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Dirichlet Problem for Degenerate Differential Equations of Fourth Order on Infinite Interval

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Abstract: In the paper we consider the following differential equation

$$Lu \equiv (t^{\alpha} u'')'' + pt^{\beta} u = f \tag{1}$$

Where $t \in (1,+\infty)$, $\alpha \neq 1$, $\alpha. \neq 3$, $p = const, f \in L_{2,-\beta}(1,+\infty)$, $\alpha, \beta \in R$. We prove some compactness results in weighted Sobolev spaces, investigate the uniqueness and existence of the generalized solution of the Dirichlet problem for the equation (1), as well as describe the spectrum of the corresponding operator for $\beta \leq \alpha - 4$. We prove that the spectrum of the operator $t^{-\beta} L$ for $\beta = \alpha - 4$ and $p = 0$ is purely continuous and coincides with the ray $\left[\frac{(\alpha-1)^2(\alpha-3)^2}{16}; +\infty\right)$.

Key words: Dirichlet problem · Weighted Sobolev spaces · Generalized solution · Spectrum of the linear operator

INTRODUCTION

We consider the generalized Dirichlet problem for the degenerate ordinary differential equation of the fourth order of the following form

$$Lu \equiv (t^{\alpha} u'')'' + pt^{\beta} u = f, \tag{1}$$

Where $t \in (1,+\infty)$, $\alpha \neq 1$, $\alpha \neq 3$, $p = const$, $f \in L_{2,-\beta}(1,+\infty)$.

First we define the weighted Sobolev space $\dot{W}^2_{\alpha}(1,+\infty)$, recall the behaviour of the functions from $\dot{W}^2_{\alpha}(1,+\infty)$ for large values of t (see [1]) and prove some compactness results. Then we define the generalized solution of the Dirichlet problem for the equation (1), prove that it exists and is unique. Moreover, we explore the spectrum of the operator $t^{-\beta} L$ and prove that its spectrum for $\beta = \alpha - 3$ and $p = 0$ is purely continuous and coincides with the ray $\left[\frac{(\alpha-1)^2(\alpha-3)^2}{16}; +\infty\right)$.

Embedding questions for the weighted spaces on infinite intervals have been studied in [2, 3]. Our approach is close to the articles [4-6], where have been regarded the Dirichlet problem for the equations of second and fourth order. In contrast to this works we consider the equation (1) for arbitrary $\alpha, \beta, \alpha \neq 1, \alpha \neq 3$ in weighted spaces $L_{2,\beta}(1,+\infty)$

§1. The Space $\dot{W}^2_{\alpha}(1,+\infty)$: Let $\dot{C}^2[1,+\infty) = \left\{u \in C^2(1,+\infty), u(1) = u'(1) = u(+\infty) = u'(+\infty) = 0\right\}$. Denote by $\dot{W}^2_{\alpha}(1,+\infty)$ the completion of $\dot{C}^2[1,+\infty)$ in the norm

$$\|u\|^2_{\dot{W}^2_{\alpha}(1,+\infty)} = \int_1^{+\infty} t^{\alpha} |u''(t)|^2 \, dt.$$

First note that for the functions $u \in \dot{W}^2_{\alpha}(1,+\infty)$ for every $t_0 \in [1,+\infty)$ exist the finite values $u(t_0)$, $u'(t_0)$ and $u(1) = u'(1) = 0$ (see [2], [7]).

Proposition 1: *For the functions $u \in \dot{W}^2_{\alpha}(1,+\infty)$ for large values of t we have the following estimates*

$$|u(t)|^2 \leq c_1 t^{3-\alpha} \|u\|^2_{\dot{W}^2_{\alpha}(1,+\infty)}, \alpha \neq 1, \alpha \neq 3, \tag{2}$$

$$|u'(t)|^2 \leq c_2 t^{1-\alpha} \|u\|^2_{\dot{W}^2_{\alpha}(1,+\infty)}, \alpha \neq 1. \tag{3}$$

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For $\alpha = 3$ in the inequality (2) we replace $t^{3-\alpha}$ with $|\ln t|$ and for $\alpha = 1$ with $t^2|\ln t|$ in (2) and $t^{1-\alpha}$ with $|\ln t|$ in (3) (see [1]).

It follows from Proposition 1 that for $\alpha > 3$ the conditions $u(+\infty) = u'(+\infty) = 0$ after completion "are retained", for $1<\alpha\leq 3$ "is retained" only the condition $u'(+\infty) = 0$, while for $\alpha \leq 1$ the values $u(+\infty)$ and $u'(+\infty)$ in general, can become infinite.

$$\text{Let} L_{2,\beta}(1,+\infty) = \left\{ f, \|f\|^2_{L_{2,\beta}(1,+\infty)} = \int_1^\infty t^\beta |f(t)|^2 dt < \infty \right\}.$$

Proposition 2. *For $\beta \leq \alpha - 4$ we have the continuous embedding*

$$\dot{W}^2_\alpha(1,+\infty) \subset L_{2,\beta}(1,+\infty), \tag{4}$$

which for $\beta < \alpha - 4$ is compact.

Proof: Indeed, using the inequality (2) for $\beta < \alpha - 4$ we obtain

$$\int_1^{+\infty} t^\beta |u(t)|^2 dt \leq c_1 \|u\|^2_{\dot{W}^2_\alpha(1,+\infty)} \cdot \int_1^{+\infty} t^\beta t^{3-\alpha} dt = c_2 \|u\|^2_{\dot{W}^2_\alpha(1,+\infty)}$$

For the proof of the embedding (4) in case of $\beta = \alpha - 4$, $\alpha \neq 1$, $\alpha \neq 3$ we use Hardy's inequality (see [2]). Since $\dot{C}^2[1,+\infty)$ is dense in $\dot{W}^2_\alpha(1,+\infty)$ it is enough to prove the embedding (4) for $u \in \dot{C}^2[1,+\infty)$. In this case we can write

$$\int_1^\infty t^{\alpha-4} |u(t)|^2 dt \leq \frac{4}{(\alpha-3)^2} \int_1^\infty t^{\alpha-2} |u'(t)|^2 dt \leq \frac{16}{(\alpha-1)^2(\alpha-3)^2} \int_1^\infty t^\alpha |u''(t)|^2 dt$$

consequently we get that $\int_1^\infty t^{\alpha-4} |u(t)|^2 dt \leq \frac{16}{(\alpha-1)^2(\alpha-3)^2} \|u\|^2_{\dot{W}^2_\alpha(1,+\infty)}$. It is important to note that here the number $16(\alpha-1)^{-3}(\alpha-1)^{-2}$ is exact. Thus, for $\beta \leq \alpha - 4$ we get the inequality

$$\|u\|_{L_{2,\beta}(1,+\infty)} \leq c\|u\|_{\dot{W}^2_\alpha(1,+\infty)} \tag{5}$$

Let $\beta < \alpha - 4$ and $\{u_n\}_{n=1}^\infty$ is a bounded in $\dot{W}^2_\alpha(1,+\infty)$ sequence $\|u_n\|_{\dot{W}^2_\alpha(1,+\infty)} \leq M$. First we choose from this sequence convergent in $L_{2,\beta}(1,2)$ subsequence $\{u_{1n}\}_{n=1}^\infty$ (see [7]). Continue this process and choose from the convergent in $L_{2,\beta}(1,k+1)$ sequence $\{u_{kn}\}_{n=1}^\infty$ convergent in $L_{2,\beta}(1,k+1)$ subsequence $\{u_{k+1,n}\}_{n=1}^\infty$. Now we construct the diagonal sequence $\{u_{nn}\}_{n=1}^\infty$ and prove that it is convergent in $L_{2,\beta}(1,+\infty)$. For this it is enough to prove that $\|u_{nn} - u_{kk}\|_{L_{2,\beta}(A,+\infty)} \to 0$ for $A \to +\infty$. Using the inequality (2) we can write

$$\|u_{nn} - u_{kk}\|^2_{L_{2,\beta}(?,+\infty)} = \int_A^{+\infty} t^\beta |u_{nn} - u_{kk}|^2 dt \leq 2\int_A^{+\infty} t^\beta \left(|u_{nn}|^2 + |u_{nn}|^2\right) dt \leq 4M \int_A^{+\infty} t^\beta t^{3-\alpha} dt = \frac{4M}{\alpha-4-\beta} A^{\beta-(\alpha-4)}.$$

The Proof Is Complete

Remark 1: For $\beta > \alpha - 4$ the embedding (4) fails.

In fact, if we take, for example, the function $u(t) = t^\gamma \cdot \varphi(t)$, where $\varphi \in C^2[1,+\infty)$, $\varphi(t) = 0$ for $t \in [1,2]$ and $\varphi(t) = 1$ for $t \in [3,+\infty)$, then for $\gamma \in [-(1+\beta)/2, -(1+(\alpha-4))/2]$ we have that $u \in \dot{W}^2_\alpha(1,+\infty)$, but $u \notin L_{2,\beta}(1,+\infty)$.

Remark 2: For $\beta = \alpha - 4$ the embedding (4) is not compact.

Indeed, for the bounded in $\dot{W}^2_\alpha(1,+\infty)$ sequence $u_n(t) = n^{-1/2} t^{(3-\alpha)/2} |\ln t|^{-1/2-1/2n} \cdot \varphi(t)$ it is easy to check that we can not choose convergent in $L_{2,\alpha-4}(1,+\infty)$ subsequence (see [6]).

§2. Dirichlet Problem

Definition 1: *The function is called the generalized solution of the Dirichlet problem for the equation (1), if for every $u \in \dot{W}^2_\alpha(1,+\infty)$ holds the following equality*

$$(t^\alpha u'', v'') + p(t^\beta u, v) = (f, v), \tag{6}$$

Where $(\cdot,\cdot)$ stands for the scalar product in $L_2(1,+\infty)$.

First we consider a particular case of the equation (1) for $p = 0$

$$Bu \equiv (t^\alpha u'')'' = f, \ f \in L_{2,-\beta}(1,+\infty), \ \beta \leq \alpha - 4. \tag{7}$$

Proposition 3: *The generalized solution of the equation (6) exists and is unique for every* $f \in L_{2,-\beta}(1,+\infty)$.

Proof: The uniqueness of the generalized solution immediately follows from Definition 1, if we put in the equality (7) $f=0$ and $v = u$. To prove the existence of the generalized solution define the functional $l_f(v) = (f,v)$ on the space $\dot{W}_\alpha^2(1,+\infty)$ and show that it is continuous. In fact, using inequality (5) we have

$$\left|l_f(v)\right| \le \|f\|_{L_{2,-\beta}(1,+\infty)} \|v\|_{L_{2,\beta}(1,+\infty)} \le c\|f\|_{L_{2,2}(1,+\infty)} \|v\|_{\dot{W}_\alpha^2(1,+\alpha)}$$

and hence the functional $l_f(v)$ is bounded. Now the existence of the generalized solution follows from the Riesz theorem on the representation of the linear continuous functional.

Define the operator B corresponding to Definition 1 of the generalized solution [6].

Definition 2: We say that the function $u \in \dot{W}_\alpha^2(1,+\infty)$ belong to the domain of definition of the operator B, if exists $f \in L_{2,-\beta}(1,+\infty)$ such that is valid the equality (6) and then we write $Bu = f$.

The operator B acts from the space $L_{2,\beta}(1,+\infty)$ to the space $L_{2,-\beta}(1,+\infty)$. To get an operator in the same space, which is necessary to explore the spectrum of the operator, represent the function $f \in L_{2,-\beta}(1,+\infty)$ in the form $f(t) = t^\beta g(t)$. Then we obtain that $g \in L_{2,\beta}(1,+\infty)$ and $\|g\|_{L_{2,\beta}(1,+\infty)} = \|f\|_{L_{2,-\beta}(1,+\infty)}$. Define an operator $\mathrm{B} : L_{2,\beta}(1,+\infty) \to L_{2,\beta}(1,+\infty)$ by the formula $\mathrm{B}u = t^{-\beta} Bu, D(\mathrm{B}) = D(B)$. Then the equation (7) can be written in the form

$$\mathrm{B}u = g,\ g \in L_{2,\beta}(1,+\infty),\ \beta \le \alpha - 4$$

Theorem 1: The operator $\mathrm{B} : L_{2,\beta}(1,+\infty) \to L_{2,\beta}(1,+\infty)$ is for $\beta \le \alpha$-4 positive and self-adjoint. The bounded inverse operator $\mathrm{B}^{-1} : L_{2,\beta}(1,+\infty) \to L_{2,\beta}(1,+\infty)$ is for $\beta < \alpha$-4 compact.

Proof: Positivity and symmetry of the operator B follow from Definition 1, consequently we $D(B) \subset D(B^*)$ get Now prove that the operator B^* is invertible. Let $v \in D(B^*)$, $B^*v = 0$. Then for every $u \in D(B)$ is valid the equality $(\mathrm{B}u, v) = (u, \mathrm{B}^* v) = 0$ (here $(\cdot,\cdot)$ stands for the scalar product in $L_{2,\beta}(1,+\infty)$). According to Proposition 3 (the image of the operator B coincides with the space $L_{2,\beta}(1,+\infty)$) we conclude that v=0. Now the self-adjointness of the operator B follows from its invertibility. From the inequality (5) and the equality (7) for $v=u$ we get

$$(t^\alpha u'', u'') = \|u\|^2_{\dot{W}_\alpha^2(1,+\infty)} = (f,u) \le \|f\|_{L_{2,-\beta}(1,+\infty)}$$
$$\|u\|_{L_{2,\beta}(1,+\infty)} \le c\|f\|_{L_{2,-\beta}(1,+\infty)} \|u\|_{\dot{W}_\alpha^2(1,+\infty)}$$

i.e. we have $\|u\|_{W_\alpha^2(1,+\alpha)} \le c\|f\|_{L_{2,-\beta}(1,+\alpha)} = c\|g\|_{L_{2,\beta}(1,+\alpha)}$. Hence, using the definition of the operator B and the inequality (5) we obtain

$$\left\|\mathrm{B}^{-1} g\right\|_{L_{2,\beta}(1,+\infty)} = \|u\|_{L_{2,\beta}(1,+\infty)} \le c\|u\|_{\dot{W}_\alpha^2(1,+\infty)} \le c_1 \|g\|_{L_{2,\beta}(1,+\infty},$$

therefrom follows the boundedness of the operator B^{-1} for $\beta \le \alpha$ - 4. Compactness of the operator B^{-1} for $\beta < \alpha$ - 4 follows from the compactness of the embedding (4), since $R(\mathrm{B}^{-1}) = D(\mathrm{B}) \subset \dot{W}_\alpha^2(1,+\infty)$

The Proof is Complete: Thus for $\beta \le \alpha$ - 4. we get that the operator B^{-1} is compact and self-adjoint, therefore is valid the following (see [8])

Corollary: The spectrum of the operator $\mathrm{B} : L_{2,\beta}(1,+\infty) \to L_{2,\beta}(1,+\infty)$ for $\beta < \alpha$-4 is discrete $\sigma(\mathrm{B}) = \sigma_p(\mathrm{B})$ and the system of the eigenfunctions is complete in $L_{2,\beta}(1,+\infty)$.

Note that for $\beta = \alpha$-4 the spectrum of the operator B is nondiscrete [9] as consequence of the noncompactness of the embedding (4). More precisely we have the following

Theorem 2: The spectrum of the operator B for $\beta = \alpha$-4 is purely continuous $\sigma(\mathrm{B}) = \sigma_c(\mathrm{B})$ and coincides with the ray $\left[\frac{(\alpha-1)^2(\alpha-3)^2}{16}; +\infty\right)$.

Proof: First note that in Proposition 2 for the operator B in case of $\beta = \alpha$-4 actually we have proved the inequality

$(\mathrm{B}u,u) \geq \frac{(\alpha-1)^2(\alpha-3)^2}{16}\|u\|^2_{L_{2,\beta}(1,+\infty)}$, i.e.

semiboundedness of the operator B. Since the operator B is positive and self-adjoint, therefrom we conclude that the points $\lambda < \frac{(\alpha-1)^2(\alpha-3)^2}{16}$ belong to the resolvent set $\rho(B)$ of the operator B To show the belonging of the points $\lambda \geq \frac{(\alpha-1)^2(\alpha-3)^2}{16}$ to the continuous spectrum it is enough to prove that the number of the solutions of the equation $(t^{\alpha}u'')'' = \lambda t^{\alpha-4}u$ (this is an Euler type equation), which belong to $L_{2,\beta}(1,+\infty)$ is less than two [10, 11]. First note that the function $t^{\gamma} \in L_{2,\beta}(1,+\infty)$ only in case when $\mathrm{Re}\,\beta < 1/2$. Characteristic equation of the above Euler type equation has the hereafter form $\gamma(\gamma-1)(\alpha+\gamma-2)(\alpha+\gamma-3)-\lambda=0$. Denote by $\mu = \gamma^2 + (\alpha-3)\gamma$. Then we obtain the quadratic equation $\mu^2 - (\alpha-2)\mu - \lambda = 0$ For $\lambda = \frac{(\alpha-1)^2(\alpha-3)^2}{16}$ we get two roots $\mu_1 = -(\alpha-3)^2/4$, $\mu_2 = (\alpha-1)^2/4$.. For the corresponding γ we obtain four roots γ = (3-α)/2 (double root) and $\gamma = (3-\alpha \pm \sqrt{2\alpha^2 - 8\alpha + 10})/2$. It is evident that only one root $\mathrm{Re}\,\beta < 1/2$, satisfies to the condition (in this case all roots are real). For $\lambda > \frac{(\alpha-1)^2(\alpha-3)^2}{16}$ we get $\mu_1 < -(\alpha-3)^2/4, \mu_2 > (\alpha-1)^2/4$. It is easy to check that in this case only for the root $\gamma = (3-\alpha-\sqrt{\alpha^2-6\alpha+9+4\mu_2})/2$ holds the condition *Re* β<1/2 therefrom immediately follows the statement of the theorem.

The Proof Is Complete: Note that the original equation (1) can be written in the form

$$\mathrm{B}u = -pu + g,\ g(t) = t^{-\beta}f(t) \in L_{2,\beta}(1,+\infty),\ \beta \leq \alpha - 4$$

i.e. the number -*P* can be regarded as a spectral parameter. Therefore we can state that the equation (1) is uniquely solvable, if -*P*∈ρ(*B*) where ρ(*B*)is the set of all regular points of the operator B.

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