A Numerical Method Based on Toeplitz-Collocation Method and Orthogonal Polynomials for Solving Nonlinear Functional Integral Equations

Hesam-Aldien Derili Gherjalar, Asghar Arzhang and Mehdi Yousefi

Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran

Abstract: A new numerical method for solving nonlinear functional integral equations of the Hammerstein type is proposed. Toeplitz matrix method with collocation method is used, as a numerical method to obtain a nonlinear system of algebraic equations. The convergence and numerical stability of the proposed method are mathematically proved. Finally, some examples with the exact solutions are given to show the efficiency of the method.

Key words: Nonlinear functional integral equation • Hammerstein type • Toeplitz matrix method • Nonlinear algebraic system • Collocation method • Orthogonal polynomials

2000 AMS Subject Classifications: 46N20 • 65R20 • 45G10 • 93C10 • 65L60

INTRODUCTION

Integral equations in general arise in physics (solid state physics, plasma physics, quantum mechanics), astrophysics (the radiative transfer being modeled with the well known Chandrasekhar integral equation), fluid dynamics (the study of water waves on liquids of infinite depth uses the Nekrasov's integral equation), cell kinetics [1], chemical kinetics, the theory of gases, mathematical economics, hereditary phenomena in biology. For these and other applications [2-4]. Also integral equations in general case, play an important role in many branches of linear and nonlinear functional analysis and their applications in the theory of elasticity, engineering, mathematical physics and contact mixed problems [5]. Therefore, many different methods are used to obtain the solution of the nonlinear integral equation. In [6], Brunner et al. have introduced a class of methods depending on some parameters to obtain the numerical solution of Abel integral equation of the second kind. In [7], Kaneko and Xu used degenerate kernel method to obtain the solution of Hammerstein integral equation. The linear multistep methods are used in [8], to obtain the numerical solution of a singular nonlinear Volterra integral equation. Also, in [9], Kilbas and Saigo have used an asymptotic method to obtain numerically the solution of nonlinear Abel-Volterra integral equation. In [10], Orsi used a Product Nyström method, as a numerical method, to obtain the solution of nonlinear Volterra integral equation, when its kernel takes a logarithmic and Carleman forms. Moreover, some methods can be found in Refs. [11-15] to discuss and obtain the solution of Hammerstein integral equation. We have applied Daubechies wavelets for solving Nonlinear Integral equations in [12]. Existence results for functional integral equations are obtained in [16] using a fixed point theorem due to Avery and Peterson. The fourth order elastic beam equation with clamped or simply supported at both ends boundary conditions is equivalent with a Hammerstein integral equation. The existence of positive solutions for this type of Hammerstein integral equation is obtained in [17] using the Krasnoselskii fixed point theorem and in [18] using the Leray-Schauder fixed point theorem. The technique from [17,18] can be extended even for functional Hammerstein integral equations. The existing results presented above for Fredholm and Hammerstein functional integral equations motivates the study of this type of functional integral equations. Our attention in this paper is focused on the numerical methods for Hammerstein functional integral equations. For the numerical solution of Fredholm and Hammerstein integral equations the existing methods are generally based on Nyström type methods, iterative methods [3, 19, 20]. But, for functional Fredholm and Hammerstein integral equations the approximation of the solution is studied in few papers. The numerical methods for functional Fredholm integral equations are based on the collocation techniques, homotopy perturbation methods, Lagrange and Chebyshev

Corresponding Author: Hesam-Aldien Derili Gherjalar, Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.
polynomials, the variational iteration method and the spline functions method [21-23]. The Hammerstein functional integral equation studied in [22, 23] is of special type. In this paper, we extended the method given by M. A. Abdou, et al [14, 15] for nonlinear functional integral equations. We give a numerical method for the following functional Hammerstein integral equation of second kind:

\[ x(t) = g(t) + \int_a^b H(t, s) f(s, x(s), x(\varphi(s))) ds, \quad x \in [a, b]. \]  

where

\[ a, b, H, f, g, \varphi : [a, b] \to [a, b] \]  

and \( H, f, g, \varphi \) are continuous.

Differentiating Eq. (1) with respect to the variable \( t \), we have

\[ \ddot{x}(t) = h(t) + \int_a^b \frac{\partial H(t, s)}{\partial t} f(s, x(s), x(\varphi(s))) ds, \]  

\[ h(t) = g'(t), \quad t \in [a, b]. \]  

The integro differential Eq. (2) is equivalent to the functional integral Eq. (1). Therefore, the same solution will satisfy both of the two equivalent equations, after neglecting the constant term. A particular case of the Hammerstein functional integral equation (2) is equivalent with two-point boundary value problems associated to second order and fourth order functional differential equations (where \( H \) is the Green function). For instance, the fourth order elastic beam equation with deviating argument and clamped ends boundary conditions can be written in an equivalent Hammerstein functional integral equation form using a corresponding Green function. The existence and uniqueness solution of the nonlinear integral Eq. (1) are discussed and proved. The Toeplitz matrix method is used, as a numerical method, to obtain a nonlinear system of algebraic equations. Also, we derive many important theorems related to the existence and uniqueness solution of the functional integral equation and its algebraic system. Finally, we give some applications that confirm the convergence and the numerical stability of the presented method.

**Theoretical Results:** Here we will be used Banach fixed point theorem as a source of existence and uniqueness solution of Eq.(1). For this purpose, we write it in the following form

\[ \tilde{G}[x(t)] = g(t) + G[x(t)], \]  

where,

\[ G[x(t)] = \int_a^b H(t, s) f(s, x(s), x(\varphi(s))) ds. \]  

for our purpose, we assume the following conditions hold

- \[ \left( \int_a^b \int_a^b |H(t, s)|^2 \, ds \right)^{1/2} = c, \quad c \text{ is a constant.} \]
- The given function \( g(t) \) is continuous in the space \( L_2[a, b] \) and its norm is defined as

\[ \|g(t)\|_2 = \left( \int_a^b |g(t)|^2 \, dt \right)^{1/2}. \]

There exist, \( a, b > 0 \) such that

\[ |f(s, u, v) - f(s, u', v)| \leq \alpha |u - u'| + \beta |v - v'| \]

for all \( s \in [a, b], (u, v), (u', v') \in \mathbb{R} \times \mathbb{R}, \)

\[ b - a < \frac{1}{K(\alpha + \beta)} \]

for all \( (t, s) \in [a, b] \times [a, b] \)

There exist \( \gamma, \delta, \mu, \rho > 0 \) such that

\[ |H(t, s) - H(t', s')| \leq \delta |t - t'| + \lambda |s - s'| \]

\[ |f(s, u, v) - f(s', u, v)| \leq \gamma |s - s'| \]

\[ |\varphi(t) - \varphi(t')| \leq \mu |t - t'|, \]

\[ |g(t) - g(t')| \leq \rho |t - t'|, \]

for all \( t, s, t', s' \in [a, b], (u, v) \in \mathbb{R} \times \mathbb{R}, \)

The known continuous function \( f(s, x(s), x(\varphi(s))) \) satisfies, for the constants \( A, A, A > P \) satisfies the following conditions

(a): \[ \left( \int_a^b |f(s, x(s), x(\varphi(s)))|^2 \, dt \right)^{1/2} \leq A \|x(t)\|_2 \]

(b): \[ |f(s, x_1(s), x_1(\varphi(s))) - f(s, x_2(s), x_2(\varphi(s)))| \leq M(s)|x_1(t) - x_2(t)|. \]

where \( \|M(s)\|_2 = P \)

Let \( f_0 : [a, b] \to \mathbb{R}, f_0(s) = f(s, x(s), x(\varphi(s))) \). Since \( f_0, g, \varphi \) are continuous we infer that \( f_0 \) is continuous on the compact set \([a, b]\) and therefore exists \( M_0 \geq 0 \) such that \( |f_0(s)| \leq M_0 \) for all \( s \in [a, b] \).
Lemma 1: Under the conditions (i)-(vii-a), the operator $\overline{G}$ defined by (3), maps the space $L_2[a,b]$ into itself.

Proof. From the equations (3), (4) we deduce that

$$
\|G[x(t)]\|_2 \leq \|g(t)\|_2 + \|G[x(t)]\|_2 \leq \|g(t)\|_2 + \|H(t,s)|f(s,x(s),x(\phi(s)))|dt\|_2.
$$

By using Cauchy-Schwarz inequality, then using the conditions (i)-(vii-a), the above inequality can be adapted to

$$
\|\overline{G}[x(t)]\|_2 \leq \zeta + \sigma \|x(t)\|_2, \quad (\sigma = \alpha d) \quad (5)
$$

The last inequality shows that, the operator $\overline{G}$ maps the ball $S_\rho$ into itself, where

$$
\rho = \frac{\zeta}{1 - \alpha d}. \quad (6)
$$

Since $\rho > 0$, $\zeta > 0$ therefore we have $\sigma < 1$. Moreover, the inequality (5) involves the boundedness of the operator $\overline{G}$ of Eq.(3), where

$$
\|G[x(t)]\|_2 \leq \sigma \|x(t)\|_2 \quad (7)
$$

Also, the inequalities (5) and (7) define the boundedness of the operator $\overline{G}$.

Lemma 2: Under the conditions (i)-(vii-b), the operator $\overline{G}$ defined by (3), is contractive in Banach space $L_2[a,b]$.

Proof: For the functions $x_1(t), x_2(t), y_1(t), y_2(t)$ in the space $L_2[a,b]$, the formulas (2) and (3) lead to:

$$
\|G[x_1(t)] - G[x_2(t)]\|_2 \leq \|H(t,s)\|f(s,x_1(s),x_1(\phi(s))) - f(s,x_2(s),x_2(\phi(s)))|dt\|_2.
$$

Applying condition (i), (vii-b), then using Cauchy-Schwarz inequality, the above inequality can be adapted to

$$
\|G[x_1(t)] - G[x_2(t)]\|_2 \leq \sigma \|x_1(t) - x_2(t)\|_2. \quad (8)
$$

The last inequality shows that, the operator $\overline{G}$ is continuous in the space $L_2[a,b]$, then $\overline{G}$ is a contraction operator under the condition $\sigma < 1$.

- Now with the above two lemmas we proof our main Theorem.

Theorem 1: Under the conditions (i)-(viii), then the Eq.(1) has a unique solution $x(t) \in L_2[a,b]$.

Proof: The proof of theorem is directly obtained from the above two lemmas. So the mentioned operator $\overline{G}$ is contractive in the Banach space $L_2[a,b]$. By using Banach fixed point theorem, the operator $\overline{G}$ has a unique fixed point which is the necessary unique solution of Eq.(1).

The Toplitz-collocation Method: In this section, we discuss the Toplitz matrix method [14, 15] to obtain the numerical solution of functional nonlinear integral equation. The idea of this method is to obtain a system of $2N+1$ nonlinear algebraic equations, where $2N+1$ is the number of collocation mesh points. The coefficient matrix is expressed as sum of two matrices, one of them is Toplitz matrix and the other is a matrix with zero elements except the first and last columns.

- Consider the functional integral equation

$$
x(t) = g(t) + \int_a^t H(t,s)f(s,x(s),x(\phi(s))) ds \quad (9)
$$

The method assumes

$$
\int_a^b H(t,s)f(s,x(s),x(\phi(s))) ds = \sum_{\alpha=0}^{2N} a^{\alpha} h^{N+\alpha} H(t,s)f(s,x(s),x(\phi(s))) ds, \quad (10)
$$

where $A_\alpha(t)$ and $B_\alpha(t)$ are arbitrary functions to be determined and $R$ is the estimate error. Putting $x$ in Eq.(10) yields a set of two equations in terms of the functions $A_\alpha(t)$ and $B_\alpha(t)$. In this case, the error $R$ will be neglected, then we obtain
\[
\int_a^{a+h} H(t,s)f(s,1,1)ds = A_n(t)f(a,1,1) + B_n(t)f(a+h,1,1),
\]
(12)

and
\[
\int_a^{a+h} H(t,s)f(s,y,y)ds = A_n(t)f(a,a,\varphi(a)) + B_n(t)f(a+h,a+h,\varphi(a+h)),
\]
(13)

By using Eqs (11) and (12), we can deduce
\[
A_n(t) = \frac{1}{h^2}[f(a+h,a+h,\varphi(a+h))I(t) - f(a+h,1,1)J(t)],
\]
(14)

\[
B_n(t) = \frac{1}{h^2}[f(a+h,a+h,\varphi(a+h))J(t) - f(a+h,1,1)I(t)],
\]
(15)

where
\[
I(t) = \int_a^{a+h} H(t,s)f(s,1,1)ds, \quad J(t) = \int_a^{a+h} H(t,s)f(s,1,1)ds
\]
and
\[
h = f(a+h,a+h,\varphi(a+h))f(a,1,1) - f(a+h,1,1)I(a,a,\varphi(a))
\]

With using Eqs.(11)-(14), the formula (9) becomes
\[
\int_a^{a+h} H(t,s)f(s,x(s),x(\varphi(s)))ds = \sum_{n=0}^{2N+1} D_n(t)\varphi(nh,x(nh),\varphi(nh)),
\]
(16)

where,
\[
D_n(t) = \begin{cases} A_0(t), & n = 0 \\ A_n(t) + B_{n-1}(t), & 0 < n < 2N + 1 \\ B_{2N}(t), & n = 2N + 1 \end{cases}
\]
(17)

Thus, the integral Eq (8) takes the form
\[
x(t) = \sum_{n=0}^{2N} D_n(t)\varphi(nh,x(nh),x(\varphi(nh))) = g(t),
\]
(18)

With using collocation nodes as \( x = mh \) in (17) and using follow notations
\[
x(mh) = x_m, \quad D_n(mh) = D_{mn}, \quad g(mh) = g_m,
\]
\[
f(nh,x(nh),x(\varphi(nh))) = f_n(x_m),
\]
(19)

we obtain the following nonlinear algebraic system
\[
x_m = \sum_{n=0}^{2N} D_{mn}f_n(x_m) = g_m, \quad 0 \leq m \leq 2N
\]
(20)

Here, we have
\[
D_{mn} = G_{mn} - E_{mn}
\]
which is a Toeplitz matrix of order \( 2N+1 \)
and,
\[
E_{mn}(t) = \begin{cases} B_{-1}(mh), & n = 0 \\ 0, & 0 < n < 2N \\ A_{2N}(mh), & n = 2N \end{cases}
\]
(21)

\[
\begin{bmatrix}
G_{mn} = A_n(mh) + B_{n-1}(mh), & 0 \leq m,n \leq 2N
\end{bmatrix}
\]

represents a matrix of order \( (2N+1) \) whose elements are zeroes except for the first and the last rows(columns).

**Error Analysis:** In this section, our aim is to prove the existence and uniqueness of the solution of nonlinear functional algebraic system (19) in Banach space \( \Gamma \). For this purpose, we write it in the operator form
\[
\bar{T}x_m = Tx_m + g_m,
\]
(23)

where
\[
Tx_m = \sum_{n=0}^{2N} D_{mn}f_n(x_m), \quad 0 \leq m \leq 2N.
\]
(24)

Then we consider the following definitions

**Definition 1:** [15] The estimate local error \( \epsilon_i \) is determined by the following equation
\[
x(t) - x_i(t) = \sum_{n=0}^{2N} D_{mn}[f(nh,x(nh),x(\varphi(nh))) - f(nh,x(nh),x(\varphi(nh)))] + R_j.
\]
(25)

where \( x(t) \) is the approximate solution of Eq.(1). Also, Eq.(25) gives
\[
R_j = \int_a^b H(t,s)f(s,x(s),x(\varphi(s)))ds - \sum_{n=0}^{2N} D_{mn}f_n(x_m,x(\varphi(nh)))
\]
(26)

**Definition 2:** [15] The Toeplitz method is said to be convergent of order \( r \) in the interval \([a,b]\), if and only if for sufficiently large \( N \) there exist a constant \( D > 0 \) independent on \( N \) such that
\[
\| x(t) - x_m(t) \| \leq DN^{-r}.
\]
(27)
**Theorem 2:** The algebraic system (19), in Banach space $\Gamma$ has a unique solution under the following conditions:

- \( \sup_{n} |g_{m}| \leq H < \infty \) \quad (H is a constant)
- \( \sup_{n} \sum_{n=0}^{2N} D_{mn} \leq E \), \quad (E is a constant)
- The known functions \( f(nh, x(nh), x(\varphi(nh))) \), for the constants \( Q > Q_1, Q > R_1 \) satisfy

  \[(a) \quad \sup_{n} |f(nh, x(nh), x(\varphi(nh)))| \leq Q_1 \| x \|_{\infty} , \]

  \[(b) \quad \sup_{n} |f(nh, x(nh), x(\varphi(nh))) - f(nh, y(nh), y(\varphi(nh)))| \leq R_1 \| X - Y \|_{\infty} , \]

where \( \| X \|_{\infty} = \sup_{n} |x_{n}| \) and \( \| Y \|_{\infty} = \sup_{n} |y_{n}| \), for each integer \( n \).

**Proof:** Let \( X \) be the set of all functions \( X = x_{m} \) in \( \Gamma \) such that \( \| X \|_{\infty} \leq B, B \) is a constant. Define the norm of the operator \( T_{X} \) in Banach space \( \Gamma \) by

\[ \| T_{X} \|_{\infty} = \sup_{m} \| T_{X} x_{m} \|, \quad (m \in \mathbb{Z}). \]  \hspace{1cm} (28)

The formula (23) and (24) lead to

\[ \| T_{X} x_{m} - T_{Y} x_{m} \| \leq \sum_{n=0}^{2N} D_{mn} |f(nh, x(nh), x(\varphi(nh))) - f(nh, y(nh), y(\varphi(nh)))| \]

Under the two conditions (ii) and (vii-b), we obtain

\[ \| T_{X} x_{m} - T_{Y} x_{m} \| \leq \sigma_{1} \| X - Y \|_{\infty} \]

The above inequality is true for each integer \( m \), hence in view of (28) we have

\[ \| T_{X} x_{m} - T_{Y} x_{m} \| \leq \sup_{m} |x_{m} - (x_{m})_{j}| \leq \sigma_{1} \| X - Y \|_{\infty} \]

Then, \( T \) is a contraction operator in banach space \( \Gamma \) under the condition \( \sigma < 1 \). Hence by Banach fixed point theorem, \( T \) has a unique fixed point which is, of course, represents the unique solution of the nonlinear algebraic system in Banach space \( \Gamma \).

**Theorem 3:** If the conditions (ii) and (vii-b) of Theorem 2 are satisfied and the sequence of functions \( f_{j}(x) = (f_{j})_{m} \) converges uniformly to the function \( f(x) = f_{m} \) in the banach space \( \Gamma \). Then the sequence of functions \( \varphi_{j} = (\varphi_{j})_{m} \) converges uniformly to the solution of (22) in the Banach space \( \Gamma \).

**Proof:** By using (20), we have

\[ |x_{m} - (x_{m})_{j}| \leq \sum_{n=0}^{2N} D_{mn} |f(nh, x(nh), x(\varphi(nh))) - f(nh, x(nh), y(\varphi(nh)))| \]

The above inequality after using condition (vii-b) holds for each integer \( m \), hence from condition (ii) we find

\[ \sup_{m} |x_{m} - (x_{m})_{j}| \leq \sigma_{1} \| X - X_{j} \|_{\infty} + \| g - g_{j} \|_{\infty} \]

Finally, the previous inequality takes the form,

\[ \| X - X_{j} \|_{\infty} \leq \frac{1}{(1 - E \theta)} \| g - g_{j} \|_{\infty} \quad (\theta < 1) \]  \hspace{2cm} (29)

Since \( \| g - g_{j} \|_{\infty} \to 0 \) as \( j \to \infty \) SO \( x - x_{j} \) \( \infty \) as \( j \to \infty \)

By using above Theorem, We have

\[ |R_{j} | \leq \sup_{m} |x_{m} - (x_{m})_{j}| + \sum_{n=0}^{2N} D_{mn} |f(nh, x(nh), x(\varphi(nh))) - f(nh, x(nh), x_{j}(\varphi(nh)))| \]

using condition (vii-b), we obtain

\[ |R_{j} | \leq \sup_{m} |x_{m} - (x_{m})_{j}| + Q \| X - X_{j} \|_{\infty} \sup_{n=0} \sum_{n=0}^{2N} D_{mn} \]

The above inequality is true for each integer \( j \), hence from condition (ii) we obtain

\[ |R_{j} | \leq (1 + E \theta) \| X - X_{j} \|_{\infty} \]

Since \( \| X - X_{j} \|_{\infty} \to 0 \) as \( j \to \infty \) than \( |R_{j} | \) \( \infty \) as \( j \to \infty \).

**Theorem 4:** If the sequence of continuous functions \( g(x) \) converges uniformly to the function \( g(x) \) in Banach space \( \Gamma \), then under the conditions of Theorem 1, the sequence of approximate solution \( x_{m}(x) \) converges uniformly to the exact solution of (1) in Banach space \( L_{2}[a, b] \).

**Proof:** The formula (1) with its approximate solution give the following inequality

\[ |x(t) - x_{j}(t)| \| H(t, s) \| f(s, x(s), x(\varphi(s))) - f(s, x_{j}(s), x_{j}(\varphi(s)))| ds |_{2} + \| g(t) - g_{j}(t) |_{2} \]
Applying condition (vi-b) and using Cauchy inequality, we have the following inequality:

\[
\|x(t) - x_j(t)\|_2 \leq \left( \int_a^b |H(t,s)|^2 \, dt \right)^{1/2} \left( \int_a^b |M(s)|^2 \, ds \right)^{1/2} \left\| x(t) - x_j(t) \right\|_2 + \left\| g(t) - g_j(t) \right\|_2
\]

and we can deduce that

\[
\|x(t) - x_j(t)\|_2 \leq \frac{1}{1 + c_4^2} \left\| g(t) - g_j(t) \right\|_2
\]

that gives required results. In other words if \( \|g(t) - g_j(t)\| \to 0 \), as \( J \to \infty \) then \( \|x(t) - x_j(t)\| \to 0 \), as \( J \to \infty \).

**Numerical Examples and Applications:** In this section, several numerical examples of the nonlinear Hammerstein functional integral equations (1) are considered to show the accuracy of presented method. In this study, all examples are solved using the method stated in Section 3 and the functional integral equations are converted to systems of nonlinear equations (19). All computations are performed using Maple 14. For all examples, the first column represents \( n \) for summation, the second column report the knots, the third column contains the values of the exact solution on these knots and fourth column devoted to the approximations on the knots. The fifth column contains the effective errors

\[
e_1 = |x(t_k) - x_n(t_k)|
\]

**Example 1:** The following functional integral equation:

\[
x(t) = \frac{t}{6} - \int_0^t G(t,s) \left( e^{-s} \left( \frac{s}{2} - 2 \right) + e^{s} \frac{x(s)}{2} \right) \, ds, \quad t \in [0,1]
\]

where the function \( G : [0,1] \times [0,1] \to \mathbb{R} \) is given by:

\[
G(t,s) = \begin{cases} 
 1 - t, & s \leq t, \\
 1 - s, & s \geq t.
\end{cases}
\]

has exact solution \( x(t) = t e^{-t} \) and this integral equation is equivalent with the two-point boundary value problem:

\[
\begin{aligned}
x'(t) &= e^{-t} (t^2/2 - 2) + e^{t} / 2, \\
x(0) &= 0, x(1) = e^{-1}, \quad t \in [0,1].
\end{aligned}
\]

Applying the presented method we get the results in Table 1.

**Example 2:** Consider the functional integral equation:

\[
x(t) = \frac{2t}{\pi} \left( \sqrt{2 + \pi^2/32} + 4(\pi/4 - t) \right) - \\
\int_0^t H(t,s) \left( 1 + 2(1 + s^2) \cos(s) - 2 \cos(s) \right) \cdot ds \quad t \in [0, \pi/4]
\]

where the function \( H : [0, \pi/4] \times [0, \pi/4] \to \mathbb{R} \) is given by:

\[
H(t,s) = \begin{cases} 
 1 - t, & 0 \leq s \leq t, \\
 1 - s, & 0 \leq t \leq s.
\end{cases}
\]

has exact solution \( x(t) = t^2 + \sin(t) + 1 \) and this functional integral equation is equivalent with the two-point boundary value problem:

\[
\begin{aligned}
x'(t) &= 1 + 2(1 + t^2) \cos(t) - 2 \cos(t) x(t) \\
x(0) &= 1, x(\pi/4) = 1 + \sqrt{2 + \pi^2/32}, \quad t \in [0, \pi/4]
\end{aligned}
\]

The results are given in Table 2.

**Example 3:** Consider the functional integral equation:

\[
x(t) = g(t) + \int_0^t H(t,s) \left( x(s) + e^{s^2/2} x(s) \right) \, ds \quad t \in [0,1]
\]

where

\[
g(t) = 3t^2 - 2s^2 + t + 1 + t^2(2 - t) \cdot e
\]

and the function \( G : [0,1] \times [0,1] \to \mathbb{R} \) is given by:

\[
G(t,s) = \begin{cases} 
 1 - t, & s \leq t, \\
 1 - s, & s \geq t.
\end{cases}
\]

has the exact solution \( x(t) = e^t \). Here

\[
H(t,s) = t^2 / 6(1 - s)^2 (s - t + 2s(1 - t))
\]

and

\[
K(t,s) = s^2 / 6(1 - t)^2 (t - s + 2t(1 - s))
\]

This functional integral equation is equivalent with the following two-point boundary value problem associated to the elastic beam functional differential equation with clamped boundary conditions:

258
Table 1:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$t_i$</th>
<th>$x_i(t)$</th>
<th>$x_{s,i}(t)$</th>
<th>$e_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>0.000000000</td>
<td>1.84173e-05</td>
<td>2.00000e-05</td>
</tr>
<tr>
<td></td>
<td>0.167</td>
<td>0.141315335</td>
<td>0.141673818</td>
<td>3.54883e-05</td>
</tr>
<tr>
<td></td>
<td>0.333</td>
<td>0.238684174</td>
<td>0.258047654</td>
<td>2.63180e-05</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.303265329</td>
<td>0.303466841</td>
<td>2.05121e-05</td>
</tr>
<tr>
<td></td>
<td>0.667</td>
<td>0.341526087</td>
<td>0.342098740</td>
<td>1.63555e-05</td>
</tr>
<tr>
<td></td>
<td>0.833</td>
<td>0.362241401</td>
<td>0.362253724</td>
<td>1.12723e-05</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>0.367879441</td>
<td>0.367979442</td>
<td>0.96145e-06</td>
</tr>
</tbody>
</table>

Table 2:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$t_i$</th>
<th>$x_i(t)$</th>
<th>$x_{s,i}(t)$</th>
<th>$e_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>9.52642-10</td>
<td>1.43567e-05</td>
</tr>
<tr>
<td></td>
<td>0.131</td>
<td>1.1392061400</td>
<td>1.1392162903</td>
<td>0.10384e-05</td>
</tr>
<tr>
<td></td>
<td>0.262</td>
<td>1.2953348160</td>
<td>1.2953537613</td>
<td>1.95601e-05</td>
</tr>
<tr>
<td></td>
<td>0.393</td>
<td>1.4050185970</td>
<td>1.4050388450</td>
<td>1.94476e-05</td>
</tr>
<tr>
<td></td>
<td>0.524</td>
<td>1.6736354800</td>
<td>1.6736582350</td>
<td>1.81606e-05</td>
</tr>
<tr>
<td></td>
<td>0.654</td>
<td>1.8222328910</td>
<td>1.8223279010</td>
<td>1.23658e-05</td>
</tr>
<tr>
<td>0.785</td>
<td>2.0149376810</td>
<td>2.01493891785</td>
<td>1.23658e-06</td>
<td></td>
</tr>
</tbody>
</table>

Table 3:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$t_i$</th>
<th>$x_i(t)$</th>
<th>$x_{s,i}(t)$</th>
<th>$e_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>1.000000000000</td>
<td>1.000003051232</td>
<td>3.05113e-06</td>
</tr>
<tr>
<td></td>
<td>0.167</td>
<td>1.181752650000</td>
<td>1.181757137550</td>
<td>2.79255e-06</td>
</tr>
<tr>
<td></td>
<td>0.333</td>
<td>1.356147280000</td>
<td>1.35616880460</td>
<td>1.35824e-05</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>1.648724710000</td>
<td>1.64877283430</td>
<td>5.60123e-05</td>
</tr>
<tr>
<td></td>
<td>0.667</td>
<td>1.948838394000</td>
<td>1.94893512570</td>
<td>1.17585e-05</td>
</tr>
<tr>
<td></td>
<td>0.833</td>
<td>2.300209027000</td>
<td>2.30020926010</td>
<td>1.17305e-06</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>2.718281280000</td>
<td>2.718281382800</td>
<td>1.24521e-06</td>
</tr>
</tbody>
</table>

5

<table>
<thead>
<tr>
<th>$n$</th>
<th>$t_i$</th>
<th>$x_i(t)$</th>
<th>$x_{s,i}(t)$</th>
<th>$e_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000000000000</td>
<td>1.000003051232</td>
<td>3.05113e-06</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>1.105179198756848</td>
<td>1.1051794847177</td>
<td>4.66472e-07</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>1.211403103458170</td>
<td>1.211403404358170</td>
<td>3.48297e-07</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>1.349859087576000</td>
<td>1.3498591186404</td>
<td>3.10166e-07</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>1.491825027082080</td>
<td>1.491825270082080</td>
<td>3.29406e-07</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>1.648723252944148</td>
<td>1.648723252944148</td>
<td>9.82244e-07</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>1.822118803506569</td>
<td>1.8221266620245</td>
<td>1.80234e-06</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>2.013752177470417</td>
<td>2.0137554697044</td>
<td>1.78265e-06</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>2.225540296246880</td>
<td>2.225540296246880</td>
<td>1.22485e-06</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>2.459603111156950</td>
<td>2.4596034334619</td>
<td>1.22485e-06</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.718281828459045</td>
<td>2.7182827899130</td>
<td>0.96145e-06</td>
<td></td>
</tr>
</tbody>
</table>
Table 4:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( t_i )</th>
<th>( x(t) )</th>
<th>( x_i(t) )</th>
<th>( e_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>0.000000000</td>
<td>0.001000000</td>
<td>1.000000e-003</td>
</tr>
<tr>
<td></td>
<td>0.167</td>
<td>0.1666666666</td>
<td>0.175296892</td>
<td>8.630326e-003</td>
</tr>
<tr>
<td></td>
<td>0.333</td>
<td>0.3333333333</td>
<td>0.3371843854</td>
<td>3.851012e-003</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.5000000000</td>
<td>0.507807082</td>
<td>7.807082e-003</td>
</tr>
<tr>
<td></td>
<td>0.667</td>
<td>0.6666666666</td>
<td>0.741754836</td>
<td>7.508817e-002</td>
</tr>
<tr>
<td></td>
<td>0.833</td>
<td>0.8333333333</td>
<td>0.908804113</td>
<td>7.546708e-002</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1.0000000000</td>
<td>1.071246406</td>
<td>7.124640e-002</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0.0000000000</td>
<td>0.000722689</td>
<td>7.226890e-04</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>0.1000000000</td>
<td>0.1006788149</td>
<td>2.076892e-04</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>0.2000000000</td>
<td>0.2006788149</td>
<td>6.788149e-04</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
<td>0.3000000000</td>
<td>0.3007165478</td>
<td>7.165790e-04</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>0.4000000000</td>
<td>0.4453448111</td>
<td>9.035667e-04</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.5000000000</td>
<td>0.5008035408</td>
<td>8.085408e-04</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>0.6000000000</td>
<td>0.6009210600</td>
<td>6.921061e-04</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>0.7000000000</td>
<td>0.7011018070</td>
<td>1.101807e-03</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>0.8000000000</td>
<td>0.8014881010</td>
<td>1.488101e-03</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>0.9000000000</td>
<td>0.9014881019</td>
<td>1.481802e-03</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1.0000000000</td>
<td>1.0016157310</td>
<td>1.615731e-03</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
    x^{(4)}(t) &= 1/2, x(t) + e^{t^2} x(t^2/2), \\
    x(0) &= 1, x(1) = e, \\
    x'(0) &= 1, x'(1) = e, \\
    t &\in [0,1].
\end{align*}
\]

The results are in Table 3.

**Example 4:** Consider the functional integral equation:

\[
x(t) = \frac{9}{16} + \frac{1}{12} \int_0^1 (t - s) \left( \frac{x(s)}{2} \right)^2 ds, \quad t \in [0,1]
\]

has exact solution \( x(t) = t \) Applying the presented algorithm We have results in the Table 4.

**CONCLUSIONS**

A new numerical method using the fixed point technique, Toeplitz matrix method and collocation method is proposed for finding the numerical solution of Hammerstein functional integral equations. The algorithm of the method has recurrent form easy to program and it is convergent and numerically stable, the main results being Theorem 3 and Theorem 4. The method can be extended even for the nonlinear Fredholm functional integral equations (Urysohn functional I.E.). The convergence and the numerical stability of the method are confirmed by the presented numerical experiments. The convergence is tested for \( n = 3 \) and \( n = 5 \). To prove the convergence of the method only Lipschitz properties are required, smoothness conditions being not necessary. These extend the applicability of the method. It is known that the existing numerical methods require high order smoothness conditions in the proof of convergence. The principle of the method gives its generality, being extensible to other types of operatorial equations with modified argument. These justify the name the Toeplitz-collocation method.

**ACKNOWLEDGEMENTS**

The authors would like to thank Islamic Azad University (Karaj Branch) for supporting this work.

**REFERENCES**