World Applied Sciences Journal 16 (12): 1830-1836, 2012
ISSN 1818-4952


# The Application of Homotopy Perturbation Method to Solve the Higher Order KdV Equations

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**Abstract:** In this paper, an analytical approximation to the solution of higher order Korteweg-de Vries (KdV) equations has been studied. The Homotopy Perturbation Method (HPM) introduced by He is employed to drive this analytical solution and the results will be compared with those of the Adomian decomposition method. Numerical results reveal that the HPM provides highly accurate numerical solutions for higher order KdV equations.

**Key words:** Homotopy perturbation method . KdV equation

## INTRODUCTION

There are many nonlinear partial differential equations which are quite useful and applicable in engineering and physics such as the well-known KdV equation [1], Modified KdV (MKdV) equation [2], Benjamin-Bona-Mahony (BBM) equation [3], Burgers equation [4], Regularized-Long-Wave (RLW) equation [5] and so on. Nonlinear partial differential equations are generally difficult to be solved and their exact solutions are difficult to be obtained. The exact and numerical solutions of this kind of equations play an important role in physical sciences and in engineering fields. Therefore, there have been attempts to develop new techniques for obtaining analytical solutions which reasonably approximate the exact solutions. In recent years, several such techniques have drawn particular attention, such as Hirota's bilinear method [6], the homogeneous balance method [7, 8], the inverse scattering method [9], the Adomian decomposition method [10], the variational iteration method [11], the homotopy analysis method [12] and the homotopy perturbation method [13].

Perturbation techniques are widely used in science and engineering to handle nonlinear problems [14]. The HPM was first proposed by He in [15] and further developed and improved by him in [16-19]. This method is based on the use of traditional perturbation method and the homotopy technique. By using this method, a rapid convergent series solution can be obtained in most cases. Usually, a small number of terms of the series solution can be used for numerical purposes with a high degree of accuracy. The applications of the HPM in nonlinear problems have been demonstrated by many researchers, cf. [20-24]. Recently, the HPM was employed for solving singular second order differential equations [25] and nonlinear population dynamics models [26]. Very recently, the standard HPM was successfully applied to the Klein-Gordon and sine-Gordon equations [27]. The applicability of the HPM has also been extended to fractional equations [28-30]. In general, this method has been successfully applied to solve many types of linear and nonlinear problems in science and engineering by many authors [31-35]. Accordingly, it can be said that He's homotopy perturbation method is a universal one and is able to solve various kinds of nonlinear functional equations.

The higher order wave equations of KdV type model strongly nonlinear long wavelength and the short amplitude waves. It is a just reason for the strongly nonlinear character and integrability of these equations attracts many researchers to study them. In recent years, a lot of attention has been devoted to the study of higher order KdV equations. For example, Abbasbandy *et al*. in [36] applied the homotopy analysis method for the fifth-order KdV equation. Ugurlu *et al*. obtained the exact and numerical solutions of the fifth order KdV equations and couple KdV system by using to direct algebraic method [37]. Alvaro *et al*. in [38] obtained the exact solutions for a third-order KdV equation with variable coefficients and forcing term. In the present study, we employ the homotopy perturbation method to obtain the solution of higher order KdV equations.

The paper is organized as follows: In section 2, the analysis of the HPM for nonlinear differential equations is explained in detail. The HPM is presented for the third-and fourth-order KdV equations in section 3 and 4, respectively. In section 5, we briefly discuss the conclusion.

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Analysis of the method: In this section, the HPM is described for the solution of nonlinear differential equations. Toward this end, we consider

$$A(u) = g(r) \quad (1)$$

where A is a general differential operator and g(r) is a known analytic function on a Hilbert space. The operator A can be decomposed as $L'+N$, where $L'$ is linear and N is nonlinear part of A. Therefore, Eq. (1) can be written as

$$L'u + Nu = g(r) \quad (2)$$

and $L'$ can be divided into G+R, where G, as an easily invertible operator, is generally taken as the highest-order derivative in order to avoid difficult integrations when complicated Green's functions would be involved and the linear remainder, which is denoted as R. Therefore, Eq. (2) may be expressed as

$$Gu + Ru + Nu = g(r) \quad (3)$$

solving Eq. (3) for G(u), we have

$$Gu = g(r) - Ru - Nu \quad (4)$$

Operating with its inverse $G^{-1}$ yields

$$u = G^{-1}g(r) - G^{-1}Ru - G^{-1}Nu \quad (5)$$

An equivalent expression is

$$u = K + G^{-1}g(r) - G^{-1}Ru - G^{-1}Nu \quad (6)$$

where K incorporates the constants of integration and satisfies GK = 0 In order to apply the HPM, we can define a homotopy H(U,p) with properties

$$H(U,0) = F(U), \quad H(U,1) = L(U) \quad (7)$$

where

$$F(U) = U - K \quad (8)$$

and

$$L(U) = U - K - G^{-1}g(r) + G^{-1}RU + G^{-1}NU \quad (9)$$

Classically, we choose a convex homotopy by

$$H(U,p) = (1-p)F(U) + pL(U) = 0 \quad (10)$$

and continuously trace an implicitly defined curve from a starting point H(U,0) to the solution function H(U,1), where u is the solution of Eq. (1). The embedding parameter p monotonically changes from zero to unity as the trivial problem F(U) = 0 is continuously deformed to the original problem L(U) = 0. If the embedding parameter p is considered as a "small parameter", applying the classical perturbation technique, we can assume that the solution of Eq. (1) can be given by a power series in p, i. e.

$$U = U_0 + pU_1 + p^2U_2 + \cdots \quad (11)$$

and setting p = 1 results in the approximate solution of Eq. (1) as

$$u(r) = \lim_{p\to 1} U = U_0 + U_1 + U_2 + \cdots \quad (12)$$

HPM for the third-order nonlinear KdV equation: In this section, we consider the third-order nonlinear KdV equation

$$u_t + u^m u_x + u_{xxx} = 0$$
$$u(x,0) = \left[A\operatorname{sech}^2(Kx - x_0)\right]^{\frac{1}{m}} \quad (13)$$

with the initial condition

$$u(x,t) = \left[A\operatorname{sech}^2(Kx - ct)\right]^{\frac{1}{m}}$$

where m, K, d and $x_0$ are constants,

$$A = \frac{2(m+1)(m+2)}{d^2}K^2$$

$$c = \frac{4K^2}{m^2}$$

and the subscripts in t and x denote partial derivatives with respect to these independent variables [39] and apply the HPM to solve it. To this end, we rewrite Eq. (13) in the form

$$G_t u + Nu + G_{3x} u = 0 \quad (14)$$

with m = 4 where the notation $Nu = u^4u_x$ symbolizes the nonlinear terms, the notations

$$G_t = \frac{\partial}{\partial t}, G_{3x} = \frac{\partial^3}{\partial x^3}$$

symbolizes the linear differential operators. We assume that the inverse of the operator $G_t^{-1}$ exists and it can conveniently be taken as the definite integral with respect to t from 0 to t. Thus, applying the inverse operator $G_t^{-1}$ to both equations (14) yields

$$G_t^{-1}G_t u = -G_t^{-1}(Nu + G_{3x}u) \quad (15)$$

Applying the initial condition

$$u(x,0) = f(x) = \left[A\operatorname{sech}^2(Kx)\right]^{\frac{1}{4}}$$

we obtain

$$u(x,t) = f(x) - G_t^{-1}(Nu + G_{3x}u) \quad (16)$$

By the homotopy technique, we construct a homotopy H(U, p) which satisfies

$$H(U,p) = (1-p)F(U) + pL(U) = 0 \quad (17)$$

where $p \in [0,1]$ is an embedding parameter. The embedding parameter p monotonically changes from zero to unity as the trivial problem F(U) = 0 is continuously deformed to the original problem L(U) = 0. If the embedding parameter p is considered as a "small parameter", applying the classical perturbation technique, we can assume that the solution of Eq. (13) can be given by a power series in p, i. e.

$$U = U_0 + pU_1 + p^2U_2 + \ldots \quad (18)$$

Clearly, from Eq. (17) we have

$$H(U,0) = F(U), \quad H(U,1) = L(U) \quad (19)$$

where

$$F(U) = U - f(x)$$
$$L(U) = U - f(x) + G_t^{-1}(NU + G_{3x}U) \quad (20)$$

By substituting Eq. (20) into Eq. (17), we obtain

$$U = f(x) - pG_t^{-1}(NU + G_{3x}U) \quad (21)$$

By replacing Eq. (18) into Eq. (21) and equating the terms with identical powers of p, we get

$$p^0 : U_0 = f(x) = A^{\frac{1}{4}}\sqrt{\operatorname{sech}[Kx]} \quad (22)$$

$$p^1 : U_1 = \frac{1}{16}A^{\frac{1}{4}}Kt(8A - 29K^2 + K^2\cosh[2Kx])$$
$$\operatorname{sech}[Kx]^{\frac{7}{2}}\sinh[Kx] \quad (23)$$

Table 1: Absolute errors of solution obtained by the HPM

| t/x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.1 | 1.11E-16 | 1.11E-16 | 1.11E-16 | 5.55E-16 | 2.55E-16 |
| 0.2 | 1.11E-16 | 1.11E-16 | 2.22E-16 | 4.44E-16 | 1.44E-15 |
| 0.3 | 0.00E-00 | 0.00E-00 | 2.22E-16 | 7.77E-16 | 2.10E-15 |
| 0.4 | 1.11E-16 | 0.00E-00 | 0.00E-00 | 2.22E-16 | 8.88E-16 |
| 0.5 | 1.11E-16 | 0.00E-00 | 0.00E-00 | 0.00E-00 | 1.11E-16 |

$$p^2 : U_2 = \frac{1}{4096}A^{\frac{1}{4}}K^2t^2$$
$$(-3328A^2 + 59488AK^2 - 179306K^4$$
$$+(2304A^2 - 48256AK^2 + 148527K^4)$$
$$\cosh[2Kx] + (2080AK^2 - 7806K^4)$$
$$\cosh[4Kx] + K^4\cosh[6Kx])$$
$$\operatorname{sech}[Kx]^{\frac{13}{2}} \quad (24)$$

in a similar manner, the components $U_n$ are calculated for n = 3,4,... but for simplicity they will not be listed here. Finally, by calculating Maclaurin series seven terms of the series solution, the approximate solution of Eq. (13) is calculated as:

$$u \cong \varphi_7 = \sum_{n=0}^{6} U_n$$
$$= (0.762199 - 8.68192\times10^{-6}t^2 + 1.15375\times10^{-10}t^4\cdots)$$
$$+(0.000771727t - 2.0511\times10^{-8}t^3 + \cdots)x$$
$$+(-0.0171495 + 1.3674\times10^{-6}t^2$$
$$-5.15478\times10^{-11}t^4 + \cdots)x^2 + (-0.0000405156t$$
$$+3.05468\times10^{-9}t^3 + \cdots)x^3 \quad (25)$$

In order to verify the efficiency of the HPM for Eq. (13), we report the absolute errors of the solution obtained by this method for $t \in [0.1, 0.5]$ relative to the exact solution with c = 0.00675, K = 0.3 and A = 0.3375 in Table 1.

Results reveal that the obtained solutions by the HPM for the thired-order nonlinear KdV equation are exactly the same as those obtained by the Adomian decomposition method [39]. In order to have a visual comparison, the solution approximant by the HPM to Eq. (13) for various m are plotted in the same system of coordinates shown in the Fig. 1.

HPM for the fourth-order nonlinear KdV equation: In this section, we consider the fourth-order nonlinear KdV equation

$$u_t + (m+1)u^m u_x + u_{xxxx} = 0, u(x,0) = A\left[\operatorname{sech}^2(Kx)\right]^{\frac{1}{m}} \quad (26)$$

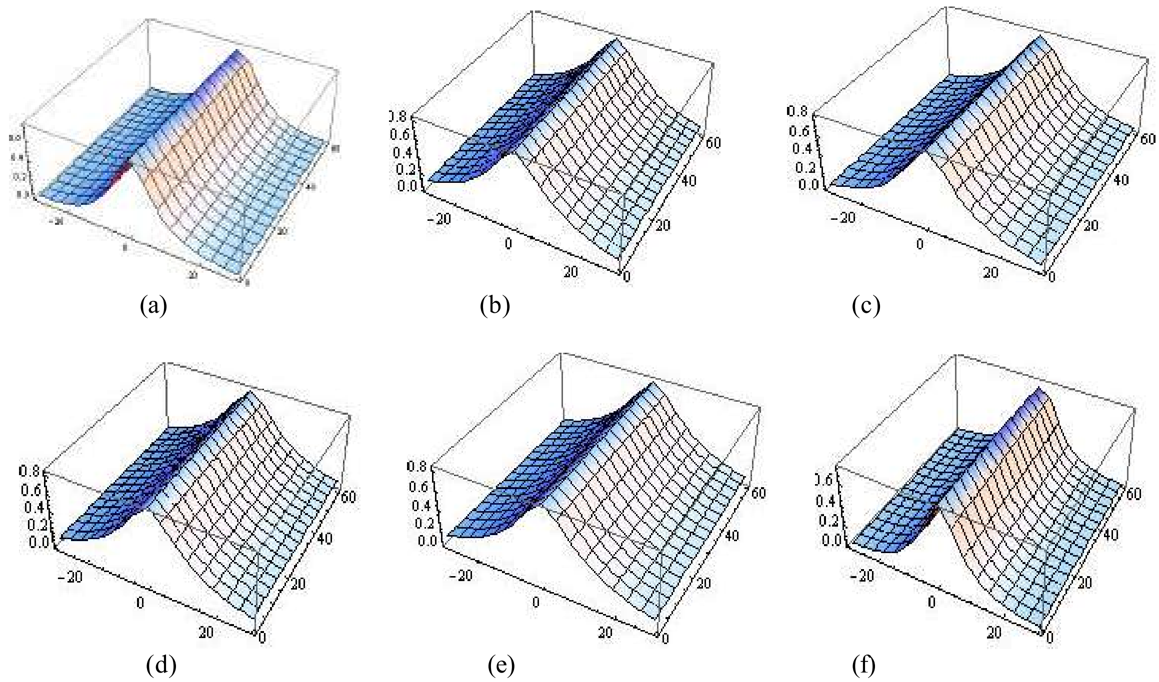


Fig. 1: The numerical results for $\varphi_l(x,t)$: (a) for p = 4 with c = 0.00675, K = 0.3 and A = 0.3375, (c) for p = 6 with c = 0.003, K = 0.3 and A = 0.28 and (e) for p = 8 with c = 0.0016875, K = 0.3 and A = 0.253125 in comparison with the analytical solutions u(x,t): (b) for p = 4, (d) for p = 6 and (f) for p = 8, for the solitary wave solutions with the initial condition of Eq. (13).

with the initial condition

$$u(x,t) = A\left[\text{sech}^2(Kx - ct)\right]^{\frac{1}{m}}$$

where m, K, c and A are constants and the subscripts in t and x denote partial derivatives with respect to these independent variables [39] and apply the HPM to solve it. To this end, we rewrite Eq. (26) in the form

$$G_t u + (m+1)Nu + G_{4x} u = 0 \tag{27}$$

with m = 4 where the notation $Nu = u^4 u_x$ symbolizes the nonlinear terms, the notations

$$G_t = \frac{\partial}{\partial t}, G_{4x} = \frac{\partial^4}{\partial x^4}$$

symbolizes the linear differential operators. We assume that the inverse of the operator $G_t^{-1}$ exists and it can conveniently be taken as the definite integral with respect to t from 0 to t. Thus, applying the inverse operator $G_t^{-1}$ to both equations (27) yields

$$G_t^{-1} G_t u = -5G_t^{-1}(Nu + G_{4x} u) \tag{28}$$

Applying the initial condition

$$u(x,0) = f(x) = A\left[\text{sech}^2(Kx)\right]^{\frac{1}{4}}$$

we obtain

$$u(x,t) = f(x) - 5G_t^{-1}(Nu + G_{4x} u) \tag{29}$$

By the homotopy technique, we construct a homotopy H(U,p) which satisfies

$$H(U,p) = (1-p)F(U) + pL(U) = 0 \tag{30}$$

where $p \in [0,1]$ is an embedding parameter. The embedding parameter p monotonically changes from zero to unity as the trivial problem F(U) = 0 is continuously deformed to the original problem L(U) = 0. If the embedding parameter p is considered as a "small parameter", applying the classical perturbation technique, we can assume that the solution of Eq. (26) can be given by a power series in p, i. e.

$$U = U_0 + pU_1 + p^2 U_2 + \dots \tag{31}$$

Clearly, from Eq. (30) we have

$$H(U,0) = F(U), \quad H(U,1) = L(U) \tag{32}$$

where

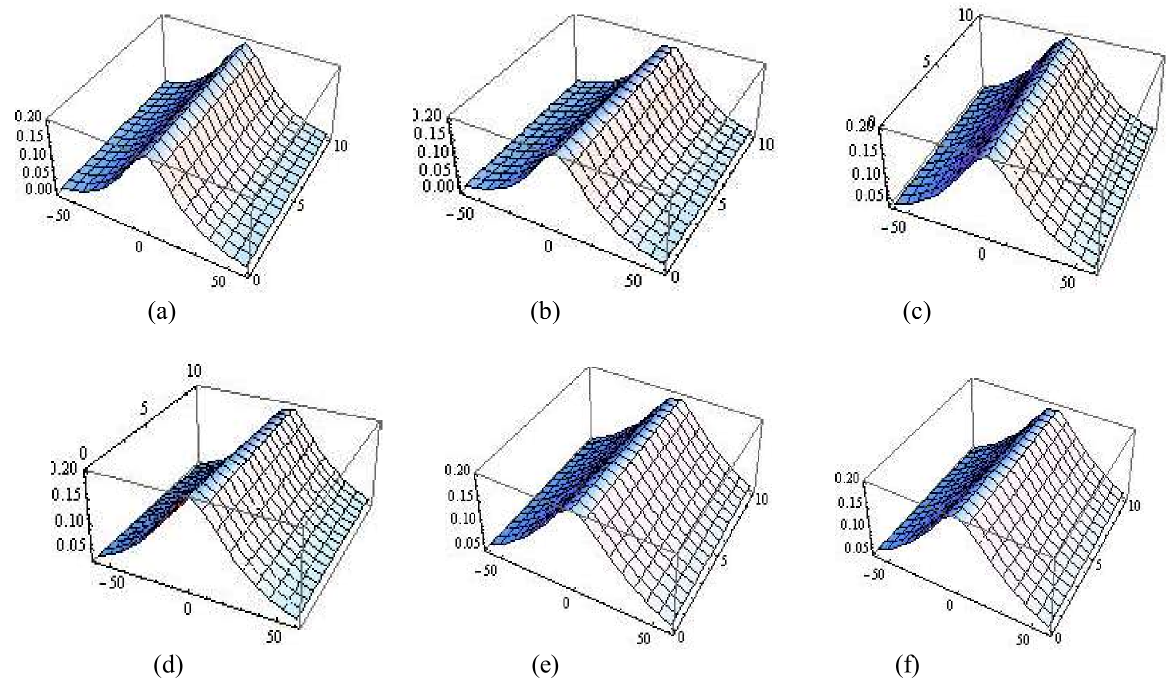


Fig. 2: The numerical results for $\varphi_l(x,t)$: (a) for p = 4, (c) for p = 6 and (e) for p = 8, in comparison with the analytical solutions u(x,t): (b) for p = 4, (d) for p = 6 and (f) for p = 8, for the solitary wave solutions with the initial condition of Eq. (26) with c = 0.05, K = 0.1 and A = 0.2

$$F(U) = U - f(x)$$
$$L(U) = U - f(x) + 5G_t^{-1}(NU + G_{4x}U) \quad (33)$$

By substituting Eq. (33) into Eq. (30), we obtain

$$U = f(x) - 5pG_t^{-1}(NU + G_{4x}U) \quad (34)$$

By substituting Eq. (31) into Eq. (34) and equating the terms with identical powers of p, we get

$$p^0 : U_0 = f(x) = A\sqrt{\mathrm{sech}[Kx]} \quad (35)$$

$$p^1 : U_1 = -\frac{1}{128} AKt\,\mathrm{sech}[Kx]^{\frac{9}{2}}$$
$$(531K^3 - 308K^3\cosh[2Kx] + K^3\cosh[4Kx] - 160A^4\sinh[2Kx]) \quad (36)$$

$$p^2 : U_2 = \frac{1}{65536} AK^2t^2\,\mathrm{sech}[Kx]^{\frac{17}{2}}$$
$$(-435200A^8 + 120851555K^6 - 8(25600A^8 + 15491243K^6)\cosh[2Kx] + 4(57600A^8 + 3620599K^6)\cosh[4Kx] - 195304K^6\cosh[6Kx] + K^6\cosh[8Kx] - 17615040A^4K^3\sinh[2Kx] + 4320000A^4K^3\sinh[4Kx] - 100800A^4K^3\sinh[6Kx]) \quad (37)$$

Table 2: Absolute errors of solution obtained by the HPM

| t/x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.1 | 5.55E-6 | 1.10E-5 | 1.65E-5 | 2.20E-5 | 2.74E-5 |
| 0.2 | 5.78E-6 | 1.15E-5 | 1.72E-5 | 2.29E-5 | 2.86E-5 |
| 0.3 | 6.01E-6 | 1.20E-5 | 1.79E-5 | 2.38E-5 | 2.97E-5 |
| 0.4 | 6.23E-6 | 1.24E-5 | 1.86E-5 | 2.47E-5 | 2.74E-5 |
| 0.5 | 6.45E-6 | 1.28E-5 | 1.92E-5 | 2.56E-5 | 2.20E-5 |

in a similar manner, the components $U_n$ are calculated for n = 3,4,… but for simplicity they will not be listed here. Finally, by calculating Maclaurin series seven terms of the series solution, the approximate solution of Eq. (26) is calculated as

$$u \cong \varphi_7 = \sum_{n=0}^{6} U_n$$
$$= (0.2 - 0.000035t + 3.10062\times10^{-7}t^2 - 1.74817\times10^{-8}t^3 + \cdots) + (8\times10^{-6}t - 9.06\times10^{-8}t^2 + 4.90812\times10^{-9}t^3 + \cdots)x + (-0.0005 + 1.7375\times10^{-6}t - 5.34139\times10^{-8}t^2 + 6.06856\times10^{-9}t^3 + \cdots)x^2 + (-1.26667\times10^{-7}t + 4.7765\times10^{-9}t^2 - 5.50174\times10^{-10}t^3 + \cdots)x^3 + \cdots \quad (38)$$

In order to verify the efficiency of the HPM for Eq. (26), we report the absolute errors of the solution obtained by this method for $t \in [0,1, 0.5]$ relative to the exact solution with $c = 0.05$, $K = 0.1$ and $A = 0.2$ in Table 2.

Results reveal that the obtained solutions by the HPM for the fourth-order nonlinear KdV equation is exactly the same as with those obtained by the Adomian decomposition method [39]. In order to have a visual comparison, the solution approximant by the HPM to Eq. (26) for various m are plotted in the same system of coordinates shown in the Fig. 2.

## CONCLUSION

The main goal of this paper has been to drive an analytical solution for the third-and fourth-order KdV equations. We have achieved this goal by applying He's homotopy perturbation method. Results are compared with those in open literature [39], revealing that the obtained solutions are exactly similar to those obtained by the Adomian decomposition method [39]. However, applying the HPM overcomes the difficulties arising in calculation of Adomian's polynomials. A clear conclusion can be draw from the numerical results that the HPM algorithm provides highly accurate numerical solutions for higher order KdV equations.

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