Non-stationary Flow of Viscoelastic Liquid Through a Gap Between Two Plates

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Abstract: Studied is a flow between two infinite planes, one of which performs periodic oscillations. Non-Newtonian fluid is described by the Voigt-Oldroyd model applicable for the motion of an emulsion of one liquid in another. Such liquid possesses relaxation properties, which give rise to a relaxation wave. The generated results may be used to determine viscosity and the relaxation constant of the liquid on the basis of physical measurements.

Key words: Viscometry · Hemodynamics · Rheology

INTRODUCTION

Refs [1, 2] report results of observation of pulsed blood motion through a capillary vessel. The existence of a wave process in capillaries where viscous forces exceed by six orders of magnitude the inertial ones requires a novel approach for a satisfactory explanation.

In attempts to find such explanation of these anomalous effects, non-Newtonian properties of blood are invoked [3, 4]. However, among the possible models of non-Newtonian liquids, most of the time only stationary ones are used. Nevertheless, it is known that the average debit under a pulsed pressure is higher than that under a stationary one [5].

Trying to explain these differences, the author of the work suggested an hypothesis, according to which blood has shear elasticity [6].

Blood represents a solution and a suspension of protein macromolecular compounds. The British scientist Oldroyd [7], in developing the Voigt model [8], has theoretically derived a dependence of the tangential stress on the shear rate for emulsions and suspensions of one Newtonian liquid in another. In contrast to the Newton law, this dependence contains not only the shear rate, but also its time derivative. In the model this author employs, he takes into account the elastic deformation energy accumulated in the process of flow due to the interfacial tension. Upon the removal of the external stress, this energy sustains the internal stress for the duration of the relaxation time. This model, as was shown in experiments, describes adequately the behaviour of solutions of certain polymers.

Because blood is an emulsion and suspension of protein macromolecular compounds, it is natural to suppose that the Voigt-Oldroyd model is also applicable to blood.

MATERIALS AND METHODS

Fig. 1 demonstrates a diagram of flow between two plates and the velocity profile at different moments in time.

Here, the following boundary conditions are taken on the lower

\[ U(t) = u(0, t) = U_0 \cos \omega t \]

and on the upper

\[ u(h, t) = 0 \]

boundaries.

The Voigt-Oldroyd law that binds the tangential tension \( \tau_{xy} \) with the value of the deformation rate \( \sigma_{xy} = \partial u / \partial y \), can be expressed in the general case as follows:

![Fig. 1: Diagram of flow between two plates and the velocity profile at different moments in time](image)
\[ \tau_{ij} = \mu(1 + T_v \frac{\partial}{\partial t} + w \frac{\partial}{\partial x}) \sigma_{ij}, \]

Where \( \mu , T_v, w \) are viscosity, relaxation constant and the flow velocity correspondingly.

In the studied case when the flow does not depend on variable \( x \), this relationship takes the following form:

\[ \tau_{xy} = \mu(1 + T_v \frac{\partial}{\partial t} \frac{\partial u}{\partial y}). \]

(3)

The momentum equation for incompressible liquids can be written as follows:

\[ \rho \frac{\partial u}{\partial t} = -\frac{\partial \tau_{xx}}{\partial y} \]

(4)

Taking into account expression (3), we can derive the following equation:

\[ \rho \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + \mu T_v \frac{\partial^2 u}{\partial t \partial y^2}. \]

(5)

For a comparative estimate of the components that enter into this equation, it is illustrative to use dimensionless dependent and independent variables \( \nu, \tau, \xi \) introduced according to the following formulæ:

\( u = U_0 \nu, \tau = T \tau, \xi = y \xi, \) where \( U_0, T, h \) - the scale coefficients of the corresponding variables.

Using the new variables, the momentum equation will take the form:

\[ \frac{\partial \nu}{\partial \tau} = \frac{\mu T v}{\rho h^2} \frac{\partial^2 \nu}{\partial \xi^2} + \frac{\mu T v}{\rho h^2} \frac{\partial \nu}{\partial \tau} \frac{\partial^2 \nu}{\partial \xi^2}. \]

(6)

It should be noted that the processes that we study (pulsed blood flow in the arterial system) are characterised by a period on the order of one second and the relaxation constant is by two orders of magnitude smaller [6]. Thus, in the first approximation, the momentum equation for a Newtonian liquid.

\[ \rho \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} \]

Will take the same form a that for a viscoelastic liquid. Furthermore, the boundary conditions (1) and (2) will also be the same.

Equation (3) Fourier-transformed with respect to the temporal variable has the form:

\[ \tau_c = \mu \left[ \frac{\partial u}{\partial y} - i T_v \omega \frac{\partial u}{\partial y} \right], \]

\[ \tau_s = \mu \left[ \frac{\partial u}{\partial y} + T_v \omega \frac{\partial u}{\partial y} \right]. \]

(7)

Where the usual designations are

\[ \tau_c = \frac{\omega}{2 \pi} \int_{-\pi/\omega}^{\pi/\omega} \tau \cos \omega \tau \int_{-\pi/\omega}^{\pi/\omega} \tau \sin \omega \tau \]

(8)

Introducing the complex functions \( \Gamma = \tau_c + i \tau_s, U(y) = u_x + i u_y \), we can reduce the two preceding relationships to one:

\[ \Gamma = \mu(1 + iT_v \omega) \frac{dU(y)}{dy} \]

(8)

Complex velocity \( U \) can be represented as the sum

\[ U = U_0(y) + (T_v / T) U_1(y), \]

Where the first term corresponds to the velocity of a Newtonian liquid with a zero relaxation time and the second term explicitly depends on the relaxation time.

Substituting this expression in formula (8) we obtain:

\[ \Gamma = \mu(1 + iT_v \omega) \frac{dU_0(y)}{dy} + (T_v / T) \frac{dU_1(y)}{dy} \]

\[ = \mu \left[ \frac{dU_0(y)}{dy} + (T_v / T) \frac{dU_1(y)}{dy} \right] + iT_v \omega \mu \left[ \frac{dU_0(y)}{dy} + (T_v / T) \frac{dU_1(y)}{dy} \right]. \]

(9)

Now we arrive to the following:

\[ \Gamma = \mu(1 + iT_v \omega) \frac{dU_0(y)}{dy} + (T_v / T) \frac{dU_1(y)}{dy} \]

\[ = \mu \left[ \frac{dU_0(y)}{dy} + (T_v / T) \frac{dU_1(y)}{dy} \right] + iT_v \omega \mu \left[ \frac{dU_0(y)}{dy} + (T_v / T) \frac{dU_1(y)}{dy} \right]. \]

(10)

\[ \tau_c = \mu \left[ \frac{dU_0(y)}{dy} + (T_v / T) \frac{dU_1(y)}{dy} \right], \]

\[ \tau_s = T_v \omega \mu \left[ \frac{dU_0(y)}{dy} + (T_v / T) \frac{dU_1(y)}{dy} \right]. \]

(11)

Here we have used the fact that function \( U_0(y) \) is real.

In the case of short relaxation time that we are interested in, these relationships can be approximated as follows:

\[ \tau_c = \mu \left[ \frac{dU_0(y)}{dy} + (T_v / T) \right], \]

\[ \tau_s = T_v \omega \mu \left[ \frac{dU_0(y)}{dy} + (T_v / T) \right]^2. \]

(12)
It should be noted that values $\tau_v, \tau_s$ have different infinitesimal orders, but their relative errors are the same and both these values are determined through the flow velocity of Newtonian liquid, which in turn is described by a boundary problem (1), (2) and (6).

**Solution of the Equations:** As it is known [9], the solution of equation (6) with boundary conditions (1), (2) has the following form:

$$ u_0(y, t) = U_0 \text{Re} e^{i\frac{\sinh k(h-y)}{\sinh kh} \exp(i\omega t)}, $$

Where

$$ k = \sqrt{\frac{\omega}{2\mu}} (1 + i) $$

This allows us to calculate the tangential tension on the upper boundary:

$$ \tau_{yx} = \mu \text{Re} e^{i\frac{1}{\partial y}(u_0 + i\omega T_v, u_0)} = \mu \text{Re} e^{i\frac{1}{\partial y}(1 + i\omega T_v)} = -\mu U_0 \text{Re} e^{i\frac{k(1 + i\omega T_v)}{shkh} \exp(i\omega t)} $$

Let's separate the real and imaginary parts of the function that enters this expression:

$$ k(1 + i\omega T_v) $$

Here we have isolated the dependence of these parameters upon the relaxation constant. The derived formula allows one to calculate the viscosity and the relaxation constant based on the measurements of the tangential tension.

In order to do that, it is necessary to represent the dependence of the tangential tension as a sum of the following terms,

$$ \tau(h, t) = \tau_v \cos(\omega t) + \tau_s \sin(\omega t) $$

in which the coefficients are the sine and cosine of the Fourier-transform of the experimental curve.

Substitution of this representation into formula (9) leads to the system of equations

$$ \tau_v = -\mu U_0 A(T_v), $$

$$ \tau_s = \mu U_0 B(T_v) $$

For small values of the gap width $h$, for which the condition $|kh| < 1$ is fulfilled, relationship (9) is simplified,

$$ \frac{1 + i\omega T_v}{h} = A(T_v) + iB(T_v) $$

from which follows that

$$ A(T_v) = \frac{1}{h}, B(T_v) = \frac{\omega T_v}{h}. $$

Equation system (12) decomposes into two equations

$$ \tau_v = -\frac{\mu U_0}{h}, \tau_s = \mu U_0 \frac{\omega T_v}{h}. $$

Which can be actually used to calculate the viscosity and the relaxation constant through the measured parameters.

It should be noted that the tangential tension of viscous and of elastic origin enters in all the equations as a superposition. However the action of viscous forces coincides in phase with the oscillations of the bottom plate, the action of elastic forces is phase-shifted by 90 degrees. This property of forces with different origin has allowed us to isolate the components. This is very important since these components have different orders of magnitude, as do their errors.

The first of formulae (13) that determines the viscosity is valid to the precision of the order of $\frac{1}{h^2}$. The exact value of the viscosity will be given by the sum of these components. The error in the value of the viscosity gives consecutively rise to an error in the value of the relaxation constant. Nevertheless, this latter error will be of the order of $\frac{1}{h^2}$. Thus, for values of different order of magnitude we have the same relative errors.

**CONCLUSION**

The functional dependencies derived in the present work allow design of devices for measurements of viscosity and relaxation constants, which can be further used in blood diagnostics.

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REFERENCES