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Quartic B-spline Interpolation Method for Linear two-point Boundary Value Problem

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Abstract: Quartic B-spline is a piecewise polynomial of degree four satisfying third order parametric continuity. In this paper, quartic B-spline is manipulated to approximate the solution for second order linear two-point boundary value problem. By presuming the spline to be the solution for the problem, an underdetermined system of linear equations of order (n+3)×(n+4), with n being the number of uniform subintervals, is built. A free variable is resulted upon solving for the unknowns and thus, adds flexibility to the approximated solution. A way to optimize the variable to achieve the best approximation is presented. This method makes use of the problem's equation to construct an error equation. Minimization of the error equation would give the value of the variable that produces the best approximation of the solution. This approach is tested on several examples and the results were compared with those of cubic B-spline and extended cubic B-spline. Quartic B-spline gives out more accurate numerical results compared to the other two for some problems.

Key words: Quartic B-spline . two-point boundary value problem . spline interpolation

INTRODUCTION

The general form of second order linear two-point boundary value problems is given as follows,

$$u''(x) + p(x)u'(x) + q(x)u(x) = r(x)$$
$$x \in [a,b], \quad u(a) = \alpha, \quad u(b) = \beta \qquad (1)$$

Generally, the exact solutions for these problems are difficult to obtain. Hence, numerical techniques are usually employed to find the approximations of the solutions. To date, there are quite a number of numerical methods available in the literature to solve the problems. Some of the common methods found in most of numerical analysis textbooks are shooting and finite difference. These methods give out accurate approximations using very little computational time. However, the solutions are produced only at the discrete points. Alternatively, some of the more recent methods such as cubic B-spsline, extended Adomian decomposition and homotopy perturbation produce the analytical approximated solution for the problems [1-3]. In other words, the solutions from these methods are laid out in a continuous fashion.

The focus of this study is on cubic B-spline method proposed in 2006 [1]. This method is a continuation of several works since 1960s [4-6]. At that time, cubic monomial spline was used as the approximated solution. Spline is a piecewise polynomial possessing some nice properties that are widely used in the field of Computer Aided Geometric Design. B-spline is a more stable representation of spline compared to the monomial spline. Caglar *et al*. manipulated B-spline of degree three, known as cubic B-spline to solve the problems. Following the approach, some works have been done by the authors using other types of spline such as cubic trigonometric B-spline, cubic Beta-spline and extended cubic B-spline in place of cubic B-spline [7-10]. This paper focuses on the use of the fourth degree B-spline or quartic B-spline in approximating the solutions for the problems.

QUARTIC B-SPLINE

Let $\{x_i\}$ be a sequence of uniform step-size h with $i \in \mathbb{Z}$. B-spline basis of order one, $N_i^1(x)$, is defined in (2). From that, B-spline basis of order k, $N_i^k(x)$ could be calculated using (3).

$$N_i^1(x) = \begin{cases} 1, & x \in [x_i, x_{i+1}], \\ 0, & otherwise. \end{cases} \qquad (2)$$

$$N_i^k(x) = \frac{x - x_i}{x_{i+k-1} - x_i} N_i^{k-1} + \frac{x_{i+k} - x}{x_{i+k} - x_{i+1}} N_{i+1}^{k-1} \qquad (3)$$

Quartic B-spline basis could be obtained by calculating the basis up to order five. The resulting quartic B-spline basis, $N_i^5(x)$ is as follows:

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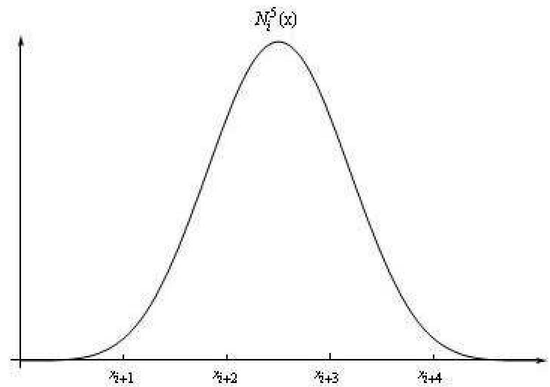

Fig. 1: Quartic B-spline basis

$$N_i^5(x) = \frac{1}{24h^4}\begin{cases} (x-x_i)^4, & x \in [x_i, x_{i+1}],\\ h^4 + 4h^3(x-x_{i+1}) + 6h^2(x-x_{i+1})^2 + 4h(x-x_{i+1})^3 + 4(x-x_{i+1})^4, & x \in [x_{i+1}, x_{i+2}],\\ 11h^4 + 12h^3(x-x_{i+2}) - 6h^2(x-x_{i+2})^2 - 12h(x-x_{i+2})^3 + 6(x-x_{i+2})^4, & x \in [x_{i+2}, x_{i+3}],\\ 11h^4 - 12h^3(x-x_{i+3}) - 6h^2(x-x_{i+3})^2 + 12h(x-x_{i+3})^3 - 4(x-x_{i+3})^4, & x \in [x_{i+3}, x_{i+4}],\\ h^4 - 4h^3(x-x_{i+4}) + 6h^2(x-x_{i+4})^2 - 4h(x-x_{i+4})^3 + (x-x_{i+4})^4, & x \in [x_{i+4}, x_{i+5}]. \end{cases}$$

It can be seen that quartic B-spline basis is a piecewise polynomial of degree four. From the theory of B-spline, the basis also satisfies parametric continuity of order (k-2), hence in this case, C^3 [11]. The plot of the basis is shown in Fig. 1.

From the basis, a curve S(x) could be generated by taking a linear combination of the quartic B-spline basis, as shown in (4).

$$S(x) = \sum_{i=-4}^{n-1} C_i N_i^5(x), \quad x \in [x_0, x_n], \quad C_i \in \mathbb{R} \tag{4}$$

This curve is called quartic B-spline. Quartic B-spline retains the properties of the basis, which are having polynomials of degree four and satisfying C^3. Quartic B-spline is a piecewise polynomial containing n pieces. By evaluating S(x) at the collocation point, x_i, (4) can be further simplified. From the definition of the basis, there are exactly four nonzero quartic B-spline basis at x_i, namely $N_{i-4}^5(x_i)$, $N_{i-3}^5(x_i)$, $N_{i-2}^5(x_i)$ and $N_{i-1}^5(x_i)$. Therefore,

$$\begin{aligned} S(x_i) &= C_{i-4}N_{i-4}^5(x_i) + C_{i-3}N_{i-3}^5(x_i) + C_{i-2}N_{i-2}^5(x_i) + C_{i-1}N_{i-1}^5(x_i) \\ &= C_{i-4}\left(\frac{1}{24}\right) + C_{i-3}\left(\frac{11}{24}\right) + C_{i-2}\left(\frac{11}{24}\right) + C_{i-1}\left(\frac{1}{24}\right) \end{aligned} \tag{5}$$

By taking the first and second derivatives of S(x) and simplifying them at x_i, (6) and (7) are resulted.

$$\begin{aligned} S'(x_i) &= C_{i-4}N_{i-4}^{5\,\prime}(x_i) + C_{i-3}N_{i-3}^{5\,\prime}(x_i) + C_{i-2}N_{i-2}^{5\,\prime}(x_i) + C_{i-1}N_{i-1}^{5\,\prime}(x_i) \\ &= C_{i-4}\left(-\frac{1}{6h}\right) + C_{i-3}\left(-\frac{1}{2h}\right) + C_{i-2}\left(\frac{1}{2h}\right) + C_{i-1}\left(\frac{1}{6h}\right) \end{aligned} \tag{6}$$

$$\begin{aligned} S''(x_i) &= C_{i-4}N_{i-4}^{5\,\prime\prime}(x_i) + C_{i-3}N_{i-3}^{5\,\prime\prime}(x_i) + C_{i-2}N_{i-2}^{5\,\prime\prime}(x_i) + C_{i-1}N_{i-1}^{5\,\prime\prime}(x_i) \\ &= C_{i-4}\left(\frac{1}{2h^2}\right) + C_{i-3}\left(-\frac{1}{2h^2}\right) + C_{i-2}\left(-\frac{1}{2h^2}\right) + C_{i-1}\left(\frac{1}{2h^2}\right) \end{aligned} \tag{7}$$

These simplifications are very useful in solving the problems using quartic B-spline.

QUARTIC B-SPLINE INTERPOLATION METHOD (QBIM)

In order to solve the problem, quartic B-spline in (4) is presumed to be its solution. Since the problem is defined on interval [a,b] let $x_0 = a$ and $x_n = b$, where n is the number of partition. Thus, (1) becomes

$$S''(x) + p(x)S'(x) + q(x)S(x) = r(x), \quad x \in [a,b], \quad S(a) = \alpha, \quad S(b) = \beta \tag{8}$$

$$x_i = a + ih, \quad h = \frac{b-a}{n}, \quad n \geq 1$$

From (4), S(x) and its derivatives have (n+4) unknowns, C_i. In order to determine the values of C_i, (8) is evaluated at x_i.

$$S''(x_i) + p(x)S'(x_i) + q(x)S(x_i) = r(x_i), \quad i = 0,1,\dots,n \tag{9}$$

Substituting (5), (6) and (7) into (9),

$$\begin{aligned} &C_{i-4}\left(\frac{1}{2h^2}\right) + C_{i-3}\left(-\frac{1}{2h^2}\right) + C_{i-2}\left(-\frac{1}{2h^2}\right) + C_{i-1}\left(\frac{1}{2h^2}\right) \\ &\quad + p(x_i)\left(C_{i-4}\left(-\frac{1}{6h}\right) + C_{i-3}\left(-\frac{1}{2h}\right) + C_{i-2}\left(\frac{1}{2h}\right) + C_{i-1}\left(\frac{1}{6h}\right)\right) \\ &\quad + q(x_i)\left(C_{i-4}\left(\frac{1}{24}\right) + C_{i-3}\left(\frac{11}{24}\right) + C_{i-2}\left(\frac{11}{24}\right) + C_{i-1}\left(\frac{1}{24}\right)\right) \\ &\quad = \left(\frac{1}{2h^2} - p(x_i)\frac{1}{6h} + q(x_i)\frac{1}{24}\right)C_{i-4} + \left(-\frac{1}{2h^2} - p(x_i)\frac{1}{2h} + q(x_i)\frac{11}{24}\right)C_{i-3} \\ &\quad + \left(-\frac{1}{2h^2} + p(x_i)\frac{1}{2h} + q(x_i)\frac{11}{24}\right)C_{i-2} + \left(\frac{1}{2h^2} + p(x_i)\frac{1}{6h} + q(x_i)\frac{1}{24}\right)C_{i-1} = r(x_i), \quad i \\ &\quad = 0,1,\dots,n. \end{aligned} \tag{10}$$

The boundary conditions are simplified using (5) resulting (11) and (12).

$$S(a) = S(x_0) = C_{-4}\left(\frac{1}{24}\right) + C_{-3}\left(\frac{11}{24}\right) + C_{-2}\left(\frac{11}{24}\right) + C_{-1}\left(\frac{1}{24}\right) = \alpha \tag{11}$$

$$S(b) = S(x_n) = C_{n-4}\left(\frac{1}{24}\right) + C_{n-3}\left(\frac{11}{24}\right) + C_{n-2}\left(\frac{11}{24}\right) + C_{n-1}\left(\frac{1}{24}\right) = \beta \tag{12}$$

Eq. (10), (11) and (12) is a system of (n+3) linear equations with (n+4) unknowns. Therefore, this is an underdetermined system resulting infinitely many solutions. By solving the system using MATLAB built-in function, the values of C_i are solved as a function of a free parameter, t. By substituting C_i into S(x) the approximated analytical solution for the problem is obtained. However, t needs to be optimized in order to produce the best approximation of the solution.

For clarity, S(x) is relabeled as S(x,t) to emphasize the free parameter, t. In order to optimize t, S(x,t) is first substituted in the problem's equation and rearranged to become

$$E(x,t) = S''(x,t) + p(x)S'(x,t) + q(x)S(x,t) - r(x) \approx 0 \tag{13}$$

The optimized t can be obtained by minimizing E(x,t). Since S(x,t) is a piecewise polynomial of n pieces,

$$E(x,t) = \begin{cases} E_1(x,t), & x \in [x_0, x_1], \\ E_2(x,t), & x \in [x_1, x_2], \\ \vdots & \vdots \\ E_n(x,t), & x \in [x_{n-1}, x_n]. \end{cases} \tag{14}$$

where $E_i(x,t)$ refers to the corresponding E(x,t) in the interval $[x_i, x_{i+1}]$, for i = 0,1,...,n-1. In this method, E(x,t) is evaluated at the midpoint of each subinterval,

$$x_i^* = \frac{x_i + x_{i+1}}{2}, \quad i = 0,1,\dots,n-1$$

The midpoint is selected because of its maximum distance to the collocation points. Logically, the error would be maximum there. Then, all the expressions are combined using the sum-squared formula,

$$D(t) = \sqrt{\sum_{i=0}^{n-1} E(x_i^*, t)^2}$$

By minimizing D(t), the optimized value of t can be obtained. However, minimizing D(t) is equivalent to minimizing

$$d(t) = \sum_{i=0}^{n-1} E(x_i^*, t)^2$$

Thus, d(t) is minimized using MATLAB built-in function, fminsearch, with 0 as the initial guess. The resulting value of t is then substituted back into S(x,t).

NUMERICAL EXAMPLES AND CONCLUSIONS

This method was tested on Problems 4.1, 4.2, 4.3 and 4.4 and the numerical results are displayed in Table 1. These experiments were run by MATLAB version 7.6.0.324.

Problem 4.1:

$$u''(x) - u'(x) = -e^{x-1} - 1, \quad x \in [0,1], \quad u(0) = u(1) = 0$$

Exact solution: $u(x) = x(1 - e^{x-1})$.

Problem 4.2:

$$u''(x) + (x+1)u'(x) - 2u(x) = (1 - x^2)e^{-x}, \quad x \in [0,1], \quad u(0) = -1, \quad u(1) = 0$$

Exact solution: $u(x) = (x - 1)e^x$

Problem 4.3:

$$u''(x) - \pi^2 u(x) = -2\pi^2 \sin(\pi x), \quad x \in [0,1], \quad u(0) = u(1) = 0$$

Exact solution: $u(x) = \sin(\pi x)$.

Problem 4.4:

$$u''(x) - u(x) = 0, \quad x \in [0,1], \quad u(0) = 0, \quad u(1) = \sinh(1)$$

Exact solution: $u(x) = \sinh(x)$.

The results are compared with those of Cubic B-spline Interpolation Method (CBIM) and Extended Cubic B-spline Interpolation Method (ECBIM) [1, 7, 8]. ECBIM also involves in optimizing a free parameter called λ. The errors were calculated using (15) and (16).

$$\text{Max-Norm} = \max_i |S(x_i) - u(x_i)|,\ i = 1,2,\ldots,n-1 \quad (15)$$

$$L^2\text{-Norm} = \sqrt{\sum_{i=1}^{n-1} |S(x_i) - u(x_i)|^2} \quad (16)$$

From Table 1, it can be seen that for Problems 4.1, 4.2 and 4.4, QBIM gives out the most accurate results out of the three methods compared. However, for Problem 4.3, ECBIM produced better results than CBIM and QBIM. Problem 4.3 is of trigonometric nature. In conclusion, QBIM produces better approximation than CBIM and ECBIM for some problems. Further studies need to be done on determining which splines suits best for various types of problems, estimating the error and analyzing the convergence of the approximated solutions as were done in [12, 13].

Table 1: Results for problems 4.1, 4.2, 4.3 and 4.4 using CBIM, ECBIM and QBIM

Problem	Method	Max-Norm	L^2-Norm	λ or t
4.1	CBIM	2.8996×10^{-4}	6.6089×10^{-4}	
	ECBIM	5.7388×10^{-6}	1.1479×10^{-5}	2.9375×10^{-3}
	QBIM	2.6097×10^{-7}	4.6631×10^{-7}	6.4313×10^{-2}
4.2	CBIM	2.3108×10^{-4}	5.2220×10^{-4}	
	ECBIM	4.9685×10^{-6}	9.9122×10^{-6}	2.9375×10^{-3}
	QBIM	3.5654×10^{-7}	7.7036×10^{-7}	4.8719×10^{-2}
4.3	CBIM	4.1088×10^{-3}	9.1875×10^{-3}	
	ECBIM	8.9601×10^{-7}	2.0035×10^{-6}	-1.6500×10^{2}
	QBIM	2.0373×10^{-5}	4.5555×10^{-5}	-1.5969×10^{1}
4.4	CBIM	5.2011×10^{-5}	1.1794×10^{-4}	
	ECBIM	6.6718×10^{-7}	1.5129×10^{-6}	1.6875×10^{-3}
	QBIM	2.5963×10^{-8}	5.8873×10^{-8}	-4.9922×10^{2}

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