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Some New Exact Traveling Wave Solutions to the (3+1)-dimensional Kadomtsev-Petviashvili equation

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Abstract: Mathematical modeling of numerous physical phenomena often leads to high-dimensional partial differential equations and thus the higher dimensional nonlinear evolution equations come into further attractive in many branches of physical sciences. In this article, we propose a new technique of the (G′/G)-expansion method combine with the Riccati equation for searching new exact traveling wave solutions of the (3+1)-dimensional Kadomtsev-Petviashvili (KP) equation. Consequently, some new solutions of the KP are successfully obtained in a unified way involving arbitrary parameters. When the parameters take special values, solitary waves are derived from the traveling waves. The obtained solutions are expressed by the hyperbolic, trigonometric and rational functions. The method can be applied to many other nonlinear partial differential equations.

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Key words: The (G′/G)-expansion method . the Riccati equation . the traveling wave solutions . the (3+1)-dimensional Kadomtsev-Petviashvili equation

INTRODUCTION

In 1834, John Scott Russell [1] first observed the solitary waves. The significant observation motivated him to conduct experiments to underline his observance and to study these solitary waves. In 1965, Zabusky and Kruskal [2] studied the interactions between the solitary waves and the reappearance of initial states and since the KdV equation was solved by Gardner *et al*. [3] by the inverse scattering method, finding the solitary wave solutions of Nonlinear Evolution Equations (NLEEs) has turned out to be one of the enthusiastic and greatly lucrative areas of research. The appearance of solitary wave in nature is rather frequent, especially in fluids, plasmas, solid state physics, condensed matter physics, optical fibers, chemical kinematics, electrical circuits, bio-genetics, elastic media etc. Therefore, the researchers conducted a huge amount of research work to investigate the exact traveling wave solutions of the phenomena. Consequently, they established many methods and techniques, such as, the Backlund transformation method [4], the Hirota's bilinear transformation method [5], the variational iteration method [6], the Adomian decomposition method [7], the tanh-function method [8], the homogeneous balance method [9], the F-expansion method [10], the Jacobi elliptic function method [11], the variable separation method [12], the Lie group symmetry method [13], the homotopy analysis method [14, 15], the homotopy perturbation method [16], the first integration method [17], the Exp-function method [18-21], the (G′/G)-expansion method [22-31] and so on.

It is significant to observe that there exist some fundamental relationships among numerous complex nonlinear partial differential equations and some basic and soluble nonlinear Ordinary Differential Equations (ODEs), such as the sine-Gordon equation, the sinh-Gordon equation, the Riccati equation, the Weierstrass elliptic equation etc. Therefore, it is natural to use the solutions of these nonlinear ODEs to construct exact solutions of various intricate nonlinear partial differential equations. Based on the relationships of complex nonlinear partial differential equations and ODEs, a number of methods, such as, the Riccati equation expansion method [32, 33], the projective Riccati equation method [34, 35], the algebraic method [36], the sinh-Gordon equation expansion method [37], the generalized F-expansion method [38, 39] etc. have been developed.

In the present article, we make use of the Riccati equation with the (G′/G)-expansion method for obtaining some new exact traveling wave solutions to

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the (3+1)-dimensional Kadomtsev-Petviashvili (KP). The Riccati equation has not been used by anybody before to solve the KP equation by (G′/G)-expansion method.

DESCRIPTION OF THE (G′/G)-EXPANSION METHOD WITH THE RICCATI EQUATION

Suppose the general nonlinear partial differential equation

$$\Phi(u, u_t, u_x, u_y, u_z, u_{tt}, u_{xx}, \cdots) = 0 \qquad (1)$$

where u = u (x,y,z,t) is an unknown function, Φ is a polynomial in u (x,y,z,t) and its partial derivatives in which the highest order partial derivatives and the nonlinear terms are involved. The main steps of the (G′/G)-expansion method combined with the Riccati equation is as follows:

Step 1: The travelling wave variable ansatz

$$u(x,y,z,t) = v(\eta), \eta = x + y + z - Vt \qquad (2)$$

where V is the speed of the traveling wave, allows us to convert the Eq. (1) into an ODE:

$$\psi(v, v', v'', \cdots) = 0 \qquad (3)$$

where the superscripts stands for the ordinary derivatives with respect to η.

Step 2: If Eq. (3) is integrable, integrate term by term one or more times, yields constant(s) of integration.

Step 3: Suppose the traveling wave solution of Eq. (3) can be expressed by a polynomial in (G′/G) as follows:

$$v(\eta) = \sum_{n=0}^{m} \alpha_n \left(\frac{G'}{G}\right)^n, \alpha_m \neq 0 \qquad (4)$$

where $G = G(\eta)$ satisfies the Riccati equation,

$$G' = h_1 + h_2 G^2, h_2 \neq 0 \qquad (5)$$

where α_n = (n = 0,1,2,…,m), h_1 and h_2 are arbitrary constants to be determined later.

The Riccati Eq. (5) plays important role in manipulating nonlinear equations to get exact solutions by the (G′/G)-expansion method. It has the following twenty one exact solutions [40].

Family 1: When h_1 and h_2 have same sign and $h_1 h_2 \neq 0$, the solutions of Eq. (5) are:

$$G_1 = \frac{1}{h_2}\left[\sqrt{h_1 h_2}\tan\left(\sqrt{h_1 h_2}\,\eta\right)\right]$$

$$G_2 = -\frac{1}{h_2}\left[\sqrt{h_1 h_2}\cot\left(\sqrt{h_1 h_2}\,\eta\right)\right]$$

$$G_3 = \frac{1}{h_2}\left[\sqrt{h_1 h_2}\left(\tan\left(2\sqrt{h_1 h_2}\,\eta\right) \pm \sec\left(2\sqrt{h_1 h_2}\,\eta\right)\right)\right]$$

$$G_4 = -\frac{1}{h_2}\left[\sqrt{h_1 h_2}\left(\cot\left(2\sqrt{h_1 h_2}\,\eta\right) \pm \csc\left(2\sqrt{h_1 h_2}\,\eta\right)\right)\right]$$

$$G_5 = \frac{1}{2h_2}\left[\sqrt{h_1 h_2}\left(\tan\left(\frac{1}{2}\sqrt{h_1 h_2}\,\eta\right) - \cot\left(\frac{1}{2}\sqrt{h_1 h_2}\,\eta\right)\right)\right]$$

$$G_6 = \frac{\sqrt{h_1 h_2}}{h_2}\left[\frac{\sqrt{(M^2 - N^2)} - M\cos\left(2\sqrt{h_1 h_2}\,\eta\right)}{M\sin\left(2\sqrt{h_1 h_2}\,\eta\right) + N}\right]$$

$$G_7 = \frac{\sqrt{h_1 h_2}}{h_2}\left[\frac{\sqrt{(M^2 - N^2)} + M\sin\left(2\sqrt{h_1 h_2}\,\eta\right)}{M\cos\left(2\sqrt{h_1 h_2}\,\eta\right) + N}\right]$$

where M and N are two non-zero real constants and satisfies the condition M^2-N^2>0.

$$G_8 = \frac{-h_1\cos\left(2\sqrt{h_1 h_2}\,\eta\right)}{\sqrt{h_1 h_2}\sin\left(2\sqrt{h_1 h_2}\,\eta\right) \pm \sqrt{h_1 h_2}}$$

$$G_9 = \frac{h_1\sin\left(2\sqrt{h_1 h_2}\,\eta\right)}{\sqrt{h_1 h_2}\cos\left(2\sqrt{h_1 h_2}\,\eta\right) \pm \sqrt{h_1 h_2}}$$

$$G_{10} = \frac{2h_1\sin\left(\frac{1}{2}\sqrt{h_1 h_2}\,\eta\right)\cos\left(\frac{1}{2}\sqrt{h_1 h_2}\,\eta\right)}{2\sqrt{h_1 h_2}\cos^2\left(\frac{1}{2}\sqrt{h_1 h_2}\,\eta\right) - \sqrt{h_1 h_2}}$$

Family 2: When h_1 and h_2 possess opposite sign and $h_1 h_2 \neq 0$, the solutions of Eq. (5) are:

$$G_{11} = -\frac{1}{h_2}\left[\sqrt{-h_1 h_2}\tanh\left(\sqrt{-h_1 h_2}\,\eta\right)\right]$$

$$G_{12} = -\frac{1}{h_2}\left[\sqrt{-h_1 h_2}\coth\left(\sqrt{-h_1 h_2}\,\eta\right)\right]$$

$$G_{13}=-\frac{1}{h_2}\left[\sqrt{-h_1h_2}\left(\tanh\left(2\sqrt{-h_1h_2}\eta\right)\pm i\,\mathrm{sech}\left(2\sqrt{-h_1h_2}\eta\right)\right)\right]$$

$$G_{14}=-\frac{1}{h_2}\left[\sqrt{-h_1h_2}\left(\coth\left(2\sqrt{-h_1h_2}\eta\right)\pm \mathrm{csch}\left(2\sqrt{-h_1h_2}\eta\right)\right)\right]$$

$$G_{15}=-\frac{1}{2h_2}\left[\sqrt{-h_1h_2}\left(\tanh\left(\frac{1}{2}\sqrt{-h_1h_2}\eta\right)+\coth\left(\frac{1}{2}\sqrt{-h_1h_2}\eta\right)\right)\right]$$

$$G_{16}=\frac{\sqrt{-h_1h_2}}{h_2}\left[\frac{\sqrt{(M^2+N^2)}-M\cosh\left(2\sqrt{-h_1h_2}\eta\right)}{M\sinh\left(2\sqrt{-h_1h_2}\eta\right)+N}\right]$$

$$G_{17}=-\frac{\sqrt{-h_1h_2}}{h_2}\left[\frac{\sqrt{N^2-M^2}+M\sinh\left(2\sqrt{-h_1h_2}\eta\right)}{M\cosh\left(2\sqrt{-h_1h_2}\eta\right)+N}\right]$$

where M and N are two non-zero real constants and satisfies the condition N^2-M^2>0.

$$G_{18}=\frac{h_1\cosh\left(2\sqrt{-h_1h_2}\eta\right)}{\sqrt{-h_1h_2}\sinh\left(2\sqrt{-h_1h_2}\eta\right)\pm i\sqrt{-h_1h_2}}$$

$$G_{19}=\frac{h_1\sinh\left(2\sqrt{-h_1h_2}\eta\right)}{\sqrt{-h_1h_2}\cosh\left(2\sqrt{-h_1h_2}\eta\right)\pm\sqrt{-h_1h_2}}$$

$$G_{20}=\frac{2h_1\sinh\left(\frac{1}{2}\sqrt{-h_1h_2}\eta\right)\cosh\left(\frac{1}{2}\sqrt{-h_1h_2}\eta\right)}{2\sqrt{-h_1h_2}\cosh^2\left(\frac{1}{2}\sqrt{-h_1h_2}\eta\right)-\sqrt{-h_1h_2}}$$

Family 3: When $h_2\neq0$ but $h_1=0$, the solution of Eq. (5) is:

$$G_{21}=-\frac{1}{h_2\eta+d}$$

where d is an arbitrary constant.

The above solutions help to generate various traveling wave solutions, including solitary, periodic and rational solutions, in elementary functions.

Step 4: To determine the positive integer m, put Eq. (4) along with Eq. (5) into Eq. (3) and consider the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (3).

Step 5: Substituting Eq. (4) together with Eq. (5) into Eq. (3) along with the value of m obtained in step 4, we obtain polynomials in G^i and G^{-i} $(i = 0,1,2,3\ldots)$. Setting each coefficient of the resulted polynomial to zero, yields a set of algebraic equations for α_n, h_1, h_2 and V.

Step 6: Suppose the value of the constants α_n h_1, h_2 and V can be obtained by solving the set of algebraic equations obtained in step 5. Since the general solutions of Eq. (5) are known (arranged in step 3), substituting α_n h_1, h_2 and V into Eq. (4), we obtain new exact traveling wave solutions of the nonlinear evolution Eq. (1).

APPLICATION OF THE METHOD

In this section, we apply the proposed approach of the (G′/G)-expansion method to construct new exact traveling wave solutions to the Kadomtsev-Petviashvili (KP) equation which is an important nonlinear equation in mathematical physics.

Let us consider the (3+1)-dimensional KP equation,

$$(u_t+6uu_x+u_{xxx})_x+3u_{yy}+3u_{zz}=0 \quad (6)$$

We investigate solutions the KP equation by the method described in section 2. Utilizing the traveling wave variable ansatz organized in Eq. (2), we obtain

$$(-Vv'+6vv'+v''')'+6v''=0 \quad (7)$$

Eq. (7) is integrable, therefore, integrating twice, we obtain

$$(6-V)v+3v^2+v''+C=0 \quad (8)$$

where C is a constant of integration.

According to step 3, the solution of Eq. (8) can be expressed by a polynomial in (G'/G) as follows:

$$v(\eta) = \alpha_0 + \alpha_1(G'/G) + \alpha_2(G'/G)^2 + \cdots + \alpha_m(G'/G)^m, \alpha_m \neq 0 \quad (9)$$

where α_n, $(n = 0,1,2,\ldots,m)$ are constants to be determined and $G = G(\eta)$ satisfies the Riccati Eq. (5). Considering the homogeneous balance between the highest order derivative v'' and the nonlinear term v^2 we obtain $m = 2$.

Therefore, solution Eq. (9) become

$$v(\eta) = \alpha_0 + \alpha_1(G'/G) + \alpha_2(G'/G)^2, \alpha_2 \neq 0 \quad (10)$$

By means of Eq. (5), Eq. (10) can be rewritten as,

$$v(\eta) = \alpha_0 + \alpha_1(h_1 G^{-1} + h_2 G) + \alpha_2(h_1 G^{-1} + h_2 G)^2 \quad (11)$$

Substituting Eq. (11) into Eq. (8), the left hand side of the equation is converted into polynomials in G^i and G^{-i}, $(i = 0,1,2,\ldots)$. Setting each coefficient of these polynomials to zero, we obtain an over-determined set of algebraic equations (we will omit to display them for simplicity) for α_0, α_1, α_2, h_1, h_2, V and C.

Solving the over-determined set of algebraic equations by using the symbolic computation software, such as Maple, we obtain

$$\alpha_2 = 2, \alpha_1 = 0, \alpha_0 = \alpha_0$$

$$V = 6 - 16 h_1 h_2 - 6\alpha_0 \text{ and } C = 3\alpha_0^2 + 16 h_1 h_2 \quad (12)$$

where α_0, h_1 and h_2 are arbitrary constants.

Now on the basis of the solutions of the Riccati Eq. (5), we obtain the following cluster of traveling wave solutions of Eq. (6).

Cluster 1: When h_1 and h_2 have same sign and $h_1 h_2 \neq 0$, the periodic form solutions of Eq. (6) are,

$$u_1 = \alpha_0 + 8 h_1 h_2 \csc^2(2\sqrt{h_1 h_2}\,\eta)$$

where $\eta = x+y+z-(6-16h_1h_2-6\alpha_0)t$ and α_0, h_1, h_2 are arbitrary constants.

$$u_3 = \alpha_0 + 8 h_1 h_2 \sec^2(2\sqrt{h_1 h_2}\,\eta$$

$$u_6 = \alpha_0 + 2\left(\frac{2\sqrt{h_1 h_2}\, M\left\{M + N\sin(2\sqrt{h_1 h_2}\,\eta) - \sqrt{M^2 - N^2}\cos(2\sqrt{h_1 h_2}\,\eta)\right\}}{\left\{M\sin(2\sqrt{h_1 h_2}\,\eta) + N\right\}\left\{M\cos(2\sqrt{h_1 h_2}\,\eta) - \sqrt{M^2 - N^2}\right\}}\right)^2$$

$$u_7 = \alpha_0 + 2\left(\frac{2\sqrt{h_1 h_2}\, M\left\{M + N\cos(2\sqrt{h_1 h_2}\,\eta) + \sqrt{M^2 - N^2}\sin(2\sqrt{h_1 h_2}\,\eta)\right\}}{\left\{M\cos(2\sqrt{h_1 h_2}\,\eta) + N\right\}\left\{M\sin(2\sqrt{h_1 h_2}\,\eta) + \sqrt{M^2 - N^2}\right\}}\right)^2$$

where M and N are two non-zero real constants satisfies the condition $M^2 - N^2 > 0$.

$$u_{10} = \alpha_0 + 2\left(\frac{\sqrt{h_1 h_2}}{2\sin((\sqrt{h_1 h_2}\,\eta)/2)\cos((\sqrt{h_1 h_2}\,\eta)/2)\left\{2\cos^2((\sqrt{h_1 h_2}\,\eta)/2) - 1\right\}}\right)^2$$

The solutions corresponding to G_2, G_4, G_5 and G_9 are identical to the solution u_1 and the solution corresponding to G_8 is identical to the solution u_3.

Cluster 2: When h_1 and h_2 possess opposite sign and $h_1 h_2 \neq 0$, the soliton and soliton-like solutions of Eqs. (6) are,

$$u_{11} = \alpha_0 - 8 h_1 h_2 \operatorname{csch}^2(2\sqrt{-h_1 h_2}\,\eta) \text{ where } \eta = x + y + z - (6 - 16 h_1 h_2 - 6\alpha_0)t$$

and α_0, h_1, h_2 are arbitrary constants.

$$u_{13} = \alpha_0 + 8\,h_1 h_2\,\mathrm{sech}^2(2\sqrt{-h_1 h_2}\,\eta)$$

$$u_{16} = \alpha_0 + 2\left(\frac{2\sqrt{-h_1h_2}\,M\left\{M - N\sinh(2\sqrt{-h_1h_2}\,\eta) - \sqrt{M^2+N^2}\cosh(2\sqrt{-h_1h_2}\,\eta)\right\}}{\left\{M\sinh(2\sqrt{-h_1h_2}\,\eta) + N\right\}\left\{M\cosh(2\sqrt{-h_1h_2}\,\eta) - \sqrt{M^2+N^2}\right\}}\right)^2$$

$$u_{17} = \alpha_0 + 2\left(\frac{2\sqrt{-h_1h_2}\,M\left\{M + N\cosh(2\sqrt{-h_1h_2}\,\eta) - \sqrt{N^2-M^2}\sinh(2\sqrt{-h_1h_2}\,\eta)\right\}}{\left\{M\cosh(2\sqrt{-h_1h_2}\,\eta) + N\right\}\left\{M\sinh(2\sqrt{-h_1h_2}\,\eta) + \sqrt{N^2-M^2}\right\}}\right)^2$$

where M and N are two non-zero real constants and satisfies the condition M^2-N^2>0.

$$u_{20} = \alpha_0 + 2\left(\frac{\sqrt{-h_1h_2}}{2\sinh((\sqrt{-h_1h_2}\,\eta)/2)\cosh((\sqrt{-h_1h_2}\,\eta)/2)\left\{2\cosh^2((\sqrt{-h_1h_2}\,\eta)/2) - 1\right\}}\right)^2$$

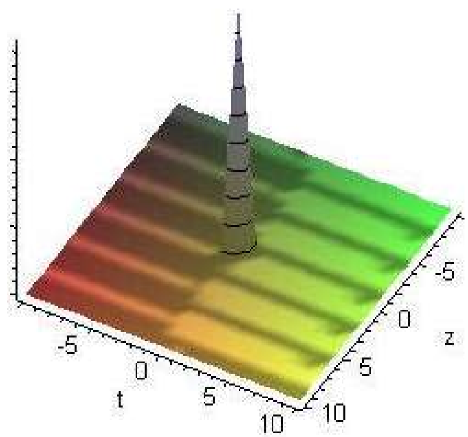

Fig. 1: Periodic solution corresponding to u_1 for $\alpha_0 = 1$, $h_1 = 2$, $h_2 = 1$

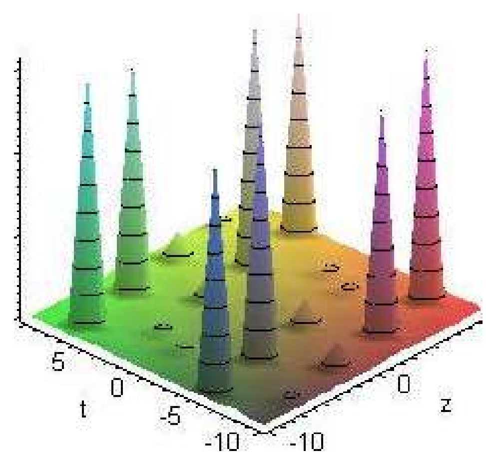

Fig. 2: Periodic solution corresponding to u_3 for $\alpha_0 = 3$, $h_1 = 2$, $h_2 = 2$

The solutions corresponding to G_{12}, G_{14}, G_{15} and G_9 are identical to the solution u_{11} and the solution corresponding to G_8 is identical to the solution u_{13}.

Cluster 3: When h_1=0 but h_2≠0, the solution of Eq. (6) is,

$$u_{21} = \alpha_0 + 2\left(\frac{h_2}{h_2\,\eta + d}\right)^2$$

where d is an arbitrary constant.

Because of the arbitrary constants α_0, h_1, h_2 and V, in the above obtained solutions, the physical quantity u might possess physically significant rich structures.

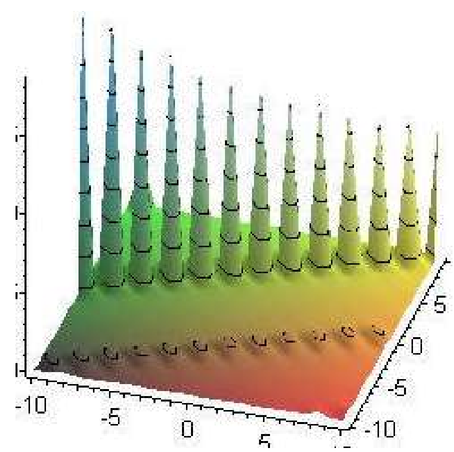

Fig. 3: Periodic solution corresponding to u_6 for $\alpha_0 = 5$, $h_1 = 5$, r = 5, M = 2 and N = 1

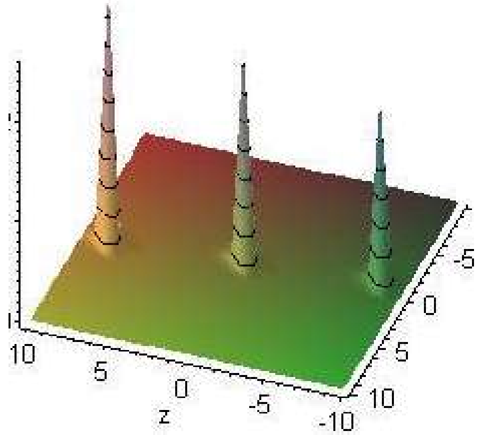

Fig. 4: Soliton solution corresponding to u_{11} for $\alpha_0 = 10$, $h_1 = -2$, $h_2 = 2$

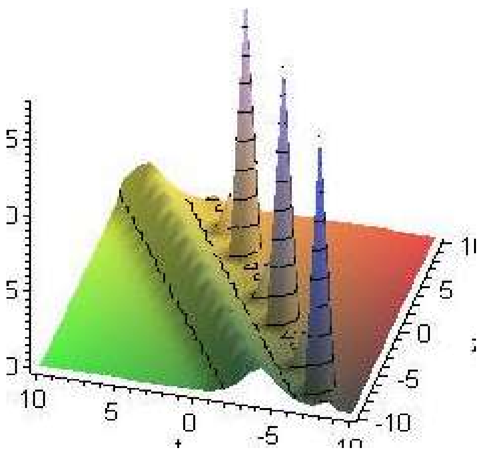

Fig. 5: Soliton solution corresponding to u_{17} for $\alpha_0 = 1$, $h_1 = 0.1$, $h_2 = -1$, M = 1 and N = 5

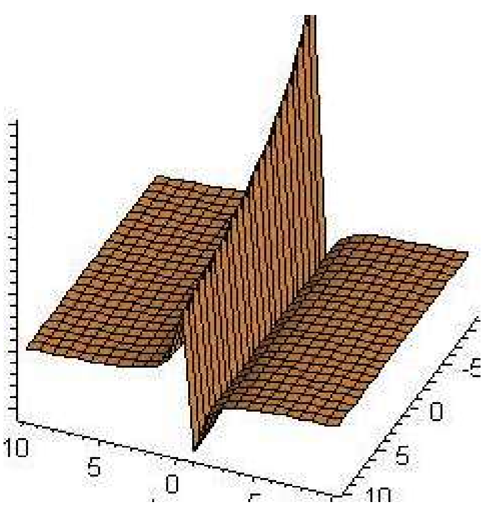

Fig. 6: Soliton solution corresponding to u_{21} for $\alpha_0 = 1$, $h_2 = 5$ and d = 100

GRAPHICAL REPRESENTATIONS

Graph is an influential tool for communication and it illustrates clearly the solutions of the problems. We consider the evolutions of the soliton, periodic and rational-like solutions u_1, u_3, u_6, u_{11}, u_{17} and u_{21} along $x = 0$ and $y = 0$. The graphs readily have shown the periodic and solitary wave forms of the solutions.

CONCLUSION

The (G′/G)-expansion method is an advance mathematical tool for investigating exact solutions of nonlinear partial differential equations associated with complex physical phenomena wherein, in general the second order linear ordinary differential equation is employed as an auxiliary equation. But, in this article, we utilize the Riccati equation as an auxiliary equation; as a result, some new explicit solutions of the Kadomtsev-Petviashvili equation are obtained in a unified way. The obtained exact solutions might be important and significant in the field of water waves of long wavelength with weakly nonlinear restoring forces and frequency dispersion. The algorithm presented in this article is effective and more powerful than the original (G′/G)-expansion method and it can be applied for other kind of nonlinear evolution equations in mathematical physics.

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