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Unexpected Results on the Integral Form of the Boundary Layer Momentum Equation

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Abstract: In this article, a further investigation on the integral solution of the boundary layer momentum equation, which is originally concerned by von Karman and Pohlhausen, is done. According to various profiles that we suggest, an analysis on the errors in values of the boundary layer thickness, δ and friction coefficient, C_f, is drawn. Although it is expected that as the chosen velocity profile becomes closer to the exact one comparing to the Pohlhausen's profile, the errors of the values for δ and C_f must become smaller, that is usually occurred, but examples may be found that treat in the opposite manner.

Key words: Boundary layer · Blasius solution · Momentum integral equation · Pohlhausen method · Boundary layer thickness · Friction coefficient

INTRODUCTION

The most important application of viscous fluid theory is the boundary layer theory, in which one is usually confronted with the two curve boundary value problem, i.e. one set of conditions is given at the surface and the other at infinity. Also, it is accepted that a boundary layer is the layer of fluid in the immediate vicinity of a bounding surface where the effects of viscosity are significant.

Many natural and industrial ?ow problems can be simpli?ed using the boundary layer concepts. Moreover, use of these concepts allows scientists and engineers to identify the most dominant parameters governing the ?ow process. Since the mid-1920s, work aimed at advancing, extending and applying boundary-layer theory has increased exponentially. The first serious industrial application of boundary-layer theory occurred in the late 1920s when designers began to use the theory's results to predict skin-friction drag on airships and airplanes. Prior to that time, they had been limited to using empirical data obtained primarily from wind tunnels. Furthermore, until the late 1920s, wind-tunnel data were notoriously inaccurate and the designers, conservative by nature, were reluctant to hinge their designs on them. But since the late 1920s, when the accuracy and value of skin-friction formulas obtained from boundary layer theory became more appreciated, the results of the theory have become a standard tool of the airplane designer.

Despite its simplicity, parallel flow over a flat plate occurs in numerous engineering applications. Assuming steady, incompressible, laminar flow with constant fluid properties and negligible body forces and pressure gradient, the boundary layer equations reduce to

Continuity:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

Momentum:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

Where u and v are velocity components and ν is kinematic viscosity.

For solving these partial differential equations, with an ingenious coordinate transformation, Blasius showed that the dimensionless velocity profile $\frac{u}{u_\infty}$ is a function only of the single composite dimensionless variable $\eta = y\left(\frac{u_\infty}{\nu x}\right)^{1/2}$ [1].

The Blasius solution is termed a similarity solution and η is a similarity variable. This terminology is used because, despite growth of the boundary layer with

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distance x from the leading edge, the velocity profile $\frac{u}{u_\infty}$ remains geometrically similar. This similarity is of the functional form $\frac{u}{u_\infty} = f\left(\frac{y}{\delta}\right)$ where δ is the boundary layer thickness. The final results of the Blasius solution lead to the following [1]:

$$\delta = \frac{5x}{\mathrm{Re}_x^{1/2}} \text{ and } C_{f,x} = \frac{0.664}{\mathrm{Re}_x^{1/2}} \tag{3}$$

Which Re_x denotes local Reynolds number based on x.

An alternative approach to solving the boundary layer equations involves the use of an approximate integral method. The approach was originally proposed by von Kárman and applied by Pohlhausen. It is without the mathematical complications inherent in the exact method; yet it can be used to obtain reasonably accurate results for the key boundary layer parameters (δ, C_f). To use the method, the boundary layer equations, must be cast in integral form. These forms are obtained by integrating the Eqs. (1) and (2) in the y-direction across the boundary layer that leads to Eq. (4):

$$\frac{d}{dx}\left[\int_0^\delta (u_\infty - u)u\ dy\right] = \nu \left.\frac{\partial u}{\partial y}\right|_{y=0} \tag{4}$$

Which is the integral form of the boundary layer momentum equation.

This integral equation can be used to obtain approximate boundary layer solutions. The procedure involves first assuming reasonable functional form for the unknowns u in terms of the corresponding (unknown) boundary layer thickness. The assumed form must satisfy appropriate boundary conditions. Substituting this form into the integral equation, expression for the boundary layer thickness may be determined and the assumed functional form may then be completely specified. Although this method is approximate, it frequently leads to accurate results for the surface parameters [2].

Consider the hydrodynamic boundary layer, for which appropriate boundary conditions are

$$u(y=0) = 0 \tag{5a}$$

$$u(y=\delta) = u_\infty \tag{5b}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0 \tag{5c}$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} = 0 \tag{5d}$$

For showing the ability of this integral method, in the classical literature occasionally a simple linear profile such as $\frac{u}{u_\infty} = \frac{y}{\delta}$ is chosen, which can satisfy only boundary conditions (5a) and (5b). The result of this crude assumption for the values of δ and C_f are as following [3]

$$\delta = \frac{3.46x}{\mathrm{Re}_x^{1/2}} \text{ and } C_f = \frac{0.577}{\mathrm{Re}_x^{1/2}} \tag{6}$$

It is hereby relied upon that this simple analysis leads us to a quite good coefficient and an accurate relationship to Re_x. To be realistic, the errors 30.8% on δ and 16.1% on C_f may be not plausible even for engineering applications. Of course, the correct relationship with Re_x is the inherent natures of the solutions that originate from smart perception of Blasius based on the similarities of velocity profiles. In other words, the selection of $\frac{u}{u_\infty} = f\left(\frac{y}{\delta}\right)$ always creates this correct relationships.

Pohlhausen chose third-degree polynomial, which was the simplest option for satisfying the boundary conditions (5), of the form

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \tag{7}$$

Using this assumption in Eq. (4) the following results were obtained [1]:

$$\delta = \frac{4.64x}{\mathrm{Re}_x^{1/2}} \text{ and } C_f = \frac{0.646}{\mathrm{Re}_x^{1/2}} \tag{8}$$

Despite the approximate nature of the foregoing procedure, compare quite well with results obtained from the exact solution. So that, the errors of δ and C_f values are 7.2% and 2.7%, respectively, that are better conclusions compare with the previous option.

Description of the Problem: This paper emphasis on some chosen velocity profiles and their effects on the values of δ and C_f.

The question we concentrate to answer is that “if each velocity profile which satisfies the boundary conditions (5) and has smaller difference with the Blasius’s velocity profile, is able to provide plausible estimations for δ and C_f ?” On the other hands, each difference between suggested velocity profile and exact one is origin of the errors in the values of δ and C_f. And it is anticipated as the proposed velocity distribution is closer to the Blasius one, better approximations for these parameters can be obtained. But, we want to see really does it happen?

Such a question is of fundamental interest in many industrial and engineering systems but has not been studied previously. The new results presented here are believed to be useful addition to the literature on boundary-layer theory in fluid mechanics.

In this article, our criterion for divergence between exact and proposed velocity profiles is sum of squares of difference for these profiles, i.e. $\underline{E}$, as is defined in Eq. (9).

$$E = \int_0^{\delta} \left(\frac{u}{u_\infty}\left(\frac{y}{\delta}\right)\Big|_{exact} - \frac{u}{u_\infty}\left(\frac{y}{\delta}\right)\Big|_{suggested} \right)^2 dy \qquad (9)$$

On the first step, let us consider two following profiles that are proposed by Keys [4] and Schets [3], respectively, for evaluating the integral method.

$$\frac{u}{u_\infty} = \sin\left(\frac{\pi y}{2\delta}\right) \text{ (Keys's suggested profile)} \qquad (10)$$

$$\frac{u}{u_\infty} = \tanh\left(\frac{2.65y}{\delta}\right) \text{ (Schets's suggested profile)} \qquad (11)$$

By using these equations into Eq. (4) and by considering that the value of δ is zero in the leading edge $x = 0$ we obtain the following results $\delta = \frac{4.79x}{\mathrm{Re}_x^{1/2}}$ and $C_f = \frac{0.656}{\mathrm{Re}_x^{1/2}}$ for Keys's profile that have 4.2% and 1.2% errors, which shows better estimations compare with Pohlhausen's profile, $\delta = \frac{3.46x}{\mathrm{Re}_x^{1/2}}$ and $C_f = \frac{0.577}{\mathrm{Re}_x^{1/2}}$ for schetz's profile that have 36.4% and 17% errors, which shows worse estimations compare with Pohlhausen's profile.

These conclusions are expected when we compare these profiles with the Blasius's and Pohlhausen's profiles (Fig. 1) and when we consider the values of E for these optional distributions that are calculated as following:

$E = 1625\times10^{-6}$ For Pohlhausen's profile,
$E = 736\times10^{-6}$ For Keys's profile,
$E = 10820\times10^{-6}$ For Schetz's profile.

Now, we propose three following profiles by conformity the boundary conditions (5). The main idea for choosing these profiles is satisfying the B.Cs (5).

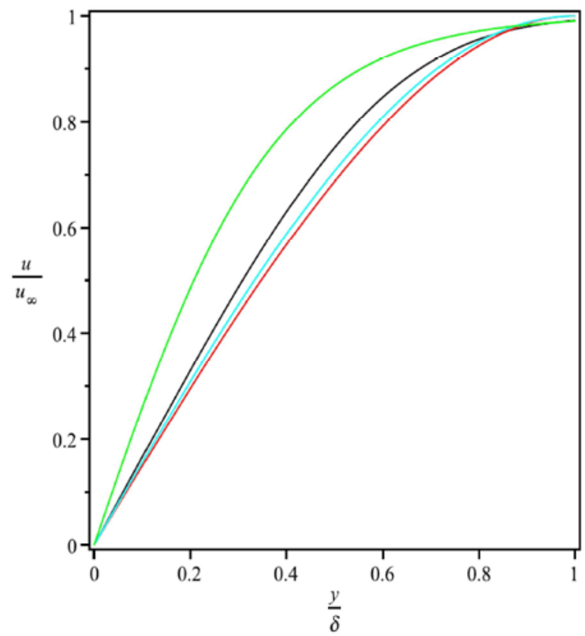

Fig. 1: Comparison of velocity profiles; Pohlhausen (red), Keys (blue), Schetz (green) with Blasius (black)

Besides, although it is common in literature that these profiles are considered in polynomial forms, we think about the functions that have the terms of other primary functions such as logarithm and triangle. Several options are found by try and error that agree with necessary boundary conditions, but for future purpose three of them are represented as following:

$$\frac{u}{u_\infty} = \frac{4}{\pi - 2}\left(Arc\tan\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)\right) \qquad (12)$$

$$\frac{u}{u_\infty} = \frac{1}{1-\ln 2}\left(\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \ln\left(\frac{y}{\delta}+1\right)\right) \qquad (13)$$

$$\frac{u}{u_\infty} = \frac{5}{3}\left(\frac{y}{\delta}\right) - \frac{2}{3}\left(\frac{y}{\delta}\right)^3 - \frac{1}{3\pi}\sin\left(\frac{\pi y}{\delta}\right) \qquad (14)$$

That we name them as No. 1, 2 and 3, respectively. By similar procedure that was done for Eqs. (10) and (11), we obtain the following results: $\delta = \frac{5.17x}{\mathrm{Re}_x^{1/2}}$ and $C_f = \frac{0.677}{\mathrm{Re}_x^{1/2}}$ for profile No. 1 that have 3.4% and 2 % errors, which shows better estimations compare with Pohlhausen's profile. $\delta = \frac{4.89x}{\mathrm{Re}_x^{1/2}}$ and $C_f = \frac{0.666}{\mathrm{Re}_x^{1/2}}$ for profile No. 2 that have 2.2% and 0.3% errors, which shows better

Table 1: Calculated results for all velocity profiles

Used profile	$E\times10^6$	Error of δ%	Error of C_f%
Pohlhausen	1625	7.2	2.7
Keys	736	4.2	1.2
Schetz	10820	36.4	17
No. 1	51	3.4	2
No. 2	442	2.2	0.3
No. 3	4920	14.4	6.5

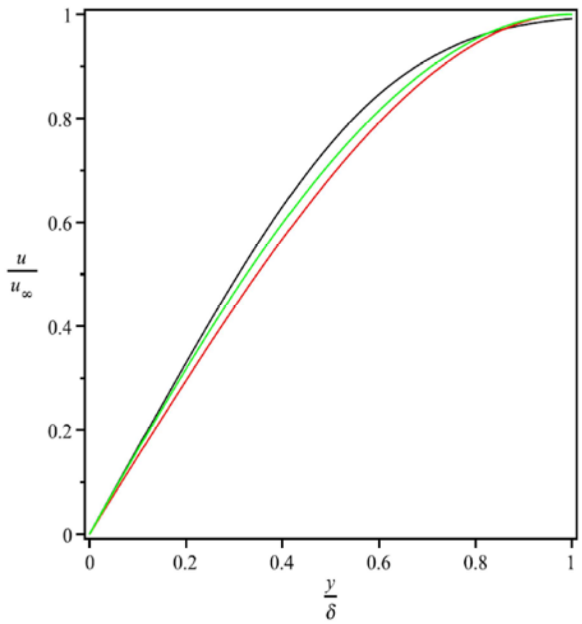

Fig. 2: Comparison of velocity profiles; Pohlhausen (red), No. 2. (green) with Blasius (black)

estimations compare with Pohlhausen's profile. $\delta = \frac{4.28x}{\mathrm{Re}_x^{1/2}}$ and $C_f = \frac{0.621}{\mathrm{Re}_x^{1/2}}$ for profile No. 3 that have 14.4% and 6.5% errors, which shows worse estimations compare with Pohlhausen's profile.

Values of E for these profiles are calculated as following:

$E = 51\times10^{-6}$ for profile No. 1,
$E = 442\times10^{-6}$ for profile No. 2,
$E = 4920\times10^{-6}$ for profile No. 3.

We summarize the aforementioned results in Table 1 for easier referral.

By comparison between profile No. 2 and 3 as shown in Figs. 2 and 3, respectively and from the values of E for these two profiles, it is expected that

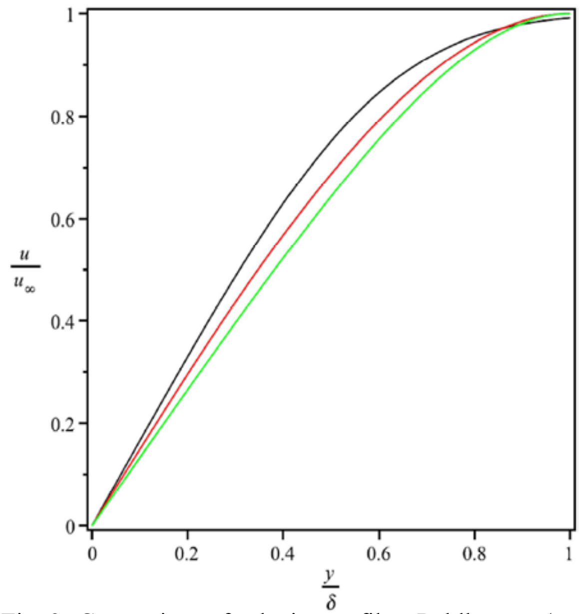

Fig. 3: Comparison of velocity profiles; Pohlhausen (red), No. 3. (green) with Blasius (black)

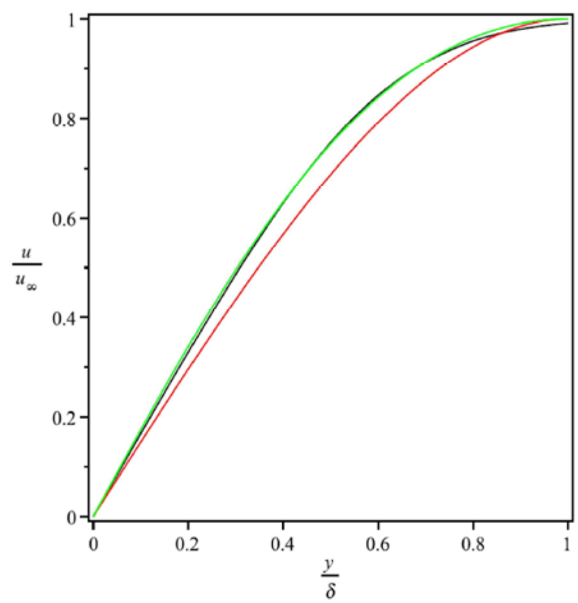

Fig. 4: Comparison of velocity profiles; Pohlhausen (red), No. 1. (green) with Blasius (black)

the errors for the values δ and C_f be lower and upper, respectively, in comparison of these values for Pohlhausen's profile, that is really occurred.

Although, we would naturally claim that closer velocity profiles to the Blasius's profile leads to more accurate results for the values of δ and C_f, the surprise is that even if the value of E for an specific profile is smaller comparing the others, it could lead to larger values for errors in the values of δ and C_f, as it occurs for profile No. 1. In other words, although the profile No. 1 is the best estimation for real profile in comparing the other ones (Fig. 4), but the precision of values for δ and C_f in this option is not the best.

In fact when using the integral methods, we should anticipate that our conclusions for calculation of desired parameters have considerable errors, despite of the used profiles satisfy the whole of necessary boundary conditions and they show a very good agreement with the exact profile. Therefore the obtained conclusions from this method must compare with results of experimental data or exact solutions for determining the order of precision estimations.

CONCLUSION

New investigation on the integral solution of momentum equation, which is introduced by von Kárman, was drawn. By various profiles that we suggested, an analysis on the errors for the values of the boundary layer thickness, δ and friction coefficient, C_f, was carried out. The problematic conclusion is that, may be a chosen velocity profile that is nearly lies to the exact one, creates bigger errors in the values of δ and C_f comparing with the velocity profile that has more derivate from the exact profile.

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