World Applied Sciences Journal 18 (12): 1870-1882, 2012
ISSN 1818-4952

DOI: 10.5829/idosi.wasj.2012.18.12.2596

Ideals Redefined to Characterize Hemirings

[1]*Tahir Mahmood and* [2]*Muhammad Shabir*

[1]Department of Mathematics, International Islamic Univesity, Islamabad, Pakistan
[2]Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan

Abstract: In this paper we define $(\in_m, \in_m \vee q_n)$-fuzzy h-ideals, $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideals and $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideals. We also characterize h-hemiregular and h-intra-hemiregular hemirings by the properties of their $(\in_m, \in_m \vee q_n)$-fuzzy h-ideals, $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideals and $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideals.

2010 mathematics subject classification: 16Y60 . 08A72 . 03G25 . 03E72

Key words: $(\in_m, \in_m \vee q_n)$-fuzzy h-ideals . $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideals . $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideals . h-hemiregular hemirings . h-intra-hemiregular hemirings

INTRODUCTION

As a generalization of associative rings and distributive lattices semirings are algebraic structures with two binary operations and were introduced by Vandiver [21]. In recent times it is found that semirings have been deeply studied, especially in relation with applications [9] and for solving many problems in optimization theory, graph theory, theory of discrete event, dynamical systems, matrices, determinants, generalized fuzzy computation, theory of automata, formal language theory, coding theory, analysis of computer programmes [8, 13, 22]. Semirings with commutative addition and zero element are called hemirings.

Ideals of hemirings and semirings play a cruial role in the structure theory and are useful for many purposes. However, in general, they do not coincide with the usual ring ideals. Many results in rings apparantly have no analogues in hemirings using only ideals. In order to overcome this deficiency Henriksen [10] defined a more restricted class of ideals in semirings, called k-ideals, with the property that if a semiring R is ring, then a subset of R is a k-ideal if and only if it is a ring ideal.

Multi-valued logic has been considered by model phenomena due to the reason that uncertainity and vagueness are involved. To handle such situations the most famous theory of fuzzy sets was first developed by Zadeh [25] in 1965. Since then it has been used extensively in different fields of life and has also been applied to many branches in Mathematics. The fuzzification of algebraic structures was initiated by Rosenfeld [16] and he introduced the notion of fuzzy subgroups. In [3] J. Ahsan initiated the study of fuzzy semirings [2]. The fuzzy algebraic structures play an important role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces [1, 9, 15].

The notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets proposed and discussed in [14, 15]. Many authors used these concepts to generalize some concepts of algebra, for example [4-6, 11]. In [7, 12] (α,β)-fuzzy ideals of hemirings are defined. In [19] authors characterized hemirings by the properties of their $(\in, \in \vee q_k)$-fuzzy ideals. In this paper we generalize these concepts and by using the concept given in [24] we define $(\in_m, \in_m \vee q_n)$-fuzzy h-ideals, $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideals and $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideals. We also characterize h-hemiregular and h-intra-hemiregular hemirings by the properties of their $(\in_m, \in_m \vee q_n)$-fuzzy h-ideals, $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideals and $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideals.

PRELIMINARIES

For undefined notions we refer to [9, 23]. A semiring is a non-empty set R together with two associative binary operations addition "+" and multiplication "." such that "." is distributive over "+" in R. An element $0 \in R$ is called a zero or additive

Corresponding Author: Tahir Mahmood, Department of Mathematics, International Islamic Univesity, Islamabad, Pakistan

identity of the semiring $(R,+,\cdot)$ if $0x = x0 = 0$ and $0+x = x+0 = x$ for all $x\in R$. An additively commutative semiring with zero is called a hemiring. An element 1 of a hemiring R is called the identity or unity of R if $1.x = x.1 = x$ for all $x\in R$. A hemiring with commutative multiplication is called a commutative hemiring. A non-empty subset A of a hemiring R is called a sub hemiring of R if it contains zero and is closed with respect to the addition and multiplication of R. A non-empty subset I of a hemiring R is called a left (right) ideal of R if I is closed under addition and $RI\subseteq I$ ($RI\subseteq I$). A non-empty subset I of a hemiring R is called an ideal of R if it is both a left ideal and a right ideal of R. A subset Q of a hemiring R is called a quasi-ideal of R if Q is closed under addition and $RQ\cap QR\subseteq Q$. A sub hemiring B of a hemiring R is called a bi-ideal of R if $BRB\subseteq B$. Every one sided ideal of a hemiring R is a quasi-ideal and every quasi-ideal is a bi-ideal but the converse is not true.

A left (right) ideal I of a hemiring R is called a left (right) h-ideal if for all $x,z\in R$ and for any $a,b\in I$ from $x+a+z = b+z$ it follows $x\in I$. A bi-ideal B of a hemiring R is called an h-bi-ideal of R if for all $x,z\in R$ and $a,b\in B$ from $x+a+z = b+z$ it follows $x\in B$ [23].

The h-closure $\overline{A}$ of a non-empty subset A of a hemiring R is defined as

$$\overline{A}=\{x\in R \mid x+a+z=b+z \text{ for some } a,b\in A, z\in R\}$$

A quasi-ideal Q of a hemiring R is called an h-quasi-ideal of R if $\overline{RQ}\cap\overline{QR}\subseteq Q$ and $x+a+z = b+z$ implies $x\in Q$, for all $x,z\in R$ and $a,b\in Q$ [23]. Every left (right) h-ideal of a hemiring R is an h-quasi-ideal of R and every h-quasi-ideal is an h-bi-ideal of R. However, the converse is not true in general.

A fuzzy subset f of a universe X is a function from X into the unit closed interval [0,1], that is f: $X\to[0,1]$. A fuzzy subset f of a universe X of the form

$$f(y)=\begin{cases} t\in(0,1] & \text{if } y=x \\ 0 & \text{if } y\neq x\end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t. For a fuzzy point x_t and a fuzzy set f of a set X, Pu and Liu [15] gave meaning to the symbol $x_t\alpha f$, where $\alpha\in\{\in,q,\in\vee q,\in\wedge q\}$. A fuzzy point x_t is said to belong to (resp. be quasi-coincident with) a fuzzy set f written $x_t\in f$ $(\text{resp. } x_t q f)$ if $f(x)\geq t$ $(\text{resp. } f(x)+t>1)$ and in this case, $x_t\in\vee qf$ $(\text{resp. } x_t\in\wedge qf)$ means that $x_t\in f$ or $x_t q f$ $(\text{resp. } x_t\in f \text{ and } x_t q f)$. To say that $x_t\overline{\alpha} f$ means that $x_t\alpha f$ does not hold. For any two fuzzy subsets f and g of X, $f\leq g$ means that, for all $x\in X$, $f(x)\leq g(x)$. The symbols $f\wedge g$ and $f\vee g$ will mean the following fuzzy subsets of X

$$(f\wedge g)(x)=\max\{f(x),g(x)\}$$

$$(f\vee g)(x)=\min\{f(x),g(x)\}$$

Definition [23]: Let f and g be two fuzzy subsets of a hemiring R. The h-intrinsic product of f and g is defined by

$$(f\otimes g)(x)=\begin{cases}\displaystyle\bigvee_{x+\Sigma_{i=1}^{m}a_ib_i+z=\Sigma_{j=1}^{n}a'_jb'_j+z}\left\{\left(\bigwedge_{i=1}^{p}f(a_i)\right)\wedge\left(\bigwedge_{i=1}^{p}g(b_i)\right)\wedge\left(\bigwedge_{j=1}^{q}f(a'_j)\right)\wedge\left(\bigwedge_{j=1}^{q}g(b'_j)\right)\right\} \\ 0 \qquad \text{if } x \text{ cannot be expressed as } x+\Sigma_{i=1}^{p}a_ib_i+z=\Sigma_{j=1}^{q}a'_jb'_j+z\end{cases}$$

Definition: A fuzzy subset f of a hemiring R is called a fuzzy h-subhemiring of R if for all $x,y,z,a,b\in R$, we have

(i) $f(x+y)\geq\min\{f(x),f(y)\}$

(ii) $f(xy)\geq\min\{f(x),f(y)\}$

(iii) $x+a+z=b+z\Rightarrow f(x)\geq\min\{f(a),f(b)\}$

Definition: A fuzzy subset f of a hemiring R is called a fuzzy left (right) h-ideal of R if for all $x,y,z,a,b\in R$, we have

(i) $f(x+y)\geq\min\{f(x),f(y)\}$

(ii) $f(xy)\geq f(y)$ $(f(xy)\geq f(x))$

(iii) $x+a+z=b+z\Rightarrow f(x)\geq\min\{f(a),f(b)\}$

A fuzzy subset f of R is called a fuzzy h-ideal of R if it is both a fuzzy left and a fuzzy right h-ideal of R.

Definition [23]: A fuzzy subset f of a hemiring R is called a fuzzy h-bi-ideal of R if for all $x,y,z,a,b \in R$, we have

(i) $f(x+y) \geq \min\{f(x), f(y)\}$

(ii) $f(xy) \geq \min\{f(x), f(y)\}$

(iii) $f(xyz) \geq \min\{f(x), f(z)\}$

(iv) $x+a+z = b+z \Rightarrow f(x) \geq \min\{f(a), f(b)\}$

Definition [23]: A fuzzy subset f of a hemiring R is called a fuzzy h-quasi-ideal of R if for all $x,y,z,a,b \in R$, we have

(i) $f(x+y) \geq \min\{f(x), f(y)\}$

(ii) $(f \otimes \mathbf{R}) \wedge (\mathbf{R} \otimes f) \leq f$

(iii) $x+a+z = b+z \Rightarrow f(x) \geq \min\{f(a), f(b)\}$

where R is the fuzzy subset of R mapping every element of R on 1.

Note that if f is a fuzzy left h-ideal (right h-ideal, h-bi-ideal, h-quasi-ideal) , then $f(0) \geq f(x)$ for all $x \in R$.

Definition [23]: A hemiring R is said to be h-hemiregular if for each $x \in R$, there exist $a,b,z \in R$ such that

$$x + xax + z = xbx + z$$

Definition [23]: A hemiring R is said to be h-intra-hemiregular if for each $x \in R$,, there exist $a_i, a_i', b_j, b_j', z \in R$ such that

$$x + \sum_{i=1}^{m} a_i x^2 a_i' + z = \sum_{j=1}^{n} b_j x^2 b_j' + z$$

Lemma [23]: If I and L are respectively, right and left h-ideals of a hemiring R, then $\overline{IL} \subseteq I \cap L$.

Lemma [23]: A hemiring R is h-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R we have $\overline{IL} = I \cap L$.

Lemma [23]: A hemiring R is h-intra-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R we have $I \cap L \subseteq \overline{LI}$.

Lemma [23]: Let R be a hemiring. Then the following conditions are equivalent.

(i) R is h-hemiregular.

(ii) $B = \overline{BRB}$ for every h-bi-ideal B of R.

(iii) $Q = \overline{QRQ}$ for every h-quasi-ideal Q of R.

Lemma [23]: The following conditions are equivalent for a hemiring R.

(i) R is both h-hemiregular and h-intra-hemiregular.

(ii) $B = \overline{B^2}$ for every h-bi-ideal B of R.

(iii) $Q = \overline{Q^2}$ for every h-quasi-ideal Q of R.

$(\in_m, \in_m \vee q_n)$-FUZZY h-IDEALS

In this section we will discuss $(\in_m, \in_m \vee q_n)$-fuzzy h-ideals, $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideals and $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideals of R.

Let $m,n \in [0,1]$ and $m<n$ and x_t is a fuzzy subset of R. Then,

(i) $x_t \in_m f \Rightarrow f(x) \geq t > m$

(ii) $x_t q_n f \Rightarrow f(x) + t > 2n$

(iii) $x_t \in_m \vee q_n f \Rightarrow x_t \in_m f$ or $x_t q_n f$

(iv) $x_t \in_m \wedge q_n f \Rightarrow x_t \in_m f$ and $x_t q_n f$

(v) $x_t \overline{\alpha} f$ means that $x_t \alpha f$ does not hold, where $\alpha \in \{\in_m, q_n, \in_m \vee q_n, \in_m \wedge q_n\}$

for all fuzzy subsets f of R, [24].

Definition: Let f,g be fuzzy subsets of R and $m,n \in [0,1]$ with $m<n$. Then on the set of all fuzzy subsets of R an ordered relation "$[\in q]_n^m$" is defined as follows:

"$f[\in q]_n^m g$ means $x_t \in_m f \Rightarrow x_t \in_m \vee q_n g$", for all $x \in R$.

Lemma: Let f,g be fuzzy subsets of R and $m,n \in [0,1]$ with $m<n$ Then the relation "?" on the set of all fuzzy subsets of R defined by

$$\text{"}f ? g \Leftrightarrow f[\in q]_n^m g \text{ and } g[\in q]_n^m f\text{"}$$

is an equivallance relation.

Lemma: Let f,g be fuzzy subsets of R and $m,n \in [0,1]$ with $m<n$. Then $f[\in q]_n^m g$ if and only if $\max\{g(x),m\} \geq \min\{f(x),n\}$.

Definition: Let f be fuzzy subset of R and $m,n\in[0,1]$ with m<n. Then f is said to be $(\in_m,\in_m\vee q_n)$-fuzzy h-subhemiring of R if

(1) $(f+f)[\in q]_n^m f$

(2) $(f\otimes f)[\in q]_n^m f$

(3) $x+a+z=b+z$ and $a_t,b_r\in_m f\Rightarrow x_{t\wedge r}\in_m\vee q_n f$, for all $a,b,x,z\in R$ and $t,r\in(0,1]$.

Definition: Let f be a fuzzy subset of R and $m,n\in[0,1]$ with m<n. Then f is said to be $(\in_m,\in_m\vee q_n)$-fuzzy left (right) h-ideal of R if it satisfies (1),(3) and

(4) $(\mathbf{R}\otimes f)[\in q]_n^m f$ (resp.

(5) $(f\otimes\mathbf{R})[\in q]_n^m f$)

A fuzzy subset f of R is said to be an $(\in_m,\in_m\vee q_n)$-fuzzy h-ideal of R if it is both $(\in_m,\in_m\vee q_n)$-fuzzy left h-ideal and $(\in_m,\in_m\vee q_n)$-fuzzy right h-ideal of R.

Definition: Let f be a fuzzy subset of R and $m,n\in[0,1]$ with m<n. Then f is said to be an $(\in_m,\in_m\vee q_n)$-fuzzy h-quasi-ideal of R if it satisfies (1),(3) and

(6) $((f\otimes\mathbf{R})\wedge(\mathbf{R}\otimes f))[\in q]_n^m f$

Definition: Let f be a fuzzy subset of R and $m,n\in[0,1]$ with m<n. Then f is said to be an $(\in_m,\in_m\vee q_n)$-fuzzy h-bi-ideal of R if it satisfies (1),(2),(3) and

(7) $(f\otimes\mathbf{R}\otimes f)[\in q]_n^m f$

Theorem: Let f be a fuzzy subset of R and $m,n\in[0,1]$ with m<n. Then f is an $(\in_m,\in_m\vee q_n)$-fuzzy h-subhemiring of R if and only if f satisfies the following:

(1′) $\max\{f(x+y),m\}\geq\min\{f(x),f(y),n\}$

(2′) $\max\{f(xy),m\}\geq\min\{f(x),f(y),n\}$

(3′) $x+a+z=b+z\Rightarrow\max\{f(x),m\}\geq\min\{f(a),f(b),n\}$

for all $a,b,x,y,z\in R$.

Proof: Assume f is an $(\in_m,\in_m\vee q_n)$-fuzzy h-subhemiring of R. To prove (1′),(2′) and (3′) hold. Suppose (1′) does not hold. Then there exists $x,y\in R$ and $t\in(0,1]$, such that

$$\max\{f(x+y),m\}<t<\min\{f(x),f(y),n\}$$

Then

$$f(x+y)<t<n,f(x)>t>m,f(y)>t>m$$
$$\Rightarrow(x+y)_t\overline{\in_m\vee q_n}f \qquad (*)$$

But

$$(f+f)(x+y)=\bigvee_{(x+y)+(a_1+b_1)+z=(a_2+b_2)+z}\{f(a_1)\wedge f(a_2)\wedge f(b_1)\wedge f(b_2)\}$$
$$\geq\{f(0)\wedge f(x)\wedge f(y)\} \quad (\text{because } (x+y)+0+z=x+y+z)$$
$$\geq\{f(x)\wedge f(y)\wedge n\} \quad (\text{because } f(0)\geq\min\{f(x),n\})\geq t\wedge n=t$$

$$\Rightarrow(f+f)(x+y)\geq t>m\Rightarrow(x+y)_t\in_m f+f$$

Then by hypothesis $(x+y)_t\in_m\vee q_n f$ (**) .

Then (*) and (**) gives contradiction. Hence (1′) holds. Similarly (2′) and (3′) hold.

Conversely assume (1′),(2′) and (3′) hold. To prove f is the $(\in_m,\in_m\vee q_n)$-fuzzy h-subhemiring of R. Suppose on the contrary (1) does not hold and for some $x\in R$ and $t\in(0,1],x_t\in_m f+f$ but $x_t\overline{\in_m\vee q_n}f$. Then $(f+f)(x)\geq t>m$ and $f(x)<t,f(x)<n$. Then for any $x,z,a,b,a',b'\in R$ such that $x+a+b+z=a'+b'+z$ and

$$\Rightarrow f(x)\geq f(a)\wedge f(b)\wedge f(a')\wedge f(b')$$

Then

$$t\leq(f+f)(x)=\bigvee_{x+(a+b)+z=(a'+b')+z}\{f(a)\wedge f(a')\wedge f(b)\wedge f(b')\}=f(x)$$

$\Rightarrow f(x)\geq t$. Which is a contradiction. So (1) holds. Similarly (2) holds.

Next suppose (3) does not hold. Then there exists $x,z,a,b\in R$ and $t,r\in(0,1]$ such that $x+a+z=b+z$ and $a_t,b_r\in_m f$ but $x_{t\wedge r}\overline{\in_m\vee q_n}f$. Then $f(a)\geq t>m,f(b)\geq r>m$ and $f(x)<\min\{t,r\}$ and $f(x)+\min\{t,r\}<2n$. Which implies

$$\max\{f(x),m\} < \min\{t,r\} \le \min\{f(a),f(b),n\}$$

Which contradicts (3′). Hence f is the $(\in_m,\in_m \vee q_n)$-fuzzy h-subhemiring of R.

By using Theorem 3.8, the following can also be proved.

Theorem: Let f be a fuzzy subset of R and $m,n\in[0,1]$ with m<n. Then f is an $(\in_m,\in_m \vee q_n)$-fuzzy left (right) h-ideal of R if and only if f satisfies (1′),(3′) and

(4′) $\max\{f(xy),m\} \ge \min\{f(y),n\}$

(5′) $\max\{f(xy),m\} \ge \min\{f(x),n\}$

for all $x,y\in R$.

Theorem: Let f be a fuzzy subset of R and $m,n\in[0,1]$ with m<n. Then f is an $(\in_m,\in_m \vee q_n)$-fuzzy h-quasi-ideal of R if and only if f satisfies (1′),(3′) and

(6′) $\max\{f(x),m\} \ge \min\{(f\otimes \mathbf{R})(x),(\mathbf{R}\otimes f))(x),n\}$

for all $x\in R$.

Theorem: Let f be a fuzzy subset of R and $m,n\in[0,1]$ with m<n. Then f is an $(\in_m,\in_m \vee q_n)$-fuzzy h-bi-ideal of R if and only if f satisfies (1′),(2′),(3′) and

(7′) $\max\{f(xzy),m\} \ge \min\{f(x),f(y),n\}$

for all $x,y,z\in R$.

Remark: If f is an $(\in_m,\in_m \vee q_n)$-fuzzy h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of R, then

(i) for m = 0 and n = 1, f is the fuzzy h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of R, which are discussed in [23] .
(ii) for m = 0 and n = 0.5, f is the $(\in,\in \vee q)$-fuzzy h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of R, which are discussed in [12, 17].
(iii) for m = 0.5 and n = 1, f is the $(\bar{\in},\bar{\in} \vee \bar{q})$-fuzzy h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of R, which are discussed in [18].
(iv) for m = 0 and $n=\frac{1-k}{2}$, f is the $(\in,\in \vee q_k)$-fuzzy h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of R, which are discussed in [19].
(v) for $m=\frac{1-k}{2}$ and n = 1, f is the $(\bar{\in},\bar{\in} \vee \overline{q_k})$-fuzzy h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of R, which are discussed in [20].

Lemma: Let f be a fuzzy subset of R and $m,n\in[0,1]$ with $n=\frac{1+m}{2}$. Then f is an $(\in_m,\in_m \vee q_n)$-fuzzy h-subhemiring of R if and only if non empty $U_n^m(f;t)=\{x\in R : x_t \in_m \vee q_n f\}$ is h-subhemiring of R for all $t\in(m,n]$.

Proof: Suppose f is an $(\in_m,\in_m \vee q_n)$-fuzzy h-subhemiring of R and let $x,y\in U_n^m(f;t)$ for some $t\in(m,n]$. Then $x_t \in_m \vee q_n f$ and $y_t \in_m \vee q_n f$. Now $x_t \in_m \vee q_n f \Rightarrow f(x)\ge t > m$ or $f(x)+t>2n$. Which means $f(x)\ge t>m$ or $f(x)>2n-t>2n-1=m$. Similarly $f(y)\ge t>m$ or $f(y)>2n-t>2n-1=m$. Further as f is an $(\in_m,\in_m \vee q_n)$-fuzzy h-subhemiring of R, so by Theorem 3.8,

$$\max\{f(x+y),m\} \ge \min\{f(x),f(y),n\} > \min\{m,n\} = m$$
$$\Rightarrow f(x+y) \ge \min\{f(x),f(y),n\}$$
$$\Rightarrow (x+y)_t \in_m \vee q_n f \Rightarrow x+y \in U_n^m(f;t)$$

Similarly $xy\in U_n^m(f;t)$. Next let $a,b,x,z\in R$ such that x+a+z = b+z and $a,b\in U_n^m(f;t)$ for some $t\in(m,n]$. Then $a_t,b_t \in_m \vee q_n f$. Now $a_t \in_m \vee q_n f \Rightarrow f(a)\ge t>m$ or $f(a)+t>2n$. Which means $f(a)\ge t>m$ or $f(a)>2n-t>2n-1=m$. Similarly $f(b)\ge t>m$ or $f(b)>2n-t>2n-1=m$. Further as f is an $(\in_m,\in_m \vee q_n)$-fuzzy h-subhemiring of R, so by Theorem 3.8,

$$\max\{f(x),m\} \ge \min\{f(a),f(b),n\} > \min\{m,n\} = m$$
$$\Rightarrow f(x) \ge \min\{f(a),f(b),n\}$$
$$\Rightarrow x_t \in_m \vee q_n f \Rightarrow x\in U_n^m(f;t)$$

Hence $U_n^m(f;t)$ is h-subhemiring of R.

Conversely, assume that $U_n^m(f;t)\ne\phi$ is h-subhemiring of R for all $t\in(m,n]$. To prove f is an $(\in_m,\in_m \vee q_n)$-fuzzy h-subhemiring of R. For this we prove (1′),(2′) and (3′) hold. Suppose on the contrary that there exists $x,y\in R$ such that

$$\max\{f(x+y),m\} < \min\{f(x),f(y),n\} = t$$

Then $x,y\in U_n^m(f;t)$ but $x+y\notin U_n^m(f;t)$, which is a contradiction, so for all $x,y\in R$ and $t\in(m,n]$, (1′) is satisfied. Similarly (2′) is satisfied. Again suppose that there exist $x,z,a,b\in R$ such that x+a+z = b+z and

$$\max\{f(x),m\} < \min\{f(a),f(b),n\} = t$$

Then $t \in (m,n]$ and $a,b \in U_n^m(f;t)$ but $x \notin U_n^m(f;t)$, which is a contradiction, so (3′) is satisfied. Hence f is an $(\in_m, \in_m \vee q_n)$-fuzzy h-subhemiring of R.

Lemma: Let f be a fuzzy subset of R and $m,n \in [0,1]$ with $n = \frac{1+m}{2}$. Then f is an $(\in_m, \in_m \vee q_n)$-fuzzy h-ideal (h-quasi-ideal, h-bi-ideal) of R if and only if non empty $U_n^m(f;t) = \{x \in R : x_t \in_m \vee q_n f\}$ is h-ideal (h-quasi-ideal, h-bi-ideal) of R for all $t \in (m,n]$.

Proof: Proof is similar to the proof of Lemma 3.13.

Proofs of the following results are straightforward, hence omitted.

Theorem: A non-empty subset A of a hemiring R is h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of R if and only if the characteristic function C_A is an $(\in_m, \in_m \vee q_n)$-fuzzy h-subhemiring (h-ideal, h-quasi-ideal, h-bi-ideal) of R of R for all $m,n \in [0,1]$ and $m<n$.

Theorem: Let f be an $(\in_m, \in_m \vee q_n)$-fuzzy h-subhemiring (h-ideal, h-bi-ideal, h-quasi-ideal) of R. Then $f \wedge n$ is $(\in_m, \in_m \vee q_n)$-fuzzy h-subhemiring (h-ideal, h-bi-ideal, h-quasi-ideal) of R, where $(f \wedge n)(x) = f(x) \wedge n$ for all $x \in R$.

Definition: Let f,g be fuzzy subsets of R. Then the fuzzy subsets $f \wedge_m^n g$, $f \vee_m^n g$ and $f \otimes_m^n g$ of R are defined as following:

$$(f \wedge_m^n g)(x) = \{(f \wedge g)(x) \wedge n\} \vee m$$
$$(f \vee_m^n g)(x) = \{(f \vee g)(x) \wedge n\} \vee m$$
$$(f \otimes_m^n g)(x) = \{(f \otimes g)(x) \wedge n\} \vee m, \text{ for all } x \in R$$

Definition: Let f,g be fuzzy subsets of R. Then the fuzzy subset $f +_m^n g$ of R is defined as following:

$$(f +_m^n g)(x) = \left[\left\{\bigvee_{x+(a+b)+z=(a'+b')+z} \begin{Bmatrix} f(a) \wedge f(a') \\ \wedge g(b) \wedge g(b') \end{Bmatrix}\right\} \wedge n\right] \vee m$$

for all possible expressions of $x,z,a,b,a',b' \in R$.

Lemma: Let $A,B \subseteq R$. Then

$$(C_A +_m^n C_B)(x) = (C_{\overline{A+B}}(x) \wedge n) \vee m$$

Proof: Proof is straightforward.

Lemma: Let f be a fuzzy subset of R. Then f satisfies conditions (1′) and (3′) if and only if it satisfies condition

(8′) $f +_m^n f \leq (f \wedge n) \vee m$

Proof: Suppose f satisfies conditions (1′) and (3′). Let

$$x + (a+b) + z = (a' + b') + z$$

for some $a,a',b,b',x,z \in R$. Then

$$(f +_m^n f)(x) = \left[\left(\bigvee_{x+(a+b)+z=(a'+b')+z} \{f(a) \wedge f(a') \wedge f(b) \wedge f(b')\}\right) \wedge n\right] \vee m$$
$$= \left[\left(\bigvee_{x+(a+b)+z=(a'+b')+z} \{(f(a) \wedge f(b) \wedge n) \wedge (f(a') \wedge f(b') \wedge n)\}\right) \wedge n\right] \vee m$$
$$\leq \left[\left(\bigvee_{x+(a+b)+z=(a'+b')+z} \{(f(a+b) \vee m) \wedge (f(a'+b') \vee m)\}\right) \wedge n\right] \vee m$$

(by condition (1′))

$$= \left[\left(\bigvee_{x+(a+b)+z=(a'+b')+z} \{(f(a+b) \wedge f(a'+b')) \vee m)\}\right) \wedge n\right] \vee m = \left[\left(\bigvee_{x+(a+b)+z=(a'+b')+z} \{(f(a+b) \wedge f(a'+b')) \wedge n)\}\right) \vee m\right] \vee m$$
$$= \left[\left(\bigvee_{x+(a+b)+z=(a'+b')+z} \{(f(a+b) \wedge f(a'+b')) \wedge n)\}\right) \wedge n\right] \vee m \leq ((f(x) \vee m) \wedge n) \vee m$$

(by condition (3′))

$$= (f(x) \wedge n) \vee m$$

Thus $f +_m^n f \leq (f \wedge n) \vee m$.

Conversely, assume that $(f +_m^n f)(x) \leq ((f \wedge n)(x)) \vee m$. Then for each x,z∈R we have 0+x+z = x+z. Thus

$$f(0) \vee m \geq ((f \wedge n)(0)) \vee m \geq (f +_m^n f)(0) = \left[\left(\bigvee_{0+(a+b)+z=(a'+b')+z} \{f(a) \wedge f(a') \wedge f(b) \wedge f(b')\}\right) \wedge n\right] \vee m$$

$$\geq \left(\bigvee_{0+(a+b)+z=(a'+b')+z} \{f(a) \wedge f(a') \wedge f(b) \wedge f(b')\}\right) \wedge n \geq f(x) \wedge n$$

This implies $f(0) \vee m \geq f(x) \wedge n$. (*)

Let x,y,z∈R. Then for all a,a′, b,b′∈R such that

$$(x+y)+(a+b)+z = (a'+b')+z$$

we have

$$\max\{f(x+y),m\} \geq \max\{(f \wedge n)(x+y),m\} \geq (f +_m^n f)(x+y)$$

$$= \left[\left(\bigvee_{(x+y)+(a+b)+z=(a'+b')+z} \{f(a) \wedge f(a') \wedge f(b) \wedge f(b')\}\right) \wedge n\right] \vee m \geq \left[\{f(0) \wedge f(x) \wedge f(0) \wedge f(y)\} \wedge n\right] \vee m$$

(because $(x+y)+(0+0)+z = (x+y)+z$)

$$\geq \min\{f(x),f(y),n\} \text{ by } (*)$$

Thus *f* satisfies condition (1′).

Now let a,b,x,z∈R such that x+a+z = b+z. Then for all possible a,a′, b,b′∈R satisfying the identity x+(a+b)+z = (a′+b′)+z we have

$$\max\{f(x),m\} \geq \max\{(f \wedge n)(x),m\} \geq (f +_m^n f)(x)$$

$$= \left[\left(\bigvee_{x+(a+b)+z=(a'+b')+z} \{f(a) \wedge f(a') \wedge f(b) \wedge f(b')\}\right) \wedge n\right] \vee m \geq \left[f(a) \wedge f(b) \wedge n\right]$$

because $x+a+z = b+z = \min\{f(a),f(b),n\}$

Thus *f* satisfies condition (3′).

Theorem: Let *f* be a fuzzy subset of R. Then *f* is an $(\in_m, \in_m \vee q_n)$-fuzzy left (resp. right) h-ideal of R if and only if *f* satisfies conditions

(8′) $f +_m^n f \leq (f \wedge n) \vee m$

(9′) $\mathbf{R} \otimes_m^n f \leq (f \wedge n) \vee m \quad \left(\text{resp. } f \otimes_m^n \mathbf{R} \leq (f \wedge n) \vee m\right)$

Proof: Working on the same lines as in Lemma 3.20, proof follows directly.

Theorem: Let *f* be a fuzzy subset of R. Then *f* is an $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideal of R if and only if *f* satisfies conditions (6′) and (8′).

Proof: Proof is straightforward because by Lemma 3.20, conditions (1′) and (3′) are equivalent to condition (8′).

Lemma: Every $(\in_m, \in_m \vee q_n)$-fuzzy left (right) h-ideal of R is an $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideal of R.

Proof: Proof is straightforward.

Lemma: Every $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideal of R is an $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideal of R.

Proof: Proof is straightforward.

Remark: Converses of the Lemma 3.23 and Lemma 3.24 are not true in general.

Example: Let $Z_0 = Z^+ \cup \{0\}$,

$$R = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in Z_0\right\}$$

and

$$A = \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z_0\right\}$$

Then R is a hemiring under the usual operations of addition and multiplication of matrices and A is a h-quasi-ideal of R but A is not left (right) h-ideal of R. Then by Theorem 3.15, C_A is an $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideal of R and C_A is not an $(\in_m, \in_m \vee q_n)$-fuzzy left (right) h-ideal of R.

Example: Let Z^+ and R^+ be the sets of all positive integers and positive real numbers, respectively. And

$$R = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b \in R^+, c \in Z^+\right\}$$

$$I = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b \in R^+, c \in Z^+, a < b\right\}$$

$$J = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b \in R^+, c \in Z^+, b > 3\right\}$$

Then R is a hemiring under the usual operations of addition and multiplication of matrices and I is a right h-ideal and J is a left h-ideal of R. Further the product IJ is a h-bi-ideal of R and it is not a h-quasi-ideal of R Then by Theorem 3.15, C_{IJ} is an $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideal of R and is not an $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideal of R.

Lemma: If f, g are $(\in_m, \in_m \vee q_n)$-fuzzy right and left h-ideals of R respectively, then $f \otimes_m^n g \le f \wedge_m^n g$.

Proof: Let $x \in R$. If $(f \otimes_m^n g)(x) = m$. Then $f \otimes_m^n g \le f \wedge_m^n g$. Otherwise

$$(f \otimes_m^n g)(x) = \left[\bigvee_{x+\Sigma_{i=1}^p a_i b_i + z = \Sigma_{j=1}^q a_j' b_j' + z} \left\{\begin{matrix} \left(\bigwedge_{i=1}^p f(a_i)\right) \wedge \left(\bigwedge_{i=1}^p g(b_i)\right) \wedge \\ \left(\bigwedge_{j=1}^q f(a_j')\right) \wedge \left(\bigwedge_{j=1}^q g(b_j')\right) \end{matrix}\right\} \wedge n\right] \vee m$$

$$= \left[\bigvee_{x+\Sigma_{i=1}^p a_i b_i + z = \Sigma_{j=1}^q a_j' b_j' + z} \left\{\begin{matrix} \left(\bigwedge_{i=1}^p ((f(a_i) \wedge n)\right) \wedge \left(\bigwedge_{i=1}^p (g(b_i) \wedge n)\right) \wedge \\ \left(\bigwedge_{j=1}^q (f(a_j') \wedge n)\right) \wedge \left(\bigwedge_{j=1}^q (g(b_j') \wedge n)\right) \end{matrix}\right\} \wedge n\right] \vee m$$

$$\le \left[\bigvee_{x+\Sigma_{i=1}^p a_i b_i + z = \Sigma_{j=1}^q a_j' b_j' + z} \left\{\begin{matrix} \left(\bigwedge_{i=1}^p ((f(a_i b_i) \vee m)\right) \wedge \\ \left(\bigwedge_{i=1}^p (g(a_i b_i) \vee m)\right) \wedge \\ \left(\bigwedge_{j=1}^q (f(a_j' b_j') \vee m)\right) \wedge \\ \left(\bigwedge_{j=1}^q (g(a_j' b_j') \vee m)\right) \end{matrix}\right\} \wedge n\right] \vee m = \left[\bigvee_{x+\Sigma_{i=1}^p a_i b_i + z = \Sigma_{j=1}^q a_j' b_j' + z} \left\{\begin{matrix} \left(\bigwedge_{i=1}^p ((f(a_i b_i) \wedge n)\right) \wedge \\ \left(\bigwedge_{i=1}^p (g(a_i b_i) \wedge n)\right) \wedge \\ \left(\bigwedge_{j=1}^q (f(a_j' b_j') \wedge n)\right) \wedge \\ \left(\bigwedge_{j=1}^q (g(a_j' b_j') \wedge n)\right) \end{matrix}\right\} \wedge n\right] \vee m$$

$$\le \left[\bigvee_{x+\Sigma_{i=1}^p a_i b_i + z = \Sigma_{j=1}^q a_j' b_j' + z} \left\{\begin{matrix} \left(\bigwedge_{i=1}^p ((f(a_i b_i) \wedge n) \wedge (g(a_i b_i) \wedge n)\right) \wedge \\ \left(\bigwedge_{j=1}^q (f(a_j' b_j') \wedge n) \wedge (g(a_j' b_j') \wedge n)\right) \end{matrix}\right\} \wedge n\right] \vee m$$

$$\le \left[\bigvee_{x+\Sigma_{i=1}^p a_i b_i + z = \Sigma_{j=1}^q a_j' b_j' + z} \left\{\max\left(\begin{matrix} f\left(\sum_{i=1}^p a_i b_i\right), g\left(\sum_{i=1}^p a_i b_i\right), \\ f\left(\sum_{j=1}^q a_j' b_j'\right), g\left(\sum_{j=1}^q a_j' b_j'\right), m \end{matrix}\right)\right\} \wedge n\right] \vee m \le (f \wedge_m^n g)(x)$$

h-HEMIREGULAR HEMIRINGS

In this section we characterize h-hemiregular hemirings by the properties of their $(\in_m, \in_m \vee q_n)$-fuzzy h-ideals, $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideals and $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideals of R.

Theorem: For a hemiring R the following conditions are equivalent.

(i) R is h-hemiregular.

(ii) $f \wedge_m^n g = f \otimes_m^n g$ for every $(\in_m, \in_m \vee q_n)$-fuzzy right and left h-ideals *f* and g of R, respectively.

Proof: (i)⇒(ii) Let x∈R. Then there exists a,a′,z∈R such that $x + xax + z = xa'x + z$. Now

$$(f \otimes_m^n g)(x) = \left[\bigvee_{x+\Sigma_{i=1}^p a_i b_i + z = \Sigma_{j=1}^q a_j' b_j' + z} \left\{ \begin{array}{l} \left(\bigwedge_{i=1}^p f(a_i) \right) \wedge \left(\bigwedge_{i=1}^p g(b_i) \right) \wedge \\ \left(\bigwedge_{j=1}^q f(a_j') \right) \wedge \left(\bigwedge_{j=1}^q g(b_j') \right) \end{array} \right\} \wedge n \right] \vee m \geq \left[f(xa) \wedge f(xa') \wedge g(x) \wedge n \right] \vee m$$

because

$$x + xax + z = xa'x + z \geq \left[f(x) \wedge g(x) \wedge n \right] \vee m = (f \wedge_m^n g)(x)$$

Thus $(f \otimes_m^n g)(x) \geq (f \wedge_m^n g)(x)$.

But by Lemma Le9g, $(f \otimes_m^n g)(x) \leq (f \wedge_m^n g)(x)$ hence $f \wedge_m^n g = f \otimes_m^n g$

(ii)⇒(i) Let I and L be right and left h-ideals of R, respectively . Then by Theorem Th9g, C_I is an $(\in_m, \in_m \vee q_n)$-fuzzy right and C_L is an $(\in_m, \in_m \vee q_n)$-fuzzy left h-ideal of R. Then by hypothesis

$$C_I \otimes_m^n C_L = C_I \wedge_m^n C_L \Rightarrow (C_{\overline{IL}} \wedge n) \vee m = (C_{I \cap L} \wedge n) \vee m \Rightarrow I \cap L = \overline{IL} \Rightarrow R$$

is h-hemiregular.

Theorem: For a hemiring R, the following conditions are equivalent.

(i) R is h-hemiregular.

(ii) $f \wedge n) \vee m \leq (f \otimes_m^n \mathbf{R} \otimes_m^n f)$ for every $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideal *f* of R.

(iii) $(f \wedge n) \vee m \leq (f \otimes_m^n \mathbf{R} \otimes_m^n f)$ for every $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideal *f* of R.

Proof: (i)⇒(ii) Let x∈R, then there exists a,a′∈R such that x+xax = xa′x. Now

$$(f \otimes_m^n \mathbf{R} \otimes_m^n f)(x) = \left[\bigvee_{x+\Sigma_{i=1}^p a_i b_i + z = \Sigma_{j=1}^q a_j' b_j' + z} \left\{ \begin{array}{l} \left(\bigwedge_{i=1}^p (f \otimes_m^n \mathbf{R})(a_i) \right) \wedge \left(\bigwedge_{i=1}^p f(b_i) \right) \wedge \\ \left(\bigwedge_{j=1}^q (f \otimes_m^n \mathbf{R})(a_j') \right) \wedge \left(\bigwedge_{j=1}^q f(b_j') \right) \end{array} \right\} \wedge n \right] \vee m$$

$$\geq \left[\left\{ \left(\bigwedge_{i=1}^p (f \otimes_m^n \mathbf{R})(xa) \right) \wedge \left(\bigwedge_{j=1}^q (f \otimes_m^n \mathbf{R})(xa') \right) \wedge f(x) \right\} \wedge n \right] \vee m$$

$$= \left[\left\{ \bigwedge_{i=1}^p \left(\left[\bigvee_{xa+\Sigma_{i=1}^p a_i b_i = \Sigma_{j=1}^q a_j' b_j'} \left\{ \begin{array}{l} \left(\bigwedge_{i=1}^p f(a_i) \right) \wedge \\ \left(\bigwedge_{j=1}^q f(a_j') \right) \end{array} \right\} \wedge n \right] \vee m \right) \wedge \left(\bigwedge_{j=1}^q \left[\bigvee_{xa'+\Sigma_{i=1}^p a_i b_i = \Sigma_{j=1}^q a_j' b_j'} \left\{ \begin{array}{l} \left(\bigwedge_{i=1}^p f(a_i) \right) \wedge \\ \left(\bigwedge_{j=1}^q f(a_j') \right) \end{array} \right\} \wedge n \right] \vee m \right) \wedge f(x) \right\} \wedge n \right] \vee m$$

$$\geq \left[\left\{ \begin{array}{l} \left(\left[\min\{f(xax), f(xa'x)\} \wedge n \right] \vee m \right) \wedge \\ \left(\left[\min\{f(xax), f(xa'x)\} \wedge n \right] \vee m \right) \end{array} \right\} \wedge n \right] \vee m$$

$$\left(xa + xaxa + z = xa'xa + z \text{ and } xa' + xaxa' + z = xa'xa' + z \right) \geq (f(x) \wedge n) \vee m$$

(ii)⇒(iii) This is straightforward by using Lemma 3.24.

(iii)⇒(i) Let Q be any h-quasi ideal of R. Then by Theorem 3.15, C_Q is an $(\in_m, \in_m \vee q_n)$-fuzzh h-quasi-ideal of R. Now by given condition

$$(C_Q \wedge n) \vee m \leq (C_Q \otimes_m^n \mathbf{R} \otimes_m^n C_Q) = C_{\overline{QRQ}} \Rightarrow Q \subseteq \overline{QRQ}$$

Also $\overline{QRQ} \subseteq \overline{RQ} \cap \overline{QR} = Q$. Thus $Q = \overline{QRQ}$, Therefore by Lemma 2.11, R is h-hemiregular.

Theorem: For a hemiring R, the following conditions are equivalent.

(i) R is h-hemiregular.

(ii) $\left(f \wedge_m^n g \right) \leq \left(f \otimes_m^n g \otimes_m^n f \right)$ for every $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideal *f* and $(\in_m, \in_m \vee q_n)$-fuzzy h-ideal g of R.

(iii) $\left(f \wedge_m^n g \right) \leq \left(f \otimes_m^n g \otimes_m^n f \right)$ for every $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideal *f* and $(\in_m, \in_m \vee q_n)$-fuzzy h-ideal g of R.

Proof: (i)⇒(ii) Let *f* be any $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideal and g any $(\in_m, \in_m \vee q_n)$-fuzzy h-ideal of R. Since R is h-hemiregular, so for any $a \in R$ there exist $x_1, x_2, z \in R$ such that $a + ax_1a + z = ax_2a + z$. Now

$$\left(f \otimes_m^n g \otimes_m^n f \right)(a) = \left[\bigvee_{a + \Sigma_{i=1}^p a_i b_i + z = \Sigma_{j=1}^q a_j' b_j' + z} \left(\begin{array}{l} \left(\left(\bigwedge_{i=1}^p (f \otimes_m^n g)(a_i) \right) \wedge \left(\bigwedge_{i=1}^p f(b_i) \right) \wedge \\ \left(\bigwedge_{j=1}^q (f \otimes_m^n g)(a_j') \right) \wedge \left(\bigwedge_{j=1}^q f(b_j') \right) \end{array} \right) \wedge n \right] \vee m$$

$$\geq \left[\{ (f \otimes_m^n g)(ax_1) \wedge f(a) \wedge (f \otimes_m^n g)(ax_2) \} \wedge n \right] \vee m$$

(because $a + ax_1a + z = ax_2a + z$)

$$\geq \{ f(a) \wedge g(x_1ax_1) \wedge g(x_1ax_2) \wedge g(x_2ax_1) \wedge g(x_2ax_2) \wedge n \} \vee m$$

(because $ax_1 + ax_1ax_1 + z = ax_2ax_1 + z$ and

$$ax_2 + ax_1ax_2 + z = ax_2ax_2 + z) \geq \{ f(a) \wedge g(a) \wedge n \} \vee m = (f \wedge_m^n g)(a)$$

(ii)⇒(iii) This is straightforward by using Lemma 3.24.

(iii)⇒(i) Let *f* be any $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideal and R be the $(\in_m, \in_m \vee q_n)$-fuzzy h-ideal of R. Then by hypothesis,

$$\left(f \wedge_m^n \mathbf{R} \right) \leq \left(f \otimes_m^n \mathbf{R} \otimes_m^n f \right) \Rightarrow (f \wedge n) \vee m \leq \left(f \otimes_m^n \mathbf{R} \otimes_m^n f \right)$$

Then by Theorem 4.2, R is h-hemiregular.

Theorem: For a hemiring R, the following conditions are equivalent.

(i) R is h-hemiregular.

(ii) $\left(f \wedge_m^n g\right) \leq \left(f \otimes_m^n g\right)$ for every $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideal f and $(\in_m, \in_m \vee q_n)$-fuzzy left h-ideal g of R.

(iii) $\left(f \wedge_m^n g\right) \leq \left(f \otimes_m^n g\right)$ for every $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideal f and $(\in_m, \in_m \vee q_n)$-fuzzy left h-ideal g of R.

(iv) $\left(f \wedge_m^n g\right) \leq \left(f \otimes_m^n g\right)$ for every $(\in_m, \in_m \vee q_n)$-fuzzy right h-ideal f and $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideal g of R.

(v) $\left(f \wedge_m^n g\right) \leq \left(f \otimes_m^n g\right)$ for every $(\in_m, \in_m \vee q_n)$-fuzzy right h-ideal f and $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideal g of R.

(vi) $\left(f \wedge_m^n g \wedge_m^n \lambda\right) \leq \left(f \otimes_m^n g \otimes_m^n \lambda\right)$ for every $(\in_m, \in_m \vee q_n)$-fuzzy right h-ideal f, $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideal g and $(\in_m, \in_m \vee q_n)$-fuzzy left h-ideal λ of R.

(vii) $\left(f \wedge_m^n g \wedge_m^n \lambda\right) \leq \left(f \otimes_m^n g \otimes_m^n \lambda\right)$ for every $(\in_m, \in_m \vee q_n)$-fuzzy right h-ideal f, $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideal g and $(\in_m, \in_m \vee q_n)$-fuzzy left h-ideal λ of R.

Proof: Working on the same lines as in Theorem 4.3, proof is straightforward.

h-INTRA-HEMIREGULAR HEMIRINGS

In this section we characterize h-intra-hemiregular hemirings and hemirings which are both h-hemiregular and h-intra-hemiregular in terms of their $(\in_m, \in_m \vee q_n)$-fuzzy h-ideals, $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideals and $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideals.

Theorem: A hemiring R is h-intra-hemiregular if and only if $f \wedge_m^n g \leq f \otimes_m^n g$, for every $(\in_m, \in_m \vee q_n)$-fuzzy left h-ideal f and for every $(\in_m, \in_m \vee q_n)$-fuzzy right h-ideal g of R.

Proof: Let R be an h-intra-hemiregular, f be an $(\in_m, \in_m \vee q_n)$-fuzzy left h-ideal and g an $(\in_m, \in_m \vee q_n)$-fuzzy right h-ideal of R. As R is h-intra-hemiregular so for every $x \in R$, there exist $a_i, a_i', b_j, b_j', z \in R$ such that

$$x + \sum_{i=1}^{p} a_i x^2 a_i' + z = \sum_{j=1}^{q} b_j x^2 b_j' + z$$

Then

$$(f \otimes_m^n g)(x) = \left[\bigvee_{x + \Sigma_{i=1}^{p} a_i b_i + z = \Sigma_{j=1}^{q} a_j' b_j' + z} \left(\left(\bigwedge_{i=1}^{p} f(a_i) \right) \wedge \left(\bigwedge_{i=1}^{p} g(b_i) \right) \wedge \left(\bigwedge_{j=1}^{q} f(a_j') \right) \wedge \left(\bigwedge_{j=1}^{q} g(b_j') \right) \right) \wedge n \right] \vee m$$

$$\geq \left[f(a_i x) \wedge f(b_j x) \wedge g(x a_i') \wedge g(x b_j') \wedge n \right] \vee m$$

(because $x + \sum_{i=1}^{p} (a_i x)(x a_i') + z = \sum_{j=1}^{q} (b_j x)(x b_j')) + z$)

Conversely assume that A and B are left and right h-ideals of R, respectively. Then by Theorem 3.15, the characteristic functions C_A and C_B are respectively $(\in_m, \in_m \vee q_n)$-fuzzy left h-ideal and $(\in_m, \in_m \vee q_n)$-fuzzy right h-ideal of R. Then by given hypothesis

$$C_A \wedge_m^n C_B \leq C_A \otimes_m^n C_B \Rightarrow (C_{A \cap B} \wedge n) \vee m \leq (C_{\overline{AB}} \wedge n) \vee m \Rightarrow A \cap B \subseteq \overline{AB}$$

Thus by Lemma 2.10, R is h-intra-hemiregular.

Theorem: The following conditions are equivalent for a hemiring R:

(i) R is both h-hemiregular and h-intra-hemiregular.

(ii $(f \wedge n) \vee m = f \otimes_m^n f$ for every $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideal f of R.

(iii) $(f \wedge n) \vee m = f \otimes_m^n f$ for every $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideal f of R.

Proof: (i)⇒(ii) Let f be an $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideal of R and $x \in R$. Since R is both h-hemiregular and h-intra-hemiregular, there exist elements $a_1, a_2, p_i, p'_i, q_j, q'_j, z \in R$ such that

$$\begin{aligned} x &+ \sum_{j=1}^{q}(xa_2q_jx)(xq'_ja_1x) + \sum_{j=1}^{q}(xa_1q_jx)(xq'_ja_2x) + \sum_{i=1}^{p}(xa_1p_ix)(xp'_ia_1x) \\ &+ \sum_{i=1}^{p}(xa_2p_ix)(xp'_ia_2x) + z = \sum_{i=1}^{p}(xa_2p_ix)(xp'_ia_1x) + \sum_{i=1}^{p}(xa_1p_ix)(xp'_ia_2x) \\ &+ \sum_{j=1}^{q}(xa_1q_jx)(xq'_ja_1x) + \sum_{j=1}^{q}(xa_2q_jx)(xq'_ja_2x) + z \end{aligned}$$

$$\begin{aligned} (f \otimes_m^n f)(x) &= \left[\bigvee_{x+\Sigma_{i=1}^{p}a_ib_i+z=\Sigma_{j=1}^{q}a'_jb'_j+z} \left\{ \begin{array}{l} \left(\bigwedge_{i=1}^{p} f(a_i)\right) \wedge \left(\bigwedge_{i=1}^{p} f(b_i)\right) \wedge \\ \left(\bigwedge_{j=1}^{q} f(a'_j)\right) \wedge \left(\bigwedge_{j=1}^{q} f(b'_j)\right) \end{array} \right\} \wedge n \right] \vee m \\ &\geq \left[\begin{array}{l} \left(\bigwedge_{j=1}^{q} f(xa_2q_jx)\right) \wedge \left(\bigwedge_{j=1}^{q} f(xq'_ja_1x)\right) \wedge \left(\bigwedge_{j=1}^{q} f(xa_1q_jx)\right) \wedge \left(\bigwedge_{j=1}^{q} f(xq'_ja_2x)\right) \wedge \\ \left(\bigwedge_{i=1}^{p} f(xa_1p_ix)\right) \wedge \left(\bigwedge_{i=1}^{p} f(xp'_ia_1x)\right) \wedge \left(\bigwedge_{i=1}^{p} f(xa_2p_ix)\right) \wedge \left(\bigwedge_{i=1}^{p} f(xp'_ia_2x)\right) \wedge n \end{array} \right] \vee m \\ &\geq \left[\{(f(x) \wedge n) \vee m\} \wedge n \right] \vee m = (f(x) \wedge n) \vee m \end{aligned}$$

This implies that $f \otimes_m^n f \geq (f \wedge n) \vee m$

On the other hand if

$$x + \Sigma_{i=1}^{p}a_ib_i + z = \Sigma_{j=1}^{q}a'_jb'_j + z$$

we have

$$\begin{aligned} \{f(x) \wedge n\} \vee m &= [\{f(x) \wedge n\} \vee m] \vee m = [\{f(x) \vee m\} \wedge n] \vee m \geq \left[f(\Sigma_{i=1}^{p}a_ib_i) \wedge f(\Sigma_{j=1}^{q}a'_jb'_j) \wedge n \right] \vee m \\ &\geq \left[\left(\bigwedge_{i=1}^{p} f(a_ib_i)\right) \left(\bigwedge_{j=1}^{q} f(a'_jb'_j)\right) \wedge n \right] \vee m \geq \left[\left(\bigwedge_{i=1}^{p} f(a_i) \wedge f(b_i)\right) \wedge \left(\bigwedge_{i=1}^{p} f(a_i) \wedge f(b_i)\right) \wedge n \right] \vee m \end{aligned}$$

(because f is an $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideal of R)

Thus

$$(f \otimes_m^n f)(x) = \left[\bigvee_{x+\Sigma_{i=1}^{p}a_ib_i+z=\Sigma_{j=1}^{q}a'_jb'_j+z} \left\{ \begin{array}{l} \left(\bigwedge_{i=1}^{p} f(a_i)\right) \wedge \left(\bigwedge_{i=1}^{p} f(b_i)\right) \wedge \\ \left(\bigwedge_{j=1}^{q} f(a'_j)\right) \wedge \left(\bigwedge_{j=1}^{q} f(b'_j)\right) \end{array} \right\} \wedge n \right] \vee m \leq (f(x) \wedge n) \vee m.$$

Consequently $f \otimes_m^n f = (f \wedge n) \vee m$.

(ii)⇒(iii) This is straightforward because every $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideal of R is $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideal of R.

(iii)$\Rightarrow$(i) Let Q be an h-quasi-ideal of R. Then C_Q is an $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideal of R. Thus by hypothesis

$$[C_Q \wedge n] \vee m = C_Q \otimes_m^n C_Q$$
$$= [C_Q \otimes C_Q \wedge n] \vee m = [C_{\overline{Q^2}} \wedge n] \vee m$$

Then it follows $Q = \overline{Q^2}$. Hence by Lemma 2.12, R is both h-hemiregular and h-intra-hemiregular.

Theorem: The following conditions are equivalent for a hemiring R :

(i) R is both h-hemiregular and h-intra-hemiregular.

(ii) $f \wedge_m^n g \leq f \otimes_m^n g$ for all $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideals *f* and g of R.

(iii) $f \wedge_m^n g \leq f \otimes_m^n g$ for every $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideal *f* and every $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideals g of R.

(iv) $f \wedge_m^n g \leq f \otimes_m^n g$ for every $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideal *f* and every $(\in_m, \in_m \vee q_n)$-fuzzy h-bi-ideals g of R.

(v) $f \wedge_m^n g \leq f \otimes_m^n g$ for all $(\in_m, \in_m \vee q_n)$-fuzzy h-quasi-ideals *f* and g of R.

Proof: Working on the same lines as in Theorem 5.2, proof is straightforward.

REFERENCES

1. Aho, A.W. and J.D. Ullman, 1979. Introduction to Automata Theory, Languages and Computation. Addison Wesley, Reading, MA.
2. Ahsan, J., 1998. Semirings characterized by their fuzzy ideals. J. Fuzzy Math., 6: 181-192.
3. Ahsan, J., K. Saifullah and M.F. Khan, 1993. Fuzzy Semirings. Fuzzy Sets Syst., 60: 309-320.
4. Bhakat, S.K., 1999. $(\in \vee q)$-level subset. Fuzzy Sets and Systems, 103: 529-533.
5. Bhakat, S.K. and P. Das, 1996. Fuzzy subrings and ideals redefined. Fuzzy Sets and Systems, 81: 383-393.
6. Davvaz, B., 2006. $(\in, \in \vee q)$-fuzzy subnearrings and ideals. Soft Comput., 10: 206-211.
7. Dudek, W.A., M. Shabir and M. Irfan Ali, 2009. (α,β)-Fuzzy ideals of Hemirings. Comput. Math. Appl. 58: 310-321.
8. Glazek, K., 2002. A guide to litrature on semirings and their applications in mathematics and information sciences: With complete bibliography, Kluwer Acad. Publ. Nederland.
9. Golan, J.S., 1999. Semirings and their applications. Kluwer Acad. Publ.
10. Henriksen, M., 1958. Ideals in semirings with commutative addition. Amer. Math. Soc. Notices, 6: 321.
11. Jun, Y.B. and S.Z. Song, 2006. Generalized fuzzy interior ideals in semigroups. Inform. Sci., 176: 3079-3093.
12. Ma, X. and J. Zhan, 2009. Generalized fuzzy h-bi-ideals and h-quasi-ideals of hemirings. Inform. Sci., 179: 1249-1268.
13. Mordeson, J.N. and D.S. Malik, 2002. Fuzzy Automata and Languages, Theory and Applications. Computational Mathematics Series, Chapman and Hall/CRC, Boca Raton.
14. Murali, V., 2004. Fuzzy points of equivalent fuzzy subsets. Inform. Sci., 158: 277-288.
15. Pu, P.M. and Y.M. Liu, 1980. Fuzzy topology I, neighborhood structure of a fuzzy point and Moore-Smith convergence. J. Math. Anal. Appl., 76: 571-599.
16. Rosenfeld, 1971. Fuzzy groups. J. Math. Anal. Appl., 35: 512-517.
17. Shabir, M., Y. Nawaz and T. Mahmood, Characterizations of hemirings by $(\in, \in \vee q)$-fuzzy ideals, submitted.
18. Shabir, M., Y. Nawaz and T. Mahmood, Characterizations of hemirings by $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals. Neural Computing and Applications, DOI: 10.1007/s00521-011-0693-4.
19. Shabir, M. and T. Mahmood, 2011. Characterizations of Hemirings by $(\in, \in \vee q_k)$-fuzzy ideals. Comp. and Maths. with Appl., 61: 1059-1078.
20. Shabir, M. and T. Mahmood, Characterizations of Hemirings by $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy ideals, submitted.
21. Vandiver, H.S., 1934. Note on a simple type of algebra in which cancellation law of addition does not hold. Bull. Amer. Math. Soc., 40: 914-920.
22. Wechler, W., 1978. The concept of fuzziness in automata and language theory. Akademic verlog, Berlin.
23. Yunqiang Yin and Hongxing Li, 2008. The charatecrizations of h-hemiregular hemirings and h-intra-hemiregular hemirings. Inform. Sci., 178: 3451-3464.
24. Yunqiang Yin and Jianming Zhan, 2010. New types of fuzzy filters of BL-algebras. Comp. and Maths. with Appl., doi:10.1016/j.camwa.2010.07.054.
25. Zadeh, L.A., 1965. Fuzzy Sets. Information and Control, 8: 338-353.