp-function method has successfully been applied to many situations. For example, He *et al*. [2] have solved the nonlinear wave equations using the EFM. EFM has been applied to obtaining solitary solutions, periodic solutions and compacton-like solutions by Wu and He [3]. Authors of [4] have examined the Exp-function method to solve generalized solitary solution and compacton-like solution of the Jaulent-Miodek equations. Abdou [5] has solved generalized solitonary and periodic solutions for the nonlinear partial differential equations by the EFM. Boz and Bekir [6] have applied the EFM for the (3+1)-dimensional nonlinear evolution equations. The modified KdV and the generalized KdV equations using the EFM have been obtained by Manafian and *et al*. [7]. Application of the Exp-function method has been applied for solving a partial differential equation arising in biology and population genetic by Dehghan and co-authors [8]. Analytical treatment of some partial differential equations arising in mathematical physics of this method is presented in [9]. The EFM has recently been solved by Zhang [10] to high-dimensional nonlinear evolution equation. Recently, the study of nonlinear partial differential equations in modeling physical phenomena has become an important tool. The investigation of the travelling wave solutions plays an important role in nonlinear sciences. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations [11]. A variety of powerful methods has been presented, including the inverse scattering transform [11], Hirota's bilinear method [12], homotopy analysis method [13, 14], variational iteration method [15, 16], homotopy perturbation method [17], sinecosine method [18], F-expansion method [19, 20], tanh-function method [21, 22], tanh-coth method [23], Bäcklund transformation [24] and so on. The nonlinear heat equation [25] in (1+1) D is in the form

$$u_t - a(u^3)_{xx} - u + u^3 = 0 \qquad (1.1)$$

and we consider nonlinear heat conduction equation in (1+2)D in the form

$$u_t - a(u^3)_{xx} - a(u^3)_{yy} - u + u^3 = 0 \qquad (1.2)$$

Wazwaz have obtained soliton solutions with the tanh method [25]. Using the Exp-function method we obtained various solutions for the nonlinear heat equation, new results are formally developed in this article. Our aim of this paper is to obtain analytical solutions of the nonlinear heat conduction equation and to determine the accuracy of the EFM in solving these kind of problems. The remainder of the paper is organized as follows: In Section 2, a brief discussion for the Exp-function method are presented and in Section 3, exact solutions of Eq. (1.1) are obtained. In Section 4, we describe this method briefly and apply

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this technique to the nonlinear heat conduction equation in (1+2)D and exact solutions of Eq. (1.2) are obtained. Section 5 ends this report with a brief conclusion.

$$u(x,y,t) = u(\eta) \text{ and } \eta = x + y - ct$$

BASIC IDEA OF EXP-FUNCTION METHOD

We first consider the nonlinear equation of the form

$$P\big(u, u_t, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, u_{xy}, u_{tt}, u_{tx}, u_{ty}, u_{tz}, \ldots\big) = 0 \tag{2.1}$$

and introduce a transformation

$$u(x,y,t) = u(\eta), \qquad \eta = \mu(x + y - ct) \tag{2.2}$$

where c is constant to be determined later. Therefore Eq. (2.1) is reduced to an ODE as follows

$$M(u, -cu', u', u', u', u'', \ldots) \tag{2.3}$$

The EFM is based on the assumption that travelling wave solutions as in [2] can be expressed in the form

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)} \tag{2.4}$$

where c,d,p and q are positive integers which could be freely chosen, a_n's and b_m's are unknown constantsto be determined. To determine the values of c and p, we balance the lnear term of highest order in Eq. (2.3) with the highest order nonlinear term. Also to determine the values of d and q, we balance the linear term of lowest order in Eq. (2.3) with the lowest order nonlinear term.

APPLICATION OF THE NONLINEAR HEAT EQUATION IN (1+1)D

We employ the EFM to the nonlinear heat conduction equation in (1+1)D as follows

$$u_t - a(u^3)_{xx} - u + u^3 = 0 \tag{3.1}$$

and use the wave variable as follow $\eta = \mu(x-ct)$ reduces it to an ODE

$$-c\mu u' - a\mu^2 (u^3)'' - u + u^3 = 0 \tag{3.2}$$

we substitute

$$u(x,t) = v^{\frac{-1}{2}}(x,t) \tag{3.3}$$

into Eq. (3.2) to get

$$2c\mu v^2 v' + 6a\mu^2 vv'' - 15a\mu^2 (v')^2 - v^3 + v^2 = 0 \tag{3.4}$$

In order to determine the values of c and p, we balance v^2v'' with vv'' in Eq. (3.4), to get

$$vv'' = \frac{c_1 \exp[(2c+3p)\eta] + \cdots}{c_2 \exp(4p\eta) + \cdots} \tag{3.5}$$

$$v^2 v' = \frac{c_3 \exp[(3c+p)\eta] + \cdots}{c_4 \exp(4p\eta) + \cdots} = \frac{c_3 \exp[(3c+2p)\eta] + \cdots}{c_4 \exp(5p\eta) + \cdots} \tag{3.6}$$

$$2c + 3p = 3c + 2p \tag{3.7}$$

respectively. Balancing highest order of the EFM in (3.5) and (3.6), we will have which leads to the result c = p. Similarly to determine the values of d and q, for the terms v^2v' and vv'' in Eq. (3.2) by simple calculation, we obtain

$$vv'' = \frac{\ldots + d_1 \exp[-(2d+3q)\eta]}{\ldots + d_2 \exp(-5p\eta)} \tag{3.8}$$

$$v^2 v' = \frac{\ldots + d_3 \exp[-(3d+q)\eta]}{\ldots + d_4 \exp(-4p\eta)} = \frac{\ldots + d_1 \exp[-(3d+2q)\eta]}{\ldots + d_2 \exp(-5p\eta)} \tag{3.9}$$

respectively. Balancing lowest order of the EFM in (3.8) and (3.9), we get

$$-(2d+3q) = -(3d+2q) \tag{3.10}$$

which leads to the result d = q.

Case 1: p = c = 1 and d = q = 1.

For simplicity, we set $b_1 = 1$, p = c = 1 and d = q = 1. Then Eq. (2.4) reduces to

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)} \tag{3.11}$$

Substituting (3.11) into Eq. (3.4) and by using the well-known Maple software, we will have

$$\frac{1}{A}[C_4\exp(4\eta) + C_3\exp(3\eta) + C_2\exp(2\eta) + C_1\exp(\eta) + C_0 + C_{-1}\exp(-\eta) + C_{-2}\exp(-2\eta) + C_{-3}\exp(-3\eta) + C_{-4}\exp(-4\eta)] = 0, \tag{3.12}$$

where

$$A = [\exp(\eta) + b_0 + b_{-1}\exp(-\eta)]^4 \tag{3.13}$$

and C_n' are coefficients of exp (nη)'. Equating the coefficients of exp (nη) to be zero, we obtain the following set of algebraic equations for a_1, a_0, a_{-1}, b_0, b_{-1}, μ and c, as

$$\begin{cases} C_4 = 0, \quad C_3 = 0, \quad C_2 = 0, \quad C_1 = 0, \\ C_0 = 0, \\ C_{-4} = 0, \quad C_{-3} = 0, \quad C_{-2} = 0, \quad C_{-1} = 0 \end{cases} \tag{3.14}$$

Solving the system of algebraic equations with the help of Maple gives the following sets of solutions

(I) The first set:

$$a_1 = 0, \ a_0 = a_0, \ a_{-1} = 0, \ b_0 = a_0, \ b_{-1} = 0, \mu = \pm\frac{2}{3\sqrt{a}}, c = \pm\sqrt{a} \tag{3.15}$$

$$v_1(x,t) = \frac{a_0}{a_0 + \exp\left[\pm\frac{2}{3\sqrt{a}}(x \mp \sqrt{a}t)\right]} \tag{3.16}$$

Recalling that $u = v^{\frac{-1}{2}}$ and using (3.16) we have

$$u_1(x,t) = \left\{\frac{a_0}{a_0 + \exp\left[\pm\frac{2}{3\sqrt{a}}(x \mp \sqrt{a}t)\right]}\right\}^{\frac{-1}{2}} \tag{3.17}$$

If we choose $a_0 = 1$ or $a_0 =-1$, then the solution (3.17) respectively give (cf. Eqs. (54)-(57) in [25])

$$u_{1,1}(x,t) = \left\{\frac{1}{2} - \frac{1}{2}\tanh\left[\frac{1}{3\sqrt{a}}(x - \sqrt{a}t)\right]\right\}^{\frac{-1}{2}} \tag{3.18}$$

$$u_{1,2}(x,t) = \left\{\frac{1}{2} - \frac{1}{2}\coth\left[\frac{1}{3\sqrt{a}}(x - \sqrt{a}t)\right]\right\}^{\frac{-1}{2}} \tag{3.19}$$

$$u_{1,3}(x,t) = \left\{\frac{1}{2} + \frac{1}{2}tanh\left[\frac{1}{3\sqrt{a}}(x + \sqrt{a}t)\right]\right\}^{\frac{-1}{2}} \tag{3.20}$$

$$u_{1,4}(x,t) = \left\{\frac{1}{2} + \frac{1}{2}\coth\left[\frac{1}{3\sqrt{a}}(x + \sqrt{a}t)\right]\right\}^{\frac{-1}{2}} \tag{3.21}$$

Case 2: p = c = 2 and d = q = 1

For simplicity, we set $b_2 = 1$, p = c = 1 and d = q = 1. Then Eq. (2.4) reduces to

$$v(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)} \tag{3.22}$$

Substituting (3.22) into Eq. (3.2) and by using the well-known Maple software, we will have

$$\frac{1}{A}[C_8 \exp(8\eta) + C_7 \exp(7\eta) + C_6 \exp(6\eta) + C_5 \exp(5\eta) + C_4 \exp(4\eta) + C_3 \exp(3\eta) + C_2 \exp(2\eta) + C_1 \exp(\eta) + C_0 + C_{-1} \exp(-\eta) + C_{-2} \exp(-2\eta) + C_{-3} \exp(-3\eta) + C_{-4} \exp(-4\eta)] = 0, \tag{3.23}$$

where

$$A = [\exp(2\eta) + b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)]^4 \tag{3.24}$$

and C_n' are coefficients of exp (nη)'. Equating the coefficients of exp (nη) to be zero, we obtain the following set of algebraic equations for a_2, a_1, a_0, a_{-1}, b_0, b_{-1}, b_1, μ and c as

$$\begin{cases} C_8 = 0, \; C_7 = 0, C_6 = 0, \; C_5 = 0, C_4 = 0, \; C_3 = 0, C_2 = 0, \; C_1 = 0 \\ C_0 = 0, \\ C_{-4} = 0, \; C_{-3} = 0, \; C_{-2} = 0, \; C_{-1} = 0. \end{cases} \tag{3.25}$$

Solving this system of algebraic equations by using Maple, we obtain the following results

(I) The first set:

$$a_2 = 0, a_1 = 0, \; a_0 = 0, \; a_{-1} = a_{-1}, b_0 = 0, \; b_{-1} = 0, \mu = \pm\frac{2}{9\sqrt{a}}, c = \mp 3\sqrt{a} \tag{3.26}$$

$$v_1(x,t) = a_{-1} \exp\left[\mp\frac{2}{3\sqrt{a}}(x \pm 3\sqrt{a}t)\right] \tag{3.27}$$

Noting that $u = v^{\frac{-1}{2}}$ and using (3.16) we have

$$u_1(x,t) = \frac{1}{\sqrt{a_{-1}}} \exp\left[\pm\frac{1}{3\sqrt{a}}(x \pm 3\sqrt{a}t)\right] \tag{3.28}$$

If we choose $a_{-1} = 1$, then the solution (3.27) gives

$$u_{1,1}(x,t) = \cosh\left[\frac{1}{3\sqrt{a}}(x + 3\sqrt{a}t)\right] + \sinh\left[\frac{1}{3\sqrt{a}}(x + 3\sqrt{a}t)\right] \tag{3.29}$$

$$u_{1,2}(x,t) = \cosh\left[\frac{1}{3\sqrt{a}}(x - 3\sqrt{a}t)\right] - \sinh\left[\frac{1}{3\sqrt{a}}(x - 3\sqrt{a}t)\right] \tag{3.30}$$

where are new exact solutions for nonlinear heat conduction in (1+1)D system.

APPLICATION OF THE NONLINEAR HEAT EQUATION IN (1+2)D

We consider the EFM to the nonlinear heat conduction equation as follows

$$u_t - a(u^3)_{xx} - a(u^3)_{yy} - u + u^3 = 0 \tag{4.1}$$

and use the wave variable as follow η = μ(x+y-ct) reduce it to an ODE

$$-c\mu u' - 2a\mu^2(u^3)'' - u + u^3 = 0 \tag{4.2}$$

we substitute

$$u(x,y,t) = v^{\frac{-1}{2}}(x,y,t) \tag{4.3}$$

into Eq. (4.2) to get

$$2c\mu v^2 v' + 12a\mu^2 vv'' - 30a\mu^2(v')^2 - v^3 + v^2 = 0 \tag{4.4}$$

By manipulating the above procedure and using (3.5)-(3.10) we get

Case 1: p = c = 1 and d = q = 1.

For simplicity, we set $b_1 = 1$, p = c = 1 and d = q = 1. Then Eq. (2.4) reduces to

$$v(\eta) = \frac{a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(\eta) + b_0 + b_{-1}\exp(-\eta)} \tag{4.5}$$

Substituting (4.5) into Eq. (4.4) and by using the well-known Maple software, we will have

$$\frac{1}{A}[C_4\exp(4\eta) + C_3\exp(3\eta) + C_2\exp(2\eta) + C_1\exp(\eta) + C_0 + C_{-1}\exp(\ \eta) + C_{-2}\exp(\ 2\eta) + C_{-3}\exp(\ 3\eta) + C_{-4}\exp(\ 4\eta)] = 0, \tag{4.6}$$

where

$$A = [\exp(\eta) + b_0 + b_{-1}\exp(-\eta)]^4 \tag{4.7}$$

and C_n' are coefficients of exp (nη)'. Equating the coefficients of exp (nη) to be zero, we obtain the following set of algebraic equations for a_1, a_0, a_{-1}, b_0, b_{-1}, μ and c, as

$$\begin{cases} C_4 = 0, \quad C_3 = 0, C_2 = 0, \quad C_1 = 0, \\ C_0 = 0, \\ C_{-4} = 0, \quad C_{-3} = 0, \quad C_{-2} = 0, \qquad C_{-1} = 0 \end{cases} \tag{4.8}$$

Solving the system of algebraic equations with the help of Maple gives the following sets of solutions

(I) The first set:

$$a_1 = 0,\ \ a_0 = a_0,\ \ a_{-1} = 0,\ b_0 = a_0,\ \ b_{-1} = 0, \mu = \pm\frac{2}{3\sqrt{2a}}, c = \pm\sqrt{2a} \tag{4.9}$$

$$v_1(x,y,t) = \frac{a_0}{a_0 + \exp\left[\pm\frac{2}{3\sqrt{2a}}(x + y \mp \sqrt{2a}t)\right]} \tag{4.10}$$

Recalling that $u = v^{\frac{-1}{2}}$ and using (4.16) we have

$$u_1(x,y,t) = \left\{\frac{a_0}{a_0 + \exp\left[\pm\frac{2}{3\sqrt{2a}}(x + y \mp \sqrt{2a}t)\right]}\right\}^{\frac{-1}{2}} \tag{4.11}$$

If we choose $a_0 = 1$ or $a_0 = -1$, then the solution (3.17) respectively give

$$u_{1,1}(x,y,t)=\left\{\frac{1}{2}-\frac{1}{2}tanh\left[\frac{1}{3\sqrt{2a}}(x+y-\sqrt{2a}t)\right]\right\}^{\frac{-1}{2}} \tag{4.12}$$

$$u_{1,2}(x,y,t)=\left\{\frac{1}{2}-\frac{1}{2}\coth\left[\frac{1}{3\sqrt{2a}}(x+y-\sqrt{2a}t)\right]\right\}^{\frac{-1}{2}} \tag{4.13}$$

$$u_{1,3}(x,y,t)=\left\{\frac{1}{2}+\frac{1}{2}tanh\left[\frac{1}{3\sqrt{2a}}(x+y+\sqrt{2a}t)\right]\right\}^{\frac{-1}{2}} \tag{4.14}$$

$$u_{1,4}(x,y,t)=\left\{\frac{1}{2}+\frac{1}{2}\coth\left[\frac{1}{3\sqrt{2a}}(x+y+\sqrt{2a}t)\right]\right\}^{\frac{-1}{2}} \tag{4.15}$$

Case 2: p = c = 2 and d = q = 1.
For simplicity, we set $b_2 = 1$, p = c = 1 and d = q = 1. Then Eq. (2.4) reduces to

$$v(\eta)=\frac{a_2\exp(2\eta)+a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)}{\exp(\eta)+b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta)} \tag{4.16}$$

Substituting (3.22) into Eq. (3.2) and by using the well-known Maple software, we will have

$$\begin{aligned}\frac{1}{A}[&C_8\exp(8\eta)+C_7\exp(7\eta)+C_6\exp(6\eta)+C_5\exp(5\eta)+C_4\exp(4\eta)+C_3\exp(3\eta)+C_2\exp(2\eta)\\&+C_1\exp(\eta)+C_0+C_{-1}\exp(-\eta)+C_{-2}\exp(-2\eta)+C_{-3}\exp(-3\eta)+C_{-4}\exp(-4\eta)]\\&=0,\end{aligned} \tag{4.17}$$

where

$$A=[\exp(2\eta)+b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta)]^4 \tag{4.18}$$

and C_n' are coefficients of exp (nη)'. Equating the coefficients of exp (nη) to be zero, we obtain the following set of algebraic equations for a_2, a_1, a_0, a_{-1}, b_0, b_{-1}, b_1, μ and c as

$$\begin{cases}C_8=0,\ \ C_7=0,C_6=0,\ \ C_5=0,C_4=0,\ \ C_3=0,C_2=0,\ \ C_1=0\\C_0=0,\\C_{-4}=0,\ \ C_{-3}=0,\ \ C_{-2}=0,\ \ \ \ C_{-1}=0.\end{cases} \tag{4.19}$$

Solving this system of algebraic equations by using Maple, we obtain the following results
(I) The first set:

$$a_2=0,a_1=0,\ \ a_0=0,\ \ a_{-1}=a_{-1},\ b_0=0,\ \ b_{-1}=0,\ b_{-1}=0 \tag{4.20}$$

$$\mu=\pm\frac{2}{9\sqrt{2a}},c=\mp3\sqrt{2a},v_1(x,y,t)=a_{-1}\exp\left[\mp\frac{2}{3\sqrt{2a}}(x+y\pm3\sqrt{2a}t)\right] \tag{4.21}$$

Noting that $u=v^{\frac{-1}{2}}$ and using (4.21) we have

$$u_1(x,y,t)=\frac{1}{\sqrt{a_{-1}}}\exp\left[\pm\frac{1}{3\sqrt{2a}}(x+y\pm3\sqrt{2a}t)\right] \tag{4.22}$$

If we choose $a_{-1} = 1$, then the solution (4.22) gives

$$u_{1,1}(x,y,t) = \cosh\left[\frac{1}{3\sqrt{2a}}(x+y+3\sqrt{2at})\right] + \sinh\left[\frac{1}{3\sqrt{2a}}(x+y+3\sqrt{2at})\right] \quad (4.23)$$

$$u_{1,2}(x,y,t) = \cosh\left[\frac{1}{3\sqrt{2a}}(x+y-3\sqrt{2at})\right] - \sinh\left[\frac{1}{3\sqrt{2a}}(x+y-3\sqrt{2at})\right] \quad (4.24)$$

where are new exact solutions for nonlinear heat conduction in (1+2)D system. We indicate that by applying this method no such discrepancy is in paper. Also, we illustrate the accuracy and efficiency of aforementioned method by applying this method to the nonlinear heat conduction equation.

CONCLUSION

In this article we investigated the nonlinear heat equation in one and two dimensions. The EFM is a useful method for finding travelling wave solutions of nonlinear evolution equations. This method has been successfully applied to obtain some new generalized solitonary solutions to the nonlinear heat equation. The EFM is more powerful in searching for exact solutions of NLPDEs. Some of the results are in agreement with the results reported by Wazwaz [25]. Also, new results are formally developed in this article. It can be concluded that the this method is a very powerful and efficient technique in finding exact solutions for wide classes of problems.

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