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The Common Solution of the Pair of Fuzzy Matrix Equations

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Abstract: In this paper, we investigate the common solution pair of the fuzzy matrix equations. The Kronecker product and Vec-operator for transforming system of fuzzy linear matrix equation to fuzzy linear system are employed. A new iterative method for solving two non-square $m \times n$ system instead of one $2m \times 2n$ system is presented. Examples are provided to illustrate the developed theory.

Key words: Fuzzy matrix equation · Non-square fuzzy linear system · Operators · Iterative method

INTRODUCTION

The systems of linear equations play a major role in various areas such as science, engineering and economics. Many authors attempt to solve this system by several different methods. In [1], Friedman *et al*. proposed a model for solving an $n \times n$ fuzzy linear system of equation where the coefficient matrix is crisp and the right-hand side column is an arbitrary vector of fuzzy numbers. Authors such as Abbasbandy *et al*. [2], Asady *et al*. [3], Allahviranloo *et al*. [4], [5], Zheng and Wang [6], [7] and Dehghan [8] have extended the work of Friedman *et al*. [1], in dealing with fuzzy systems by using a crisp linear systems.

In general, matrix equations play crucial role in control theory, partial differential equation and block diagonalization of matrices. Recently, several authors solved matrix equation with uncertainty condition. Zengtai and Xiaobin in [9] have investigated fuzzy linear matrix equation in the form of $A\tilde{X} = \tilde{B}$. They used generalized matrix inverse and presented the least square solution to the fuzzy matrix equation. Salkuyeh [10] investigated the fuzzy Sylvester matrix equation where the crisp matrices A and B are special matrices. He transformed this system to linear system and solved it by accelerated over relaxation method (AOR). Allahviranloo *et al*. [11] presented a two stage method for solving fuzzy linear matrix equation $A\tilde{X}B = \tilde{C}$. In this paper we apply known operators to transform the system of fuzzy matrix equation to $m \times n$ fuzzy linear system. We use extended embedding approach given by Wu and Ma [12], [13] to replace the original system of fuzzy matrix equation by fuzzy linear system. In continuing, a new iteration for solving non-square fuzzy linear systems which has feasible properties will be proposed.

Throughout this note, the Kronecker product of two matrices is denoted by $A \otimes B$, the vec-operator of matrix A is presented by vec(A), the notation vec(W) = w (small and bold) is used for given matrix $W \in W^{m\times n}$ and the column space of $W \in W\ m\times n$ is denoted by $\Re(W)$. The outline of the paper is as follows: In Section 2 we will present basic definitions for solving non-square fuzzy linear system. In Section 3 we will present our result solution of the pairs of matrix equations. Iterative method will be proposed in Section 4. Numerical examples will be given in Section 5 and the conclusions are in Section 6.

Basic Concepts: In this section, we recall some of the basic concepts of fuzzy system which presented in [2, 4, 5, 11, 3, 8].

Definition 2.1: A fuzzy number is defined by an ordered pair of functions $(\underline{u}(r), \overline{u}(r)),\ 0 \le r \le 1$, which satisfy the following requirements:

- $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0, 1];
- $\overline{u}(r)$ is a bounded left continuous non-increasing function over [0, 1];
- $\underline{u}(r) \le \overline{u}(r),\ 0 \le r \le 1.$

Definition 2.2: The $m \times n$ system of linear equations

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$$\sum_{j=1}^{n} a_{ij} x_j = y_i, \quad i = 1, \ldots, m \tag{1}$$

Where the coefficient matrix $A = (a_{ij})$, is a crisp $m \times n$ matrix and $y_i \in E^1$, $i = 1,...,m$, is called a fuzzy system of linear equation.

To define a solution $(x_1,...,x_n)^t$ to the system (1), the arithmetic operation of arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and $\lambda \in W$, the arithmetic operators can be defined as:

$$\begin{gathered} u = v \ \text{iff} \ \underline{u}(r) = \underline{v}(r) \ \text{and} \ \overline{u}(r) = \overline{v}(r) \\ u + v = \left(\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)\right) \\ \lambda u = \begin{cases} \left(\lambda \underline{u}(r), \lambda \overline{u}(r)\right), & \lambda \geq 0, \\ \left(\lambda \overline{u}(r), \lambda \underline{u}(r)\right), & \lambda < 0, \end{cases} \end{gathered} \tag{2}$$

Definition 2.3: A fuzzy number vector $(x_1,...,x_n)^t$ given by

$$x_i = \left(\underline{x_i}(r), \overline{x_i}(r)\right)^t, 1 \leq i \leq n, \ 0 \leq r \leq 1$$

is called a solution of the fuzzy linear system (1) if for $i = 1,...,m$ we have

$$\begin{gathered} \min\left\{ \sum_{j=1}^{n} a_{ij} u_j : u_j \in \left[x_j\right]_r \right\} = \underline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \underline{a_{ij} x_j} = \underline{y}_i, \\ \max\left\{ \sum_{j=1}^{n} a_{ij} u_j : u_j \in \left[x_j\right]_r \right\} = \overline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \overline{a_{ij} x_j} = \overline{y}_i \end{gathered} \tag{3}$$

It should be noted that two crisp $m \times n$ linear systems for all i that can be extended to an $2m \times 2n$ crisp linear system as:

$$S\mathrm{x} = \mathrm{y} \tag{4}$$

Where

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} \underline{\mathbf{x}} \\ \overline{\mathbf{x}} \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} \underline{\mathbf{y}} \\ \overline{\mathbf{y}} \end{pmatrix}$$

Where $s_{i,j}$ can be determined by

- If $a_{ij} \geq 0$ for $i, j = 1,..., 2n$ then we can put $a_{i,j} = s_{i,j}$ and $a_{i+m,j+n} = s_{i,j}$
- If $a_{ij} < 0$ for $i,j = 1,...,2n$ then we can put $s_{i,j+n} = - a_{i,j}$ and $s_{i+m,j} = - a_{i,j}$.

Theorem 2.1: Let matrix S be in the form (4), then the matrix

$$S^{-} = \frac{1}{2}\begin{pmatrix} (S_1+S_2)^{\{1,3\}} + (S_1-S_2)^{\{1,3\}} & S_1+S_2)^{\{1,3\}} - (S_1-S_2)^{\{1,3\}} \\ (S_1+S_2)^{\{1,3\}} - (S_1-S_2)^{\{1,3\}} & S_1+S_2)^{\{1,3\}} + (S_1-S_2)^{\{1,3\}} \end{pmatrix} \tag{5}$$

is a $\{1,3\}$-inverse of the matrix S, where $(S_1 + S_2)^{\{1,3\}}$ and $(S_1 - S_2)^{\{1,3\}}$ are $\{1,3\}$-inverse of the matrices $S_1 + S_2$ and $S_1 - S_2$, respectively. In particular, the Moore-Penrose of the matrix S is:

$$S^{\dagger} = \frac{1}{2}\begin{pmatrix} (S_1+S_2)\dagger + (S_1-S_2)\dagger & (S_1+S_2)\dagger - (S_1-S_2)\dagger \\ (S_1+S_2)\dagger - (S_1-S_2)\dagger & (S_1+S_2)\dagger + (S_1-S_2)\dagger \end{pmatrix} \tag{6}$$

Theorem 2.2: For the consistent system (4) and any $\{1,3\}$-inverse $S^{\{1,3\}}$ of the coefficient matrix S, $\mathbf{x} = S^{\{1,3\}}\mathbf{y}$ is a solution to the system and therefore it admits a weak or strong fuzzy solution. In particular, if $S^{\{1,3\}}$ is nonnegative with the special structure (6), then $\mathbf{x} = S^{\{1,3\}}\mathbf{y}$ admits a strong fuzzy solution for arbitrary fuzzy vector $\mathbf{y}$.

The Pair of the Fuzzy Linear Matrix Equations: In this section a type of system of the fuzzy matrix equation will be discussed. First, we have the following definitions.

Definition 3.1: The system of linear matrix equations

$$\begin{cases} A_1 \tilde{X} + \tilde{X} B_1 = \tilde{C}_1 \\ A_2 \tilde{X} B_2 = \tilde{C}_2 \end{cases} \tag{7}$$

Where A_1 and B_1, are the known $m \times m$ and $n \times n$ crisp matrices respectively, A_2 and B_2, are the known $k \times m$ and $m \times \ell$ crisp matrices respectively and $\tilde{C}_1$ and $\tilde{C}_2$, are $m \times n$ and $k \times \ell$ known fuzzy matrices and $\tilde{X}$ is an $m \times n$ unknown fuzzy matrices called "the pair of the fuzzy linear matrix equation".

Definition 3.2: [14] Suppose that $A \in {}^{m\times m}$ and $B \in {}^{n\times n}$ are square matrices. Then the Kronecker sum of A and B, denoted $A \oplus B$, defined by $(I_n \otimes A) + (B \otimes I_m)$. Note that, in general, $A \oplus B \neq B \oplus A$.

According to the properties of Kronecker sum

$(A \oplus B^T) = (I_n \otimes A) + (B^T \otimes I_m)$
$ABC = (C^T \otimes A)\mathrm{vec}(B)$

The system (7) can be transformed to

$$\begin{cases} \left(A_1 \oplus B_1^T\right) vec(\tilde{X}) = vec(\tilde{C}_1) \\ \left(B_2^T \otimes A_2\right) vec(\tilde{X}) = vec(\tilde{C}_2) \end{cases} \tag{8}$$

In the following theorem the necessary and sufficient condition for the pair of matrix equation (7) to have a common solution is presented.

Theorem 3.1: [15] Given matrices $A_1 \in {}^{m\times m}$, $B_1 \in {}^{n\times n}$, $A_2 \in {}^{m\times n}$, $B_2 \in {}^{k\times \ell}$ are the known matrices and C_1 and C_2, are $m \times n$ and $k \times \ell$ known matrices, respectively. Additionally assume

$$\mathbf{Q} = \begin{pmatrix} A_1 \oplus B_1^T \\ B_2^T \otimes A_2 \end{pmatrix} \in \mathbb{R}^{(k\ell+mn)\times mn}, \ \mathbf{c} = \begin{pmatrix} vec(C_1) \\ vec(C_2) \end{pmatrix}, \tag{9}$$

Then the pair matrix equations (7) has a common solution if and only if $\mathbf{c} \in \Re(\mathbf{Q})$. When the condition is satisfied, a representation of the general common solution is

$$X = \text{Inv}(\mathbf{Q}^{\{1,3\}}\mathbf{c} + (I_{k\ell} - \mathbf{Q}^{\{1,3\}}\mathbf{Q})\mathbf{z}) \tag{10}$$

Where $\mathbf{z} \in {}^{k\ell}$ is an arbitrary vector and $\mathbf{Q}^{\{1,3\}}$ is an arbitrary but fixed $\{1,3\}$-inverse of $\mathbf{Q}$.

If A_j^+, $(j = 1,2)$ and B_j^+, $(j = 1,2)$ contain the positive entries of A_j, $(j = 1,2)$ and B_j, $(j = 1,2)$ respectively and A_j^-, $(j = 1,2)$ and B_j^-, $(j = 12,)$ contain the negative entries of A_j, $(j = 1,2)$ and B_j, $(j = 1,2)$ respectively, it is obvious that $A_j = A_j^+ - A_j^-$ and $B_j = B_j^+ - B_j^-$, $(j = 1,2)$. So, according to properties of Kronecker operators it can be written as

$$\begin{aligned} \left(A_1 \oplus B_1^T\right) &= \left(I \otimes A_1\right) + \left(B_1^T \otimes I\right) \\ &= \left(A_1^+ \oplus B_1^{+T}\right) + \left(A_1^- \oplus B_1^{-T}\right) \end{aligned}$$

and

$$\begin{aligned} \left(B_2^T \otimes A_2\right) &= \left(B_2^+ - B_2^-\right)^T \otimes \left(A_2^+ \otimes A_2^-\right) \\ &= \left(\left(B_2^{+T} \otimes A_2^+\right) + \left(B_2^{-T} \otimes A_2^-\right)\right) \\ &\quad - \left(\left(B_2^{+T} \otimes A_2^-\right) + \left(B_2^{-T} \otimes A_2^+\right)\right) \end{aligned}$$

Therefore, we have

$$\begin{aligned} Q &= \begin{pmatrix} A_1 \oplus B_1^T \\ B_2^T \otimes A_2 \end{pmatrix} \\ &= \begin{pmatrix} \left(A_1^+ \oplus B_1^{+T}\right) \\ \left(B_2^{+T} \otimes A_2^+\right) + \left(B_2^{-T} \otimes A_2^-\right) \end{pmatrix} \\ &- \begin{pmatrix} \left(A_1^- \oplus B_1^{-T}\right) \\ \left(B_2^{+T} \otimes A_2^-\right) + \left(B_2^{-T} \otimes A_2^+\right) \end{pmatrix} \\ &= S_1 - S_2 \end{aligned}$$

Hence, we obtain

$$S_1 = \begin{pmatrix} \left(A_1^+ \oplus B_1^{+T}\right) \\ \left(B_2^{+T} \otimes A_2^+\right) + \left(B_2^{-T} \otimes A_2^-\right) \end{pmatrix}$$

and

$$S_2 = \begin{pmatrix} \left(A_1^- \oplus B_1^{-T}\right) \\ \left(B_2^{+T} \otimes A_2^-\right) + \left(B_2^{-T} \otimes A_2^+\right) \end{pmatrix}$$

In addition, it can be concluded that

$$\mathbf{P} = S_1 + S_2 = \begin{pmatrix} \left(A_1^+ + A_1^-\right) \oplus \left(B_1^+ + B_1^-\right)^T \\ \left(B_2^+ - B_2^-\right)^T \otimes \left(A_2^+ + A_2^-\right) \end{pmatrix} \tag{11}$$

Corollary 3.1: By considering $S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}$, then the matrix

$$S^- = \frac{1}{2}\begin{pmatrix} \mathbf{P}^{\{1,3\}} + \mathbf{Q}^{\{1,3\}} & \mathbf{P}^{\{1,3\}} - \mathbf{Q}^{\{1,3\}} \\ \mathbf{P}^{\{1,3\}} - \mathbf{Q}^{\{1,3\}} & \mathbf{P}^{\{1,3\}} + \mathbf{Q}^{\{1,3\}} \end{pmatrix} \tag{12}$$

is a $\{1,3\}$-inverse of the matrix S, where $\mathbf{P}^{\{1,3\}}$ and $\mathbf{Q}^{\{1,3\}}$ are $\{1,3\}$-inverse of the matrices $\mathbf{P}$ and $\mathbf{Q}$, respectively. In particular, the Moore-Penrose of the matrix S is

$$S^\dagger = \frac{1}{2}\begin{pmatrix} \mathbf{P}^\dagger + \mathbf{Q}^\dagger & \mathbf{P}^\dagger - \mathbf{Q}^\dagger \\ \mathbf{P}^\dagger - \mathbf{Q}^\dagger & \mathbf{P}^\dagger + \mathbf{Q}^\dagger \end{pmatrix}$$

Now, we are ready to extend the system of matrix equation to the system of linear fuzzy equation. The system of fuzzy matrix equation can be written as

$$S\mathbf{x} = \mathbf{c} \tag{13}$$

Where

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} \underline{\mathbf{x}} \\ -\overline{\mathbf{x}} \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} \underline{\mathbf{c}} \\ -\overline{\mathbf{c}} \end{pmatrix}$$

Remark 3.2: More recently Allahviranloo *et al*. [16] has presented the fuzzy solution of the fuzzy linear systems, defined by Friedman *et al*. [1] may not be a fuzzy numbers vector. In other words, in this case it may be at least one of vector's components is not fuzzy number. It should be emphasized that in this work the authors considered the fuzzy solutions which are fuzzy vector numbers.

Solving Non-Square Fuzzy Linear Systems: System (13) can be solved by using direct computation of $\{1,3\}$-inverse or Moore-Penrose of S. In this section, we present new iterative method to solve fuzzy linear system (13).

Suppose that $A \in {}^{n\times n}$ is a full-rank matrix to be determined $A\mathbf{x} = \mathbf{y}$ is the linear system and γ be the step-size or convergence factor. Ding and Chen in [7] presented a family of iterative methods for solving systems as follows:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \gamma G\left(\mathbf{y} - A\mathbf{x}^{(k)}\right),\ 0 < \gamma < \frac{2}{\|A\|^2} \qquad (14)$$

If A is a non-square $m\times n$ full column-rank matrix and take $G = (A^TA)^{-1} A^T$, then the following least square iterative algorithm leads to $\lim_{k\to\infty} \mathbf{x}^{(k)} = \mathbf{x}$:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \gamma\left(A^TA\right)^{-1} A^T\left(\mathbf{y} - A\mathbf{x}^{(k)}\right),\ 0 < \gamma < 2 \qquad (15)$$

$$\begin{pmatrix} \underline{\mathbf{x}} \\ -\overline{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} \underline{\mathbf{x}} \\ -\overline{\mathbf{x}} \end{pmatrix} + \gamma \begin{pmatrix} R_1 & R_2 \\ R_2 & R_1 \end{pmatrix}\left[\begin{pmatrix} \underline{\mathbf{c}} \\ -\overline{\mathbf{c}} \end{pmatrix} - \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}\begin{pmatrix} \underline{\mathbf{x}} \\ -\overline{\mathbf{x}} \end{pmatrix}\right]$$

By considering $\left(S^TS\right)^{-1} S^T = \begin{pmatrix} R_1 & R_2 \\ R_2 & R_1 \end{pmatrix}$, we can obtain

Thus, for $k = 1,2,...$, the new iterative recursion will be introduced as

$$\begin{cases} \underline{\mathbf{x}}^{(k+1)} = \underline{\mathbf{x}}^{(k)} + \gamma\left[R_1\left(\underline{\mathbf{c}} - S_1\underline{\mathbf{x}}^{(k)} + S_2\overline{\mathbf{x}}^{(k)}\right) + R_2\left(-\overline{\mathbf{c}} - S_2\underline{\mathbf{x}}^{(k)} + S_1\overline{\mathbf{x}}^{(k)}\right)\right] \\ \overline{\mathbf{x}}^{(k+1)} = \overline{\mathbf{x}}^{(k)} + \gamma\left[R_2\left(-\underline{\mathbf{c}} + S_1\underline{\mathbf{x}}^{(k)} - S_2\overline{\mathbf{x}}^{(k)}\right) + R_1\left(\overline{\mathbf{c}} + S_2\underline{\mathbf{x}}^{(k)} - S_1\overline{\mathbf{x}}^{(k)}\right)\right] \end{cases} \qquad (16)$$

The algorithm of this iterative method is given as follows:

Algorithm 1: Iterative method for solving Eq. (13)

- Choose initial guesses $\underline{\mathbf{x}}^{(0)}$ and $\overline{\mathbf{x}}^{(0)}$;
- Compute the matrix $\begin{pmatrix} R_1 & R_2 \\ R_2 & R_1 \end{pmatrix}$;
- For k,1,2,... Do
- $\underline{\mathbf{x}}^{(k+1)} = \underline{\mathbf{x}}^{(k)} + \gamma\left[R_1\left(\underline{\mathbf{c}} - S_1\underline{\mathbf{x}}^{(k)} + S_2\overline{\mathbf{x}}^{(k)}\right) + R_2\left(-\overline{\mathbf{c}} - S_2\underline{\mathbf{x}}^{(k)} + S_1\overline{\mathbf{x}}^{(k)}\right)\right]$;
- $\overline{\mathbf{x}}^{(k+1)} = \overline{\mathbf{x}}^{(k)} + \gamma\left[R_2\left(-\underline{\mathbf{c}} + S_1\underline{\mathbf{x}}^{(k)} - S_2\overline{\mathbf{x}}^{(k)}\right) + R_1\left(\overline{\mathbf{c}} + S_2\underline{\mathbf{x}}^{(k)} - S_1\overline{\mathbf{x}}^{(k)}\right)\right]$;
- If the stopping criterion satisfied, then stop
- End do.

Note that R_1 and R_2 are the elements of matrix $(S^TS)^{-1} S^T$.

Numerical Examples: In this section, several numerical experiments are given to illustrate the theoretical results. The triangular fuzzy numbers are used in all of the following numerical examples. For a fuzzy number $u = (\underline{u}(r), \overline{u}(r)) = (a + br, c + dr)$ we define its norm similar to Wang and Zheng [7] as follows

$$\|x\| = \max\ \{|a|,|b|,|c|,|d|\}$$

In addition, the stopping criterion

$$\frac{\left\|\underline{\mathbf{x}}^{(k+1)} - \underline{\mathbf{x}}^{(k)}\right\|}{\left\|\underline{\mathbf{x}}^{(k)}\right\|} < \varepsilon \ \text{ and } \ \frac{\left\|\overline{\mathbf{x}}^{(k+1)} - \overline{\mathbf{x}}^{(k)}\right\|}{\left\|\overline{\mathbf{x}}^{(k)}\right\|} < \varepsilon$$

is used $\varepsilon = 10^{-4}$ and a zero vector is always used as the initial guess.

Example 5.1: For the first example, consider the pair of fuzzy linear matrix equations

$$\begin{cases} A_1\tilde{X} + \tilde{X}B_1 = \tilde{C}_1 \\ A_2\tilde{X}B_2 = \tilde{C}_2 \end{cases} \qquad (17)$$

Where

$$A_1 = \begin{pmatrix} 3 & -1 \\ -2 & 5 \end{pmatrix},\quad B_1 = \begin{pmatrix} 4 & -2 & -1 \\ -2 & 5 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\tilde{C}_1 = \begin{pmatrix} (-41 + 26r, 26 - 41r) & (-44 + 23r, 23 - 44r) & (-22 + 25r, 25 - 22r) \\ (-50 + 50r, 50 - 50r) & (-34 + 67r, 67 - 34r) & (-22 + 40r, 40 - 22r) \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -1 & 0 \\ -1 & 3 \end{pmatrix},\quad B_2 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ -1 & 3 \end{pmatrix}$$

$$\tilde{C}_2 = \begin{pmatrix} (-5 + 8r, 8 - 5r) & (-17 + 20r, 20 - 17r) \\ (-29 + 23r, 23 - 29r) & (-65 + 86r, 86 - 65r) \end{pmatrix}$$

The exact common solution of (17) is

$$\tilde{X} = \begin{pmatrix} (-5 + 2r, 2 - 5r) & (-4 + r, 1 - 4r) & (-3 + 3r, 3 - 3r) \\ (-4 + 4r, 4 - 4r) & (-2 + 5r, 5 - 2r) & (-1 + 4r, 4 - r) \end{pmatrix}$$

Now, the coefficient system and right hand side of the system is computed as follows

$$\mathbf{Q} = \begin{pmatrix} 7 & -1 & -2 & 0 & 0 & 0 \\ -2 & 9 & 0 & -2 & 0 & 0 \\ -2 & 0 & 8 & -1 & -1 & 0 \\ 0 & -2 & -2 & 10 & 0 & -1 \\ -1 & 0 & -1 & 0 & 5 & -1 \\ 0 & -1 & 0 & -1 & -2 & 7 \\ -1 & 0 & 0 & 0 & 1 & -3 \\ -1 & 3 & 0 & 0 & 1 & -3 \\ -2 & 0 & 1 & 0 & -3 & 0 \\ -2 & 6 & 1 & -3 & -3 & 9 \end{pmatrix},$$

$$\mathbf{c} = \begin{pmatrix} (-41 + 26r, 26 - 41r) \\ (-50 + 50r, 50 - 50r) \\ (-44 + 23r, 23 - 44r) \\ (-34 + 67r, 67 - 34r) \\ (-22 + 25r, 25 - 22r) \\ (-22 + 40r, 40 - 22r) \\ (-5 + 8r, 8 - 5r) \\ (-29 + 23r, 23 - 29r) \\ (-17 + 20r, 20 - 17r) \\ (-65 + 86r, 86 - 65r) \end{pmatrix},$$

Thus, the coefficient of the extended 20×12 system $S\mathbf{x} = \mathbf{c}$ can be written as $S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}$ where S_1, S_2 is computed. Now the extended system $S\mathbf{x} = \mathbf{c}$ can be solved by utilizing Algorithm 1. By taking the step size, $\gamma = 0.0068$ after 302 iteration we reach to

$$\tilde{X} = \begin{pmatrix} (-4.9993 + 1.9997r, 1.9997 - 4.9993r) & (-3.9994 + 0.9999r, 0.9999 - 3.9994r) & (-2.9996 + 2.9996r, 2.9996 - 2.9996r) \\ (-3.9994 + 3.9994r, 3.9994 - 3.9994r) & (-1.9997 + 4.9993r, 4.9993 - 1.9997r) & (-0.9999 + 3.9994r, 3.9994 - 0.9999r) \end{pmatrix}$$

The difference between the exact solution and the approximated solution can be compared in Figure 1. Also, the relation between the number of iteration and the step size γ can be seen in Figure 2.

Example 5.2: For the second example, consider the pair of the fuzzy matrix equations such that

$$\begin{cases} A_1\tilde{X} + \tilde{X}B_1 = \tilde{C}_1 \\ A_2\tilde{X}B_2 = \tilde{C}_2 \end{cases} \quad (18)$$

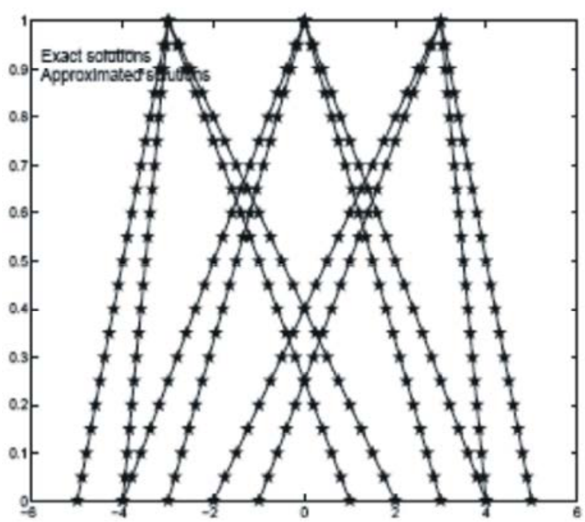

Fig. 1: Exact solutions and approximated solutions in Example 5.1

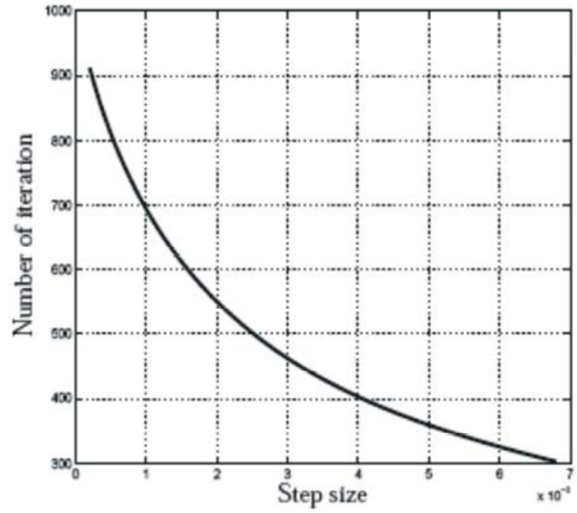

Fig. 2: The relation between step size γ and the number of iterations in Example 5.1

Where

$$A_1 = \begin{pmatrix} 3 & -1 \\ -2 & 5 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 4 & -2 \\ -2 & 5 \end{pmatrix}$$

$$\tilde{C}_1 = \begin{pmatrix} (-26 + 26r, 26 - 26r) & (-37 + 37r, 37 - 37r) \\ (-42 + 42r, 42 - 42r) & (-26 + 26r, 26 - 26r) \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 3 \\ 9 & 6 \end{pmatrix}$$

$$\tilde{C}_2 = \begin{pmatrix} (-72 + 72r, 72 - 72r) & (-60 + 60r, 60 - 60r) \\ (-63 + 63r, 63 - 63r) & (-84 + 84r, 84 - 84r) \end{pmatrix}$$

The exact common solution of (18) is

$$\tilde{X} = \begin{pmatrix} (-2 + 2r, 2 - 2r) & (-4 + 4r, 4 - 4r) \\ (-4 + 4r, 4 - 4r) & (-1 + r, 1 - r) \end{pmatrix}$$

After computing S_1, S_2 and right hand side of the system, the extended system $S\mathbf{x} = \mathbf{c}$ can be solved by utilizing Algorithm 1. By considering $\gamma = 0.0012$, after 658 iteration we obtain

$$\tilde{X} = \begin{pmatrix} (-1.9998 + 1.9998r, 1.9998 - 1.9998r) & (-3.9997 + 3.9997r, 3.9997 - 3.9997r) \\ (-3.9997 + 3.9997r, 3.9997 - 3.9997r) & (-0.9999 + 0.9999r, 0.9999 - 0.9999r) \end{pmatrix}$$

The relation between the number of iterations in Figure 3 and also the difference between the exact solution and the approximated solution in Figure 4 can be observed.

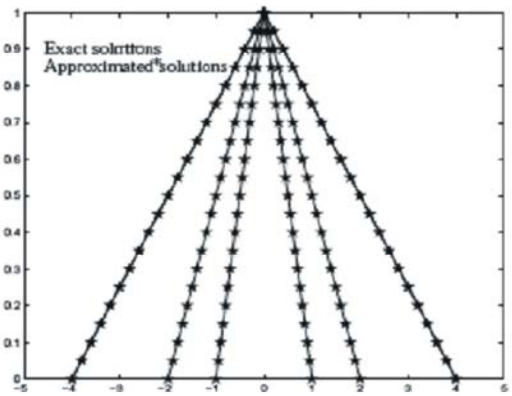

Fig. 3: Exact solutions and approximated solutions in Example 5.2

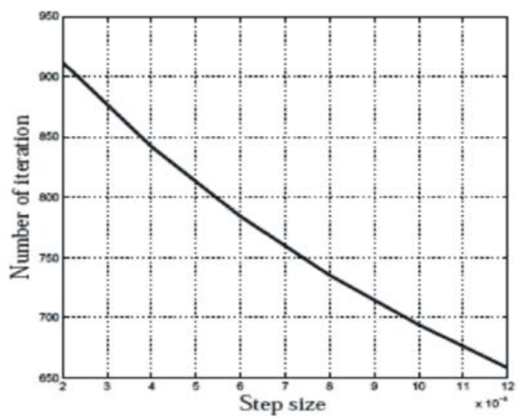

Fig. 4: The relation between step size ã and the number of iterations in Example 5.2

CONCLUSION

In this work, we describe an approach for solving the pair of fuzzy matrix equation. The original system of fuzzy matrix equation is replaced by a crisp linear system by employing some operators. A numerical iterative method was proposed for solving fuzzy linear system. The important advantage of this iteration that is each $m \times n$ non square system can be solved efficiently instead of $2m \times 2n$ system. As we know, the used computational complexity in this case is less than solving one $2m \times 2n$ system.

Appendix A: Let A be an $m \times n$ matrix. We recall that a generalized inverse G of A is an $m \times n$ matrix which satisfies one or more of the Penrose equations:

- $AGA = A$,
- $GAG = G$,
- $(AG)^T = AG$,
- $(GA)^T = GA$

For a subset {1,2,3} of set {1,2,3,4}, the set of $n \times m$ matrices satisfying the equations contained in $\{i,j,k\}$ is denoted by $A\{i,j,k\}$. A matrix in $A\{i,j,k\}$ is called an $\{i,j,k\}$-inverse of A and is denoted by $A^{\{i,j,k\}}$. In particular, the matrix A is called a {1}-inverse or a g-inverse of A if it satisfies (1). As usual, the g-inverse of A is denoted by A^{-}. If G satisfies (2) then it is called a {2}-inverse and If G satisfies (1) and (2) then it is called a reflexive inverse or a {1,2}-inverse of A. The Moore-Penrose inverse of A is the matrix G which satisfies (1)-(4). Any matrix A admits a unique Moore-Penrose inverse is denoted by $A^{\dagger}$.

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