Ideals Redefined to Characterize Hemirings

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Abstract: In this paper we define \((e_1, e_2, \ldots, e_n)\)-fuzzy \(h\)-ideals, \((e_1, e_2, \ldots, e_n)\)-fuzzy \(h\)-bi-ideals and \((e_1, e_2, \ldots, e_n)\)-fuzzy \(h\)-quasi-ideals. We also characterize \(h\)-hemiregular and \(h\)-intra-hemiregular hemirings by the properties of their \((e_1, e_2, \ldots, e_n)\)-fuzzy \(h\)-ideals, \((e_1, e_2, \ldots, e_n)\)-fuzzy \(h\)-bi-ideals and \((e_1, e_2, \ldots, e_n)\)-fuzzy \(h\)-quasi-ideals.

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INTRODUCTION

As a generalization of associative rings and distributive lattices semirings are algebraic structures with two binary operations and were introduced by Vandiver [21]. In recent times it is found that semirings have been deeply studied, especially in relation with applications [9] and for solving many problems in optimization theory, graph theory, theory of discrete event, dynamical systems, matrices, determinants, generalized fuzzy computation, theory of automata, formal language theory, coding theory, analysis of computer programmes [8, 13, 22]. Semirings with commutative addition and zero element are called hemirings.

Ideals of hemirings and semirings play a crucial role in the structure theory and are useful for many purposes. However, in general, they do not coincide with the usual ring ideals. Many results in rings apparently have no analogues in hemirings using only ideals. In order to overcome this deficiency Henriksen [10] defined a more restricted class of ideals in semirings, called k-ideals, with the property that if a semiring \(R\) is ring, then a subset of \(R\) is a k-ideal if and only if it is a ring ideal.

Multi-valued logic has been considered by model phenomena due to the reason that uncertainty and vagueness are involved. To handle such situations the most famous theory of fuzzy sets was first developed by Zadeh [25] in 1965. Since then it has been used extensively in different fields of life and has also been applied to many branches in Mathematics. The fuzzification of algebraic structures was initiated by Rosenfeld [16] and he introduced the notion of fuzzy subgroups. In [3] J. Ahsan initiated the study of fuzzy semirings [2]. The fuzzy algebraic structures play an important role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces [1, 9, 15].

The notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets proposed and discussed in [14, 15]. Many authors used these concepts to generalize some concepts of algebra, for example [4-6, 11]. In [7, 12] \((\alpha, \beta)\)-fuzzy ideals of hemirings are defined. In [19] authors characterized hemirings by the properties of their \((e_1, e_2, \ldots, e_n)\)-fuzzy ideals. In this paper we generalize these concepts and by using the concept given in [24] we define \((e_1, e_2, \ldots, e_n)\)-fuzzy \(h\)-ideals, \((e_1, e_2, \ldots, e_n)\)-fuzzy \(h\)-bi-ideals and \((e_1, e_2, \ldots, e_n)\)-fuzzy \(h\)-quasi-ideals. We also characterize \(h\)-hemiregular and \(h\)-intra-hemiregular hemirings by the properties of their \((e_1, e_2, \ldots, e_n)\)-fuzzy \(h\)-ideals, \((e_1, e_2, \ldots, e_n)\)-fuzzy \(h\)-bi-ideals and \((e_1, e_2, \ldots, e_n)\)-fuzzy \(h\)-quasi-ideals.

PRELIMINARIES

For undefined notions we refer to [9, 23]. A semiring is a non-empty set \(R\) together with two associative binary operations addition "\(+\)" and multiplication "\(\cdot\)" such that "\(+\)" is distributive over "\(\cdot\)" in \(R\). An element \(0 \in R\) is called a zero or additive
identity of the semiring \((R, +, \cdot)\) if \(0x = x0 = 0\) and \(0x = x+0 = x\) for all \(x \in R\). An additively commutative semiring with zero is called a hemiring. An element \(1\) of a hemiring \(R\) is called the identity or unity of \(R\) if \(1x = x\) for all \(x \in R\). A hemiring with commutative multiplication is called a commutative hemiring. A non-empty subset \(A\) of a hemiring \(R\) is called a subhemiring of \(R\) if it contains zero and is closed with respect to the addition and multiplication of \(R\). A non-empty subset \(I\) of a hemiring \(R\) is called a left (right) ideal of \(R\) if for all \(x, z \in R\) it follows \(x+z \in \sigma_{I}R\) (\(zR\)). A left (right) ideal of \(R\) is called a bi-ideal of \(R\) if for all \(x, z \in R\) it follows \(x+z \in \sigma_{I}R\) (\(zR\)).

A fuzzy subset \(A\) of a hemiring \(R\) is called a subhemiring of \(R\) if for all \(x, y, z, a, b \in R\) it follows \(x'y'z' = x'z' = x'z = x\) for all \(x \in R\). A hemiring with commutative multiplication is called a commutative hemiring. A non-empty subset \(A\) of a hemiring \(R\) is called a subhemiring of \(R\) if it contains zero and is closed with respect to the addition and multiplication of \(R\). A non-empty subset \(I\) of a hemiring \(R\) is called a left (right) ideal of \(R\) if for all \(x, z \in R\) it follows \(x+z \in \sigma_{I}R\) (\(zR\)). A left (right) ideal of \(R\) is called a bi-ideal of \(R\) if for all \(x, z \in R\) it follows \(x+z \in \sigma_{I}R\) (\(zR\)).

A fuzzy subset \(A\) of a hemiring \(R\) is called a fuzzy left (right) h-ideal of \(R\) if for all \(x, y, z, a, b \in R\) it follows \(x'+z' = x'z' = x'z = x\) for all \(x \in R\). A fuzzy subset \(B\) of a hemiring \(R\) is called a fuzzy bi-ideal of \(R\) if for all \(x, y, z, a, b \in R\) it follows \(x'+z' = x'z' = x'z = x\) for all \(x \in R\). A fuzzy subset \(B\) of a hemiring \(R\) is called a fuzzy quasi-ideal of \(R\) if it is both a left ideal and a right ideal of \(R\). A fuzzy subset \(Q\) of a hemiring \(R\) is called a quasi-ideal of \(R\) if \(Q \subseteq \sigma_{Q}R\) and \(R \subseteq Q\). A fuzzy subset \(Q\) of a hemiring \(R\) is called a quasi-ideal of \(R\) if \(Q \subseteq \sigma_{Q}R\) and \(R \subseteq Q\). A fuzzy subset \(Q\) of a hemiring \(R\) is called a quasi-ideal of \(R\) if \(Q \subseteq \sigma_{Q}R\) and \(R \subseteq Q\).

A fuzzy subset \(f\) of a universe \(X\) is a function from \(X\) into the unit closed interval \([0,1]\), that is \(f: X \rightarrow [0,1]\).

A fuzzy subset \(f\) of a universe \(X\) of the form

\[
\begin{cases}
1 \in (0,1] & \text{if } y = x \\
0 & \text{if } y \neq x
\end{cases}
\]

is said to be a fuzzy point with support \(x\) and value \(t\) and is denoted by \(x_t\). For a fuzzy point \(x_t\) and a fuzzy set \(f\) of a set \(X\), Pu and Liu [15] gave meaning to the symbol \(x_t\alpha f\), where \(\alpha \in \{+,\vee, \wedge, \wedge \wedge, \wedge \vee\}\). A fuzzy point \(x_t\) is said to belong to (resp. be quasi-coincident with) a fuzzy set \(f\) written \(x_t \in f\) (resp. \(x_t \in qf\)) if \(f(x) \geq t\) (resp. \(f(x) + t > 1\)) and in this case, \(x_t \in qf\) (resp. \(x_t \in qf\)) means that \(x_t \in f\) or \(x_t \in qf\) (resp. \(x_t \in f\) and \(x_t \in qf\)). To say that \(x_t \alpha f\) means that \(x_t \alpha f\) does not hold. For any two fuzzy subsets \(f\) and \(g\) of \(X\), \(fg\) means that, for all \(x \in X\), \(f(x) \leq g(x)\). The symbols \(f \wedge g\) and \(f \vee g\) will mean the following fuzzy subsets of \(X\):

\[
\begin{align*}
(f \wedge g)(x) &= \max\{f(x), g(x)\} \\
(f \vee g)(x) &= \min\{f(x), g(x)\}
\end{align*}
\]

**Definition [23]:** Let \(f\) and \(g\) be two fuzzy subsets of a hemiring \(R\). The \(h\)-intrinsic product of \(f\) and \(g\) is defined by

\[
(f \otimes g)(x) = \left\{ \begin{array}{ll}
\bigwedge_{x + \sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j + z = x} \left( \bigwedge_{i=1}^{p} f(a_i) \right) \wedge \left( \bigwedge_{j=1}^{q} g(b_j) \right) & \text{if } x \text{ cannot be expressed as } x + \sum_{i=1}^{p} a_i b_j + z = \sum_{j=1}^{q} b_j + z \\
0 & \text{otherwise}
\end{array} \right.
\]

**Definition:** A fuzzy subset \(f\) of a hemiring \(R\) is called a fuzzy \(h\)-subhemiring of \(R\) if for all \(x, y, z, a, b \in R\), we have

(i) \(f(x + y) \geq \min\{f(x), f(y)\}\)
(ii) \(f(xy) \geq \min\{f(x), f(y)\}\)
(iii) \(x + a + z = b + z \Rightarrow f(x) \geq \min\{f(a), f(b)\}\)

**Definition:** A fuzzy subset \(f\) of a hemiring \(R\) is called a fuzzy left (right) \(h\)-ideal of \(R\) if for all \(x, y, z, a, b \in R\), we have

(i) \(f(x + y) \geq \min\{f(x), f(y)\}\)
(ii) \(f(xy) \geq f(y) \cdot f(x)\)
(iii) \(x + a + z = b + z \Rightarrow f(x) \geq \min\{f(a), f(b)\}\)
A fuzzy subset \( f \) of \( R \) is called a fuzzy \( h \)-ideal of \( R \) if it is both a fuzzy left and a fuzzy right \( h \)-ideal of \( R \).

**Definition [23]:** A fuzzy subset \( f \) of a hemiring \( R \) is called a fuzzy \( h \)-bi-ideal of \( R \) if for all \( x, y, z, a, b \in R \), we have

\[
\begin{align*}
(i) & \quad f(x+y) \geq \min\{f(x), f(y)\} \\
(ii) & \quad f(xy) \geq \min\{f(x), f(y)\} \\
(iii) & \quad f(xyz) \geq \min\{f(x), f(z)\} \\
(iv) & \quad x + a + z = b + z \Rightarrow f(x) \geq \min\{f(a), f(b)\}
\end{align*}
\]

where \( R \) is the fuzzy subset of \( R \) mapping every element of \( R \) on 1.

**Definition [23]:** A hemiring \( R \) is said to be \( h \)-hemiregular if for each \( x \in R \), there exist \( a, b, z \in R \) such that

\[
x + xax + z = xbx + z
\]

**Definition [23]:** A hemiring \( R \) is said to be \( h \)-intra-hemiregular if for each \( x \in R \), there exist \( i, j \in R \) such that

\[
x + xax + z = xbx + z
\]

where \( R \) is the fuzzy subset of \( R \) mapping every element of \( R \) on 1.

**Lemma [23]:** If \( I \) and \( L \) are respectively, right and left \( h \)-ideals of a hemiring \( R \), then \( I \subseteq L \).

**Lemma [23]:** A hemiring \( R \) is \( h \)-hemiregular if and only if for any right \( h \)-ideal \( I \) and any left \( h \)-ideal \( L \) of \( R \), we have \( I \subseteq L \).

**Lemma [23]:** A hemiring \( R \) is \( h \)-intra-hemiregular if and only if for any right \( h \)-ideal \( I \) and any left \( h \)-ideal \( L \) of \( R \), we have \( I \subseteq L \).

**Lemma [23]:** Let \( R \) be a hemiring. Then the following conditions are equivalent.

\[
(i) \quad R \text{ is } h\text{-hemiregular.} \\
(ii) \quad B = \overline{BRB} \text{ for every } h\text{-bi-ideal } B \text{ of } R. \\
(iii) \quad Q = \overline{QRQ} \text{ for every } h\text{-quasi-ideal } Q \text{ of } R.
\]

**Lemma [23]:** The following conditions are equivalent for a hemiring \( R \).

\[
(i) \quad R \text{ is both } h\text{-hemiregular and } h\text{-intra-hemiregular.} \\
(ii) \quad B = \overline{B^2} \text{ for every } h\text{-bi-ideal } B \text{ of } R. \\
(iii) \quad Q = \overline{Q^2} \text{ for every } h\text{-quasi-ideal } Q \text{ of } R.
\]

**Lemma [23]:** Let \( R \) be a hemiring. Then the following conditions are equivalent.

\[
(i) \quad x_i \in m \Rightarrow f(x) \geq t \geq m \\
(ii) \quad x_i q_n f \Rightarrow f(x) + t \geq 2n \\
(iii) \quad x_i \in m \lor q_n f \Rightarrow x_i \in m f \text{ or } x_i q_n f \\
(iv) \quad x_i \in m \land q_n f \Rightarrow x_i \in m f \text{ and } x_i q_n f \\
(v) \quad x_i \bar{\alpha} \text{ means that } x_i \alpha \text{ does not hold, where } \alpha \in \{x_i \in m, q_n \in m \lor q_n \in m \land q_n\}
\]

**Lemma [23]:** A fuzzy subset \( f \) of \( R \) is called a fuzzy \( h \)-bi-ideal of \( R \) if for all \( x, y, z, a, b \in R \), we have

\[
\begin{align*}
(i) & \quad f(x + y) \geq \min\{f(x), f(y)\} \\
(ii) & \quad f(xy) \geq \min\{f(x), f(y)\} \\
(iii) & \quad f(xyz) \geq \min\{f(x), f(z)\} \\
(iv) & \quad x + a + z = b + z \Rightarrow f(x) \geq \min\{f(a), f(b)\}
\end{align*}
\]

**Lemma [23]:** Let \( f \) be a fuzzy subset of \( R \) mapping every element of \( R \) on 1.

Note that if \( f \) is a fuzzy left \( h \)-ideal (right \( h \)-ideal, \( h \)-bi-ideal, \( h \)-quasi-ideal), then \( f(0) \geq f(x) \) for all \( x \in R \).

**Lemma [23]:** Let \( f \) be a fuzzy subset of \( R \) and \( m, n \in [0,1] \) with \( m \leq n \) such that

\[
\begin{align*}
\text{for all fuzzy subsets } f \text{ of } R, [24].
\end{align*}
\]

**Definition:** Let \( f, g \) be fuzzy subsets of \( R \) and \( m, n \in [0,1] \) with \( m \leq n \).

Then the relation \( "\subseteq" \) on the set of all fuzzy subsets of \( R \) defined by

\[
f \subseteq g \:Leftrightarrow f(x) \leq g(x) \text{ for all } x \in R.
\]

**Lemma:** Let \( f, g \) be fuzzy subsets of \( R \) and \( m, n \in [0,1] \) with \( m \leq n \). Then the relation \( "\subseteq" \) on the set of all fuzzy subsets of \( R \) defined by

\[
\begin{align*}
f \subseteq g \Leftrightarrow f \in q_l^m g \text{ and } g \in q_r^m f
\end{align*}
\]

is an equivalence relation.
Definition: Let \( f \) be fuzzy subset of \( R \) and \( m,n \in [0,1] \) with \( m < n \). Then \( f \) is said to be \((\in_m, \in_n)\)-fuzzy h-subhemiring of \( R \) if

\[
(1) \quad (f + f)[\in] m_n f
\]

\[
(2) \quad (f \otimes f)[\in] m_n f
\]

\[
(3) \quad x + a + z = b + z \quad \text{and} \quad a_1, b_1 \in_m f \Rightarrow x_{\lor t} e_m \in_m f, \quad \text{for all} \ a, b, x, z \in R \text{ and } t, r \in (0,1].
\]

Definition: Let \( f \) be a fuzzy subset of \( R \) and \( m,n \in [0,1] \) with \( m < n \). Then \( f \) is said to be \((\in_m, \in_n)\)-fuzzy left (right) h-ideal of \( R \) if it satisfies (1),(3) and

\[
(4) \quad (R \otimes f)[\in] m_n f \text{ resp.} \quad (5) \quad (f \otimes R)[\in] m_n f
\]

A fuzzy subset \( f \) of \( R \) is said to be an \((\in_m, \in_n)\)-fuzzy h-ideal of \( R \) if it is both \((\in_m, \in_n)\)-fuzzy left h-ideal and \((\in_m, \in_n)\)-fuzzy right h-ideal of \( R \).

Definition: Let \( f \) be a fuzzy subset of \( R \) and \( m,n \in [0,1] \) with \( m < n \). Then \( f \) is said to be an \((\in_m, \in_n)\)-fuzzy h-quasi-ideal of \( R \) if it satisfies (1),(3) and

\[
(6) \quad (f \otimes R) \wedge (R \otimes f)[\in] m_n f
\]

But

\[
(f + f)(x + y) = \bigvee_{x+y = a_1 + b_1} (f(a_1) \wedge f(a_2) \wedge f(b_1) \wedge f(b_2))
\]

\[
\geq (f(0) \wedge f(x) \wedge f(y)) \quad \text{(because } (x+y) + 0+ z = x + y + z) \]

\[
\geq (f(x) \wedge f(y) \wedge n) \quad \text{(because } f(0) \geq \min\{f(x),f(y),n\} \geq t \wedge n = t)
\]

\[
\Rightarrow (f + f)(x + y) \geq t \Rightarrow (x + y)_t e_m f + f
\]

Then by hypothesis \((x + y)_t e_m v_{n_f} f\) (**).

Then (*) and (**) gives contradiction. Hence (1') holds. Similarly (2') and (3') hold.

Conversely assume (1'),(2') and (3') hold. To prove \( f \) is the \((\in_m, \in_n)\)-fuzzy h-subhemiring of \( R \). Suppose on the contrary (1) does not hold and for some \( x \in R \) and \( t \in (0,1], x_t e_m f + f \) but \( x_t e_m v_{n_f} f \). Then \((f + f)(x) \geq t \geq m \) and \( f(x) < t, f(x) < n \). Then for any \( x, z, a, b, a', b' \in R \) such that \( x + a + b + z = a' + b' + z \) and

\[
\Rightarrow f(x) \geq f(a) \wedge f(b) \wedge f(a') \wedge f(b')
\]

Then

\[
t \leq (f + f)(x) = \bigvee_{x+y = a'+b'} (f(a) \wedge f(a') \wedge f(b) \wedge f(b')) = f(x)
\]

\[
\Rightarrow f(x) \geq t. \quad \text{Which is a contradiction. So (1) holds. Similarly (2) holds.}
\]

Next suppose (3) does not hold. Then there exists \( x, z, a, b \in R \) and \( t, r \in (0,1] \) such that \( x + a + z = b + z \) and \( a_t b_t e_m f \) but \( x_t z e_m v_{n_f} f \). Then \( f(a) \geq t > m, f(b) \geq r > m \) and \( f(x) < \min\{t,r\} \) and \( f(x) + \min\{t,r\} < 2n \). Which implies.
\[
\max\{f(x),m\} < \min\{t,x\} \leq \min\{f(a),f(b),n\}
\]

Which contradicts \((3')\). Hence \(f\) is the \((\epsilon_m \in \epsilon \vee q_n)-fuzzy\) h-subhemiring of \(R\).

By using Theorem 3.8, the following can also be proved.

**Theorem:** Let \(f\) be a fuzzy subset of \(R\) and \(m,n \in [0,1]\) with \(m < n\). Then \(f\) is an \((\epsilon_m \in \epsilon \vee q_n)-fuzzy\) left (right) h-ideal of \(R\) if and only if \(f\) satisfies \((1')\),\((3')\) and

\[
(4') \quad \max\{f(xy),m\} \geq \min\{f(y),n\}
\]

\[
(5') \quad \max\{f(xy),m\} \geq \min\{f(x),n\}
\]

for all \(x,y \in R\).

**Theorem:** Let \(f\) be a fuzzy subset of \(R\) and \(m,n \in [0,1]\) with \(m < n\). Then \(f\) is an \((\epsilon_m \in \epsilon \vee q_n)-fuzzy\) h-quasi-ideal of \(R\) if and only if \(f\) satisfies \((1')\),\((3')\) and

\[
(6') \quad \max\{f(x),m\} \geq \min\{(f \otimes R)(x), (R \otimes f)(x), n\}
\]

for all \(x \in R\).

**Theorem:** Let \(f\) be a fuzzy subset of \(R\) and \(m,n \in [0,1]\) with \(m < n\). Then \(f\) is an \((\epsilon_m \in \epsilon \vee q_n)-fuzzy\) h-bi-ideal of \(R\) if and only if \(f\) satisfies \((1')\),\((2')\),\((3')\) and

\[
(7') \quad \max\{f(xzy),m\} \geq \min\{f(x),f(y),n\}
\]

for all \(x,y,z \in R\).

**Remark:** If \(f\) is an \((\epsilon_m \in \epsilon \vee q_n)-fuzzy\) h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of \(R\), then

(i) for \(m = 0\) and \(n = 1\), \(f\) is the fuzzy h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of \(R\), which are discussed in [23].

(ii) for \(m = 0\) and \(n = 0.5\), \(f\) is the \((\epsilon, e \vee q)-fuzzy\) h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of \(R\), which are discussed in [12, 17].

(iii) for \(m = 0.5\) and \(n = 1\), \(f\) is the \((\epsilon, \epsilon \vee q)-fuzzy\) h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of \(R\), which are discussed in [18].

(iv) for \(m = 0\) and \(n = \frac{1-k}{2}\), \(f\) is the \((\epsilon, \epsilon \vee q_k)-fuzzy\) h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of \(R\), which are discussed in [19].

(v) for \(m = \frac{1-k}{2}\) and \(n = 1\), \(f\) is the \((\epsilon, \epsilon \vee q_k)-fuzzy\) h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of \(R\), which are discussed in [20].

**Lemma:** Let \(f\) be a fuzzy subset of \(R\) and \(m,n \in [0,1]\) with \(n = \frac{1-m}{2}\). Then \(f\) is an \((\epsilon_m \in \epsilon \vee q_n)-fuzzy\) h-subhemiring of \(R\) if and only if non empty

\[
U^m_n(f;t) = \{x \in R : x \in \epsilon_m \in \epsilon \vee q_n f\}
\]

is h-subhemiring of \(R\) for all \(t \in [m,n]\).

**Proof:** Suppose \(f\) is an \((\epsilon_m \in \epsilon \vee q_n)-fuzzy\) h-subhemiring of \(R\) and let \(x,y \in U^m_n(f;t)\) for some \(t \in [m,n]\). Then \(x \in \epsilon_m \in \epsilon \vee q_n f\) and \(y \in \epsilon_m \in \epsilon \vee q_n f\). Now \(x \in \epsilon_m \in \epsilon \vee q_n f \Rightarrow f(x) \geq t > m\) or \(f(x) > 2n - t > 2n - 1 = m\). Similarly \(f(y) \geq t > m\) or \(f(y) > 2n - t > 2n - 1 = m\). Further as \(f\) is an \((\epsilon_m \in \epsilon \vee q_n)-fuzzy\) h-subhemiring of \(R\), so by Theorem 3.8,

\[
\max\{f(x+y),m\} \geq \min\{f(x),f(y),n\} > \min\{m,n\} = m
\]

\[
\Rightarrow f(x+y) \geq \min\{f(x),f(y),n\}
\]

\[
\Rightarrow x + y \in U^m_n(f;t)
\]

Similarly \(xy \in U^m_n(f;t)\). Next let \(a,b,x \in R\) such that \(x+a+z = b+z\) and \(a,b \in U^m_n(f;t)\) for some \(t \in [m,n]\). Then \(a \in \epsilon_m \in \epsilon \vee q_n f\) and \(b \in \epsilon_m \in \epsilon \vee q_n f\). Now \(a \in \epsilon_m \in \epsilon \vee q_n f \Rightarrow f(a) \geq t > m\) or \(f(a) > 2n - t > 2n - 1 = m\). Similarly \(f(b) \geq t > m\) or \(f(b) > 2n - t > 2n - 1 = m\). Further as \(f\) is an \((\epsilon_m \in \epsilon \vee q_n)-fuzzy\) h-subhemiring of \(R\), so by Theorem 3.8,

\[
\max\{f(x),m\} \geq \min\{f(a),f(b),n\} > \min\{m,n\} = m
\]

\[
\Rightarrow f(x) \geq \min\{f(a),f(b),n\}
\]

\[
\Rightarrow x \in \epsilon_m \in \epsilon \vee q_n f \Rightarrow x \in U^m_n(f;t)
\]

Hence \(U^m_n(f;t)\) is h-subhemiring of \(R\).

Conversely, assume that \(U^m_n(f;t) \neq \phi\) is h-subhemiring of \(R\) for all \(\epsilon(m,n)\). To prove \(f\) is an \((\epsilon_m \in \epsilon \vee q_n)-fuzzy\) h-subhemiring of \(R\). For this we prove \((1')\),\((2')\) and \((3')\) hold. Suppose on the contrary that there exists \(x,y \in R\) such that

\[
\max\{f(x+y),m\} < \min\{f(x),f(y),n\} = t
\]

Then \(x,y \in U^m_n(f;t)\) but \(x+y \notin U^m_n(f;t)\), which is a contradiction, so for all \(x,y \in R\) and \(t \in [m,n]\), \((1')\) is satisfied. Similarly \((2')\) is satisfied. Again suppose that there exist \(x,z,a,b \in R\) such that \(x+a+z = b+z\) and

\[
\max\{f(x),m\} < \min\{f(a),f(b),n\} = t
\]
Then $t \in (m, n]$ and $a, b \in U_n^m(f; t)$ but $x \notin U_n^m(f; t)$, which is a contradiction, so (3') is satisfied. Hence $f$ is an $(\forall_m \in m \lor \forall_n)$-fuzzy $h$-subhemiring of $R$.

**Lemma:** Let $f$ be a fuzzy subset of $R$ and $m, n \in [0, 1]$ with $n = \frac{1 + m}{2}$. Then $f$ is an $(\forall_m \in m \lor \forall_n)$-fuzzy $h$-ideal (h-quasi-ideal, h-bi-ideal) of $R$ if and only if non empty $U_n^m(f; t) = \{x \in R : x_t \in \forall_n f\}$ is h-ideal (h-quasi-ideal, h-bi-ideal) of $R$ for all $t \in (m, n]$.

**Proof:** Proof is similar to the proof of Lemma 3.13.

Proofs of the following results are straightforward, hence omitted.

**Theorem:** A non-empty subset $A$ of a hemiring $R$ is h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of $R$ if and only if the characteristic function $C_A$ of $A$ is an $(\forall_m \in m \lor \forall_n)$-fuzzy h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of $R$ for all $m, n \in [0, 1]$ and $m < n$.

**Definition:** Let $f, g$ be fuzzy subsets of $R$. Then the fuzzy subsets $f^+_{m, n}$, $f^\lor_{m, n}$ and $f^\otimes_{m, n}$ of $R$ are defined as following:

$$
(f^+_{m, n})(x) = \left[ \bigvee_{x = (a + b) + z} \{f(a) \lor f(a') \lor f(b) \lor f(b')\} \right] \land n \lor m
$$

$$
(f^\lor_{m, n})(x) = \left[ \bigvee_{x = (a + b) + z} \{f(a) \lor f(a') \lor f(b) \lor f(b')\} \right] \land n \lor m
$$

$$
(f^\otimes_{m, n})(x) = \left[ \bigvee_{x = (a + b) + z} \{f(a) \lor f(a') \lor f(b) \lor f(b')\} \right] \land n \lor m
$$

(by condition (1'))

$$
= \left[ \bigvee_{x = (a + b) + z} \{f(a + b) \land f(a' + b')\} \land n \lor m
$$

$$
= \left[ \bigvee_{x = (a + b) + z} \{f(a + b) \land f(a' + b')\} \land n \lor m
$$

(by condition (3'))

$$
= (f(x) \land n) \lor m
$$

**Definition:** Let $f, g$ be fuzzy subsets of $R$. Then the fuzzy subset $f^\land_{m, n}$ of $R$ is defined as following:

$$
(f^\land_{m, n})(x) = \left[ \bigvee_{x = (a + b) + z} \{f(a) \land f(a') \land f(b) \land f(b')\} \right] \land n \lor m
$$

for all possible expressions of $x, z, a, b, a', b' \in R$.

**Lemma:** Let $A, B \subseteq R$. Then

$$
(C_A^\land_{m, n} B)(x) = (C_B^\land_{m, n} A)(x) \land n \lor m
$$

**Proof:** Proof is straightforward.

**Lemma:** Let $f$ be a fuzzy subset of $R$. Then $f$ satisfies conditions (1') and (3') if and only if it satisfies condition

$$(8') f^\land_{m, n} f \leq (f \land n) \lor m$$

**Proof:** Suppose $f$ satisfies conditions (1') and (3'). Let $x + (a + b) + z = (a' + b') + z$

for some $a, a', b, b', x, z \in R$. Then

$$
= (f(x) \land n) \lor m
$$
Thus \( f +_m^n f \leq (f \land n) \lor m \).

Conversely, assume that \((f +_m^n f)(x) \leq ((f \land n)(x)) \lor m\). Then for each \(x, \in \mathbb{R}\) we have \(0 + x + z = x + z\). Thus

\[
\begin{align*}
  f(0) & \lor m \geq ((f \land n)(0)) \lor m \geq (f +_m^n f)(0) = \left[ \bigvee_{0+(a+b)+z = (a'+b') \neq z} \left\{ f(a) \land f(a') \land f(b) \land f(b') \right\} \right] \land n \lor m \\
  & \geq \left[ \bigvee_{0+(a+b)+z = (a'+b') \neq z} \left\{ f(a) \land f(a') \land f(b) \land f(b') \right\} \right] \land n \geq f(x) \land n
\end{align*}
\]

This implies \( f(0) \lor m \geq f(x) \land n \). (*).

Let \(x, y, z \in \mathbb{R}\). Then for all \(a, a', b, b' \in \mathbb{R}\) such that \((xy)(ab) \lor z = (a'b') \lor z\) we have

\[
\begin{align*}
  \max\{f(x \lor y), m\} & \geq \max\{(f \land n)(x \lor y), m\} \geq (f +_m^n f)(x \lor y) \\
  & = \left[ \bigvee_{(x+y)+(a+b)+z = (a'+b') \neq z} \left\{ f(a) \land f(a') \land f(b) \land f(b') \right\} \right] \land n \lor m \\
  & \geq \left[ \bigvee_{(x+y)+(a+b)+z = (a'+b') \neq z} \left\{ f(a) \land f(a') \land f(b) \land f(b') \right\} \right] \land n \geq f(x) \land n
\end{align*}
\]

(because \((x+y)+(0+0)+z = (x+y)+z\))

Thus \( f \) satisfies condition (1').

Now let \(a, a', b, b' \in \mathbb{R}\) such that \(x + a + z = b + z\). Then for all possible \(a, a', b, b' \in \mathbb{R}\) satisfying the identity \(x + (a + b) + z = (a' + b') + z\) we have

\[
\begin{align*}
  \max\{f(x), m\} & \geq \max\{(f \land n)(x), m\} \geq (f +_m^n f)(x) \\
  & = \left[ \bigvee_{x + (a+b)+z = (a'+b') \neq z} \left\{ f(a) \land f(a') \land f(b) \land f(b') \right\} \right] \land n \lor m \\
  & \geq \left[ \bigvee_{x + (a+b)+z = (a'+b') \neq z} \left\{ f(a) \land f(a') \land f(b) \land f(b') \right\} \right] \land n \geq f(a) \land f(b) \land n
\end{align*}
\]

because \(x + a + z = b + z = \min\{f(a), f(b), n\}\)

Thus \( f \) satisfies condition (3').

**Theorem:** Let \( f \) be a fuzzy subset of \( \mathbb{R} \). Then \( f \) is an \((\in_m \in \lor \in_q \lor n)\)-fuzzy left (resp. right) \( h \)-ideal of \( \mathbb{R} \) if and only if \( f \) satisfies conditions

\[
\begin{align*}
  (8') & \quad f +_m^n f \leq (f \land n) \lor m \\
  (9') & \quad R \otimes _m^n f \leq (f \land n) \lor m \quad \text{resp.} \quad f \otimes _m^n R \leq (f \land n) \lor m
\end{align*}
\]

**Proof:** Working on the same lines as in Lemma 3.20, proof follows directly.

**Theorem:** Let \( f \) be a fuzzy subset of \( \mathbb{R} \). Then \( f \) is an \((\in_m \in \lor \in_q \lor n)\)-fuzzy \( h \)-quasi-ideal of \( \mathbb{R} \) if and only if \( f \) satisfies conditions (6') and (8').

**Proof:** Proof is straightforward because by Lemma 3.20, conditions (1') and (3') are equivalent to condition (8').

---

** Lemma:** Every \((\in_m \in \lor \in_q \lor n)\)-fuzzy left (right) \( h \)-ideal of \( \mathbb{R} \) is an \((\in_m \in \lor \in_q \lor n)\)-fuzzy \( h \)-quasi-ideal of \( \mathbb{R} \).

**Proof:** Proof is straightforward.

**Lemma:** Every \((\in_m \in \lor \in_q \lor n)\)-fuzzy \( h \)-quasi-ideal of \( \mathbb{R} \) is an \((\in_m \in \lor \in_q \lor n)\)-fuzzy \( h \)-bi-ideal of \( \mathbb{R} \).

**Proof:** Proof is straightforward.

**Remark:** Converse of the Lemma 3.23 and Lemma 3.24 are not true in general.

**Example:** Let \( Z_0 = Z^+ \cup \{0\} \),

\[
R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z_0 \right\}
\]

and

\[
A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z_0 \right\}
\]
Then R is a hemiring under the usual operations of addition and multiplication of matrices and A is a h-quasi-ideal of R but A is not left (right) h-ideal of R.

Then by Theorem 3.15, \(C_A\) is an \((\in_m, \in_m \lor q_n)\)-fuzzy h-quasi-ideal of R and \(C_A\) is not an \((\in_m, \in_m \lor q_n)\)-fuzzy left (right) h-ideal of R.

Example: Let \(Z^+\) and \(R^+\) be the sets of all positive integers and positive real numbers, respectively. And

\[
R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b \in R^+, c \in Z^+ \right\}
\]

\[
I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b \in R^+, c \in Z^+, a < b \right\}
\]

Then R is a hemiring under the usual operations of addition and multiplication of matrices and I is a right h-ideal and J is a left h-ideal of R. Further the product \(IJ\) is a h-bi-ideal of R and it is not a h-quasi-ideal of R.

Then by Theorem 3.15, \(C_{IJ}\) is an \((\in_m, \in_m \lor q_n)\)-fuzzy h-bi-ideal of R and is not an \((\in_m, \in_m \lor q_n)\)-fuzzy h-quasi-ideal of R.

Lemma: If \(f, g\) are \((\in_m, \in_m \lor q_n)\)-fuzzy right and left h-ideals of R respectively, then

\[
f \otimes^n_m g \leq f \land^n_m g .
\]

Proof: Let \(x \in R\). If \(f \otimes^n_m g)(x) = m\). Then \(f \otimes^n_m g \leq f \land^n_m g\). Otherwise
h-HEMIREGULAR HEMIRINGS

In this section we characterize h-hemiregular hemirings by the properties of their \((e_m \in m \lor q_n)-fuzzy h\)-ideals, \((e_m \in m \lor q_n)-fuzzy h\)-quasi-ideals and \((e_m \in m \lor q_n)-fuzzy h\)-bi-ideals of \(R\).

**Theorem:** For a hemiring \(R\) the following conditions are equivalent.

(i) \(R\) is h-hemiregular.

(ii) \(f \land_n g = f \land_m g\) for every \((e_m \in m \lor q_n)-fuzzy right and left h-ideals \(f\) and \(g\) of \(R\), respectively.

**Proof:** (i)\(\Rightarrow\)(ii) Let \(x \in R\). Then there exists \(a,a' \in R\) such that \(x + xax = xa'x + z\). Now

\[
(f \land_m^n g)(x) = \bigvee_{x+\Sigma_{i=1}^p a_i b_i + z = \Sigma_{j=1}^q a'_j b'_j + z} \left( \bigwedge_{i=1}^p f(a_i) \right) \land \left( \bigwedge_{j=1}^q f(a'_j) \right) \land n \land m 
\]

because

\[
x + xax + z = xa'x + z \geq [f(x) \land g(x) \land n] \land m = (f \land_n m g)(x)
\]

Thus \((f \land_m^n g)(x) \geq (f \land_n m g)(x)\).

But by Lemma Le9g, \((f \land_m^n g)(x) \leq (f \land_n m g)(x)\) hence \(f \land_n m g = f \land_m^n g\)

(ii)\(\Rightarrow\)(i) Let \(I\) and \(L\) be right and left h-ideals of \(R\), respectively. Then by Theorem Th9g, \(C_I\) is an \((e_m \in m \lor q_n)-fuzzy right and \(C_L\) is an \((e_m \in m \lor q_n)-fuzzy left h-ideal of \(R\). Then by hypothesis

\[C_I \land_m^n C_L = (C_I \land n) \lor m = (C_l \land L \land n) \lor m \Rightarrow I \lor L = \overline{I} \lor R = R\]

is h-hemiregular.

**Theorem:** For a hemiring \(R\), the following conditions are equivalent.

(i) \(R\) is h-hemiregular.

(ii) \(f \land n) \lor m = (f \land_m^n R \land_m f) \) for every \((e_m \in m \lor q_n)-fuzzy h\)-bi-ideal \(f\) of \(R\).

(iii) \((f \land n) \lor m = (f \land_m^n R \land_m f) \) for every \((e_m \in m \lor q_n)-fuzzy h\)-quasi-ideal \(f\) of \(R\).

**Proof:** (i)\(\Rightarrow\)(ii) Let \(x \in R\), then there exists \(a,a' \in R\) such that \(x + xax = xa'x\). Now

\[
(f \land_m^n R \land_m f)(x) = \bigvee_{x+\Sigma_{i=1}^p a_i b_i + z = \Sigma_{j=1}^q a'_j b'_j + z} \left( \bigwedge_{i=1}^p f(a_i) \right) \land \left( \bigwedge_{j=1}^q f(a'_j) \right) \land n \land m 
\]

\[
= \left( \bigwedge_{i=1}^p f(a_i) \right) \land \left( \bigwedge_{j=1}^q f(a'_j) \right) \land n \land m
\]
Theorem: For a hemiring $R$, the following conditions are equivalent.

(i) $R$ is h-hemiregular.

(ii) $(f \wedge \mathcal{g}) \leq (f \mathcal{g} \otimes \mathcal{f})$ for every $(\mathcal{g} \otimes \mathcal{f})$-fuzzy h-bi-ideal $f$ and $(\mathcal{g} \otimes \mathcal{f})$-fuzzy h-ideal $g$ of $R$.

(iii) $(f \wedge \mathcal{g}) \leq (f \mathcal{g} \otimes \mathcal{f})$ for every $(\mathcal{g} \otimes \mathcal{f})$-fuzzy h-quasi-ideal $f$ and $(\mathcal{g} \otimes \mathcal{f})$-fuzzy h-ideal $g$ of $R$.

Proof: (i)$\Rightarrow$(ii) Let $f$ be any $(\mathcal{g} \otimes \mathcal{f})$-fuzzy h-bi-ideal and $g$ any $(\mathcal{g} \otimes \mathcal{f})$-fuzzy h-ideal of $R$. Since $R$ is h-hemiregular, so for any $a \in R$ there exist $x_1, x_2, z \in R$ such that $a + ax_1 + a + z = ax_2 + a + z$. Now

$$
\left( f \mathcal{g} \otimes \mathcal{f} \right)(a) = \bigvee_{a + x_1, a + x_1, z = a + x_2, a + z} \left[ \left( f \mathcal{g} \otimes \mathcal{f} \right)(a_1) \wedge \left( f \mathcal{g} \otimes \mathcal{f} \right)(a_2) \right] \wedge n \vee m
$$

(because $a + ax_1 + a + z = a + x_2$)

$$
\geq \left[ \left( f \mathcal{g} \otimes \mathcal{f} \right)(a_1) \wedge \left( f \mathcal{g} \otimes \mathcal{f} \right)(a_2) \right] \wedge n \vee m
$$

(because $a + ax_1 + a + z = a + x_2 + x_1 + z$ and $ax_2 + a + ax_1 + a + z = a + x_2 + x_1 + z$)

(ii)$\Rightarrow$(iii) This is straightforward by using Lemma 3.24.

(iii)$\Rightarrow$(i) Let $f$ be any $(\mathcal{g} \otimes \mathcal{f})$-fuzzy h-quasi-ideal and $R$ be the $(\mathcal{g} \otimes \mathcal{f})$-fuzzy h-ideal of $R$. Then by hypothesis,

$$
\left( f \mathcal{g} \otimes \mathcal{f} \right) \leq \left( f \mathcal{g} \otimes \mathcal{f} \right) \Rightarrow (f \wedge \mathcal{g}) \wedge m \leq \left( f \mathcal{g} \otimes \mathcal{f} \right)
$$

Then by Theorem 4.2, $R$ is h-hemiregular.

Theorem: For a hemiring $R$, the following conditions are equivalent.
(i) \( R \) is h-hemiregular.

(ii) \( f \wedge_m^n g \leq (f \Theta_m^n g) \) for every \( (e_m \vDash_m v \cdot q_n) \)-fuzzy h-bi-ideal \( f \) and \( (e_m \vDash_m v \cdot q_n) \)-fuzzy left h-ideal \( g \) of \( R \).

(iii) \( f \wedge_m^n g \leq (f \Theta_m^n g) \) for every \( (e_m \vDash_m v \cdot q_n) \)-fuzzy h-quasi-ideal \( f \) and \( (e_m \vDash_m v \cdot q_n) \)-fuzzy left h-ideal \( g \) of \( R \).

(iv) \( f \wedge_m^n g \leq (f \Theta_m^n g) \) for every \( (e_m \vDash_m v \cdot q_n) \)-fuzzy right h-ideal \( f \) and \( (e_m \vDash_m v \cdot q_n) \)-fuzzy h-bi-ideal \( g \) of \( R \).

(v) \( f \wedge_m^n g \leq (f \Theta_m^n g) \) for every \( (e_m \vDash_m v \cdot q_n) \)-fuzzy right h-ideal \( f \) and \( (e_m \vDash_m v \cdot q_n) \)-fuzzy h-quasi-ideal \( g \) of \( R \).

(vi) \( f \wedge_m^n g \wedge_m^n \lambda \leq (f \Theta_m^n g \Theta_m^n \lambda) \) for every \( (e_m \vDash_m v \cdot q_n) \)-fuzzy right h-ideal \( f \), \( (e_m \vDash_m v \cdot q_n) \)-fuzzy h-bi-ideal \( g \) and \( (e_m \vDash_m v \cdot q_n) \)-fuzzy left h-ideal \( \lambda \) of \( R \).

(vii) \( f \wedge_m^n g \wedge_m^n \lambda \leq (f \Theta_m^n g \Theta_m^n \lambda) \) for every \( (e_m \vDash_m v \cdot q_n) \)-fuzzy right h-ideal \( f \), \( (e_m \vDash_m v \cdot q_n) \)-fuzzy h-quasi-ideal \( g \) and \( (e_m \vDash_m v \cdot q_n) \)-fuzzy left h-ideal \( \lambda \) of \( R \).

**Proof:** Working on the same lines as in Theorem 4.3, proof is straightforward.

**h-INTRA-HEMIREGULAR HEMIRINGS**

In this section we characterize h-intra-hemiregular hemirings and hemirings which are both h-hemiregular and h-intra-hemiregular in terms of their \( (e_m \vDash_m v \cdot q_n) \)-fuzzy h-ideals, \( (e_m \vDash_m v \cdot q_n) \)-fuzzy h-quasi-ideals and \( (e_m \vDash_m v \cdot q_n) \)-fuzzy h-bi-ideals.

**Theorem:** A hemiring \( R \) is h-intra-hemiregular if and only if \( f \wedge_m^n g \leq (f \Theta_m^n g) \) for every \( (e_m \vDash_m v \cdot q_n) \)-fuzzy left h-ideal \( f \) and for every \( (e_m \vDash_m v \cdot q_n) \)-fuzzy right h-ideal \( g \) of \( R \).

**Proof:** Let \( R \) be an h-intra-hemiregular, \( f \) be an \( (e_m \vDash_m v \cdot q_n) \)-fuzzy left h-ideal and \( g \) an \( (e_m \vDash_m v \cdot q_n) \)-fuzzy right h-ideal of \( R \). As \( R \) is h-intra-hemiregular so for every \( x \in R \), there exist \( a_i, b_i, b_j , z \in R \) such that

\[
x + \sum_{i=1}^{p} a_i x^2 a_i + z = \sum_{j=1}^{q} b_j x^2 b_j + z
\]

Then

\[
(f \Theta_m^n g)(x) = \bigvee_{x + \sum_{i=1}^{p} a_i x^2 a_i + z = \sum_{j=1}^{q} b_j x^2 b_j + z} \left[ f \left( a_i x \right) \wedge f \left( b_j x \right) \wedge g \left( x \right) \wedge g \left( x \right) \wedge \right] \wedge \left( f \left( a_j x \right) \wedge f \left( b_i x \right) \wedge g \left( x \right) \wedge g \left( x \right) \wedge \right)
\]

(because \( x + \sum_{i=1}^{p} (a_i x^2 a_i) + z = \sum_{j=1}^{q} (b_j x^2 b_j) + z \))

Conversely assume that \( A \) and \( B \) are left and right h-ideals of \( R \), respectively. Then by Theorem 3.15, the characteristic functions \( C_A \) and \( C_B \) are respectively \( (e_m \vDash_m v \cdot q_n) \)-fuzzy left h-ideal and \( (e_m \vDash_m v \cdot q_n) \)-fuzzy right h-ideal of \( R \). Then by given hypothesis

\[
C_A \wedge_m^n C_B \leq C_A \Theta_m^n C_B \Rightarrow (C_A \land_v n) \lor (C_B \land_v n) \lor (A \land_v B) \leq \overline{AB}
\]

Thus by Lemma 2.10, \( R \) is h-intra-hemiregular.

**Theorem:** The following conditions are equivalent for a hemiring \( R \):
(i) $R$ is both $h$-hemiregular and $h$-intra-hemiregular.

(ii) $(f \wedge n) \vee m = f \otimes_m^n f$ for every $(e_m \in e_m \vee q_n)$-fuzzy $h$-bi-ideal $f$ of $R$.

(iii) $(f \wedge n) \vee m = f \otimes_m^n f$ for every $(e_m \in e_m \vee q_n)$-fuzzy $h$-quasi-ideal $f$ of $R$.

**Proof:** (i)$\Rightarrow$(ii) Let $f$ be an $(e_m \in e_m \vee q_n)$-fuzzy $h$-bi-ideal of $R$ and $x \in R$. Since $R$ is both $h$-hemiregular and $h$-intra-hemiregular, there exist elements $a_1, a_2, p_1, p_2, q_1, q_2, z \in R$ such that

$$x + \sum_{j=1}^q (x a q x) + \sum_{j=1}^q (x a q x) + \sum_{i=1}^p (x a p x) (x a q x) + \sum_{i=1}^q (x a p x) (x a q x) + \sum_{i=1}^q (x a p x) (x a q x) + \sum_{i=1}^q (x a p x) (x a q x) + z$$

$$= \sum_{i=1}^p (x a p x) (x a q x) + \sum_{j=1}^q (x a q x) + \sum_{i=1}^p (x a p x) (x a q x) + \sum_{i=1}^q (x a p x) (x a q x) + \sum_{i=1}^q (x a p x) (x a q x) + \sum_{i=1}^q (x a p x) (x a q x) + z$$

$$= \left[ (f(x) \wedge n) \vee m \right] \vee m = (f(x) \wedge n) \vee m$$

This implies that $f \otimes_m^n f \geq (f \wedge n) \vee m$

On the other hand if

$$x + \sum_{i=1}^p (x a_i b_i) + z = \sum_{j=1}^q (x a_j b_j) + z$$

we have

$$\{f(x) \wedge n\} \vee m \geq \{f(x) \wedge m\} \vee m \geq \left[ f(x) \wedge n \right] \vee m \geq \left[ f(x) \wedge m \right] \vee m \geq \left[ \sum_{i=1}^p (a_i b_i) \wedge f(b) \right] \wedge n \vee m \geq \left[ \sum_{i=1}^p (a_i b_i) \wedge f(b) \right] \wedge n \vee m$$

(because $f$ is an $(e_m \in e_m \vee q_n)$-fuzzy $h$-bi-ideal of $R$)

Thus

$$(f \otimes_m^n f)(x) = \left[ (f(x) \wedge n) \vee m \right] \vee m \leq (f(x) \wedge n) \vee m.$$
(iii)⇒(i) Let Q be an h-quasi-ideal of R. Then $C_Q$ is an $(\epsilon_m \in \epsilon_m \circ \bigvee g)$-fuzzy h-quasi-ideal of R. Thus by hypothesis

$$[C_Q \wedge n] \vee m = C_Q \bigotimes_m C_Q = [C_Q \otimes C_Q \wedge n] \vee m$$

Then it follows $Q = \overline{C_Q}$. Hence by Lemma 2.12, R is both h-hemiregular and h-intra-hemiregular.

**Theorem:** The following conditions are equivalent for a hemiring R:

(i) R is both h-hemiregular and h-intra-hemiregular.

(ii) $f \wedge g \leq f \bigotimes g$ for all $(\epsilon_m \in \epsilon_m \circ \bigvee g)$-fuzzy h-bi-ideals f and g of R.

(iii) $f \wedge g \leq f \bigotimes g$ for every $(\epsilon_m \in \epsilon_m \circ \bigvee g)$-fuzzy h-bi-ideal f and every $(\epsilon_m \in \epsilon_m \circ \bigvee g)$-fuzzy h-quasi-ideals g of R.

(iv) $f \wedge g \leq f \bigotimes g$ for every $(\epsilon_m \in \epsilon_m \circ \bigvee g)$-fuzzy h-quasi-ideal f and every $(\epsilon_m \in \epsilon_m \circ \bigvee g)$-fuzzy h-bi-ideals g of R.

(v) $f \wedge g \leq f \bigotimes g$ for all $(\epsilon_m \in \epsilon_m \circ \bigvee g)$-fuzzy h-quasi-ideals f and g of R.

**Proof:** Working on the same lines as in Theorem 5.2, proof is straightforward.

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