Speed of Longitudinal and Transverse Plane Elastic Waves in Rotating and Non-Rotating Anisotropic Mediums

A. Khan, S. Islam, M. Khan and I. Siddiqui

Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan

Abstract: In this article, we study the rotational effects on the speed of longitudinal and transverse wave propagates along the coordinate planes and axes. The mediums considered are isotropic, transversely isotropic, orthotropic, monoclinic and general anisotropic. It is observed that results from rotating to non-rotating mediums cannot be obtained neither results of one medium can be derived from the others. It is further observed that rotational effect decreases the number of the longitudinal and transverse waves and the number of the longitudinal and transverse waves decreases with the increase of anisotropy.

Key words: Isotropic, Transversely isotropic, Orthotropic, Monoclinic, Longitudinal and transverse waves

INTRODUCTION

Speed of longitudinal and transverse waves in elastic bodies were discussed by a number of authors from the beginning of 19th century but Schoenberg and Censor [1] have consider the effect of rotation on plane waves propagating in an isotropic medium. They showed that in a rotating isotropic medium three waves can propagate. They also showed that longitudinal or transverse wave can exist only if the direction of propagation and axis of rotation are either parallel or perpendicular, which is evident from Table 1.1 of section 1.

Propagation of waves in the transversely isotropic medium has been discussed by a number of authors/researches but Chadwick [2] has discussed in detail in non-rotating medium. He showed that three waves propagate in transversely isotropic medium. In section 2, we extend the case for rotating medium. In sections 3, 4 and 5 rotational effects on the speed of the longitudinal and transverse waves are discussed in the orthotropic, monoclinic and anisotropic mediums respectively.

ISOTROPIC MEDIUM

Wave propagation in the non-rotating isotropic medium: The constitutive equations of motion in the absence of body forces in the non-rotating medium can be written as [3]:

\[(\lambda + \mu)u_{ij,ji} + \mu u_{i,jj} = \rho \ddot{u}_i; i,j = 1,2,3\] (1.1)

where \(\rho\) is the density, \(u_i\) is the displacement vector and \(\lambda, \mu\) are Lamé’s constants. For a plane wave we assume the solution of the form:

\[u_i = A\exp[ik(xp_i - ct)]p_i\] (1.2)

where \(p_i\) and \(p_j\) are respectively the propagation and polarization vectors of the wave. \(A\) is the amplitude of the wave, \(c\) is the wave speed and \(k\) is wave number. The secular equation is

\[(\lambda + \mu)n_i n_j p_j + (\mu - \rho c^2) n_i = 0\] (1.3)

It is known that in isotropic medium longitudinal and transverse wave does propagate in all directions with speeds [3]:

\[c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} \text{ and } c_T = \sqrt{\frac{\mu}{\rho}}\]

respectively.

Wave propagation in the rotating medium

The equation of motion in the absence of body forces in the rotating medium can be written as follows [1]:

\[
\sigma_{ij} = \rho \left(\ddot{u}_i + \Omega_j \Omega_i - \Omega^2 u_i + 2\epsilon_{ijk} \Omega_k u_j\right)
\]

Assume that the body is rotating about \(x_3\)-axis. Then \(\Omega_1 = \Omega_2 = 0\), the secular equation becomes
Table 1.1: Longitudinal and transverse wave speed in a rotating isotropic medium

<table>
<thead>
<tr>
<th>Direction of propagation along</th>
<th>Speed of longitudinal wave m/s</th>
<th>Speed of transverse wave m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>x₁-axis</td>
<td>Nil</td>
<td>√(µ/ρ)</td>
</tr>
<tr>
<td>x₂-axis</td>
<td>Nil</td>
<td>√(µ/ρ)</td>
</tr>
<tr>
<td>x₃-axis (λ + 2µ/ρ) ≥ Γ²</td>
<td>√(µ/ρ) ± (µ/ρ + Γ²) ± (√(µ/ρ) - Γ²)</td>
<td>√(µ/ρ) ± (µ/ρ + Γ²) ± (√(µ/ρ) - Γ²)</td>
</tr>
<tr>
<td>x₁x₂-plane</td>
<td>c₁₂</td>
<td>√(µ/ρ)</td>
</tr>
<tr>
<td>x₂x₃-plane</td>
<td>Nil</td>
<td>Nil</td>
</tr>
<tr>
<td>x₁x₃-plane</td>
<td>Nil</td>
<td>Nil</td>
</tr>
</tbody>
</table>

\[
\left\{ (\lambda + \mu) k_i n_i + (\rho \Omega) \right\} p_i + \left\{ \mu k^2 - k^2 \rho c^2 - \rho \Omega^2 \right\} p_i - 2ikpc_e_k \Omega p_k = 0
\]  

(1.5)

Speed of the longitudinal and transverse waves is given in Table 1.1.

where \( \frac{\Omega}{k} = \Gamma \). Rotational effect allows the longitudinal waves to move only along the axis of rotation subject to \( \frac{\lambda + 2\mu}{\rho} \geq \Gamma^2 \) and perpendicular to the plane of it, while the speed of the transverse wave, propagate along the axis of rotation, depend upon \( \Gamma \) and does not propagate in the planes containing the axis of rotation. The results of Table 1.1 are agreed with Schoenberg and Censor [1].

**TRANSVERSELY ISOTROPIC MEDIUM**

**Basic equations and solution of the problem:** It is known that the generalized Hooke’s law for transversely isotropic medium, taking x₃-axis as the axis of the symmetry of the medium is:

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{33} \\
\sigma_{12}
\end{bmatrix} = 
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
c_{21} & c_{11} & c_{13} & 0 & 0 & 0 \\
c_{31} & c_{10} & c_{13} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2(c_{11} - c_{12})
\end{bmatrix} 
\begin{bmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\epsilon_{23} \\
2\epsilon_{33} \\
2\epsilon_{12}
\end{bmatrix}
\]  

(2.1)

Thus secular equation in component form is:

\[
\left\{ \begin{array}{c}
(c_{11}n_i^2 + \frac{1}{2}(c_{11} - c_{12})n_i^2 + c_{44}n_i^2 - \rho c^2) \right\} p_i + \left\{ \frac{1}{2}(c_{11} + c_{12})n_i n_j \right\} p_j + \left\{ (c_{13} + c_{44}) n_i n_j \right\} p_k = 0 \\
\frac{1}{2}(c_{11} + c_{12})n_i n_j \right\} p_i + \left\{ \frac{1}{2}(c_{11} - c_{12})n_i^2 + c_{44}n_i^2 - \rho c^2 \right\} p_j + \left\{ (c_{13} + c_{44}) n_i n_j \right\} p_k = 0 \\
\left\{ (c_{13} + c_{44}) n_i n_j \right\} p_i + \left\{ (c_{13} + c_{44}) n_i n_j \right\} p_j + \left\{ c_{44}n_i^2 + c_{44}n_i^2 + c_{44}n_i^2 - \rho c^2 \right\} p_k = 0 \\
\end{array} \right.
\]  

(2.2)

The speed of the longitudinal and transverse waves are given in Table 2.1.

It is observed that two waves propagate along the plane perpendicular to the axis of symmetry where as three waves propagate in the other directions. It is also noted that along the co-ordinate axes one longitudinal and two transverse waves exist where as in the co-ordinate planes two longitudinal and one transverse wave exist except the...
Table 2.1: Longitudinal and transverse wave speed in transversely isotropic medium

<table>
<thead>
<tr>
<th>Direction of propagation along</th>
<th>Speed of longitudinal wave m/s</th>
<th>Speed of transverse wave m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>x_1-axis</td>
<td>( \sqrt{ \frac{c_{11}}{\rho} } )</td>
<td>( c_{11} = \frac{c_{11} - c_{22}}{2\rho}, c_{12} = \frac{c_{44}}{\rho} )</td>
</tr>
<tr>
<td>x_2-axis</td>
<td>( c_1 = \sqrt{ \frac{c_{11}}{\rho} } )</td>
<td>( c_{11} = \frac{c_{11} - c_{22}}{2\rho}, c_{12} = \frac{c_{44}}{\rho} )</td>
</tr>
<tr>
<td>x_3-axis</td>
<td>( c_1 = \sqrt{ \frac{c_{11}}{\rho} } )</td>
<td>( c_{11} = \frac{c_{11} - c_{22}}{2\rho}, c_{12} = \frac{c_{44}}{\rho} )</td>
</tr>
<tr>
<td>x_1x_2-plane</td>
<td>( c_1 = \sqrt{ \frac{c_{11}}{\rho} } )</td>
<td>( c_1 = \frac{c_{11} - c_{22}}{2\rho}, c_{12} = \frac{c_{44}}{\rho} )</td>
</tr>
<tr>
<td>x_2x_3-plane</td>
<td>( c_1 = \sqrt{ \frac{c_{11} n_1^2 + (c_{11} + 2c_{44}) n_3^2}{\rho} } ), ( c_{12} = \sqrt{ \frac{c_{33} n_3^2 + (c_{11} + 2c_{44}) n_1^2}{\rho} } )</td>
<td>( c_1 = \frac{c_{11} - c_{22}}{2\rho}, c_{12} = \frac{c_{44} n_1^2 + c_{44} n_3^2}{\rho} )</td>
</tr>
<tr>
<td>x_1x_3-plane</td>
<td>( c_1 = \sqrt{ \frac{c_{11} n_1^2 + (c_{11} + 2c_{44}) n_3^2}{\rho} } ), ( c_{13} = \sqrt{ \frac{c_{33} n_3^2 + (c_{11} + 2c_{44}) n_1^2}{\rho} } )</td>
<td>( c_1 = \frac{c_{11} - c_{22}}{2\rho}, c_{13} = \frac{c_{44} n_1^2 + c_{44} n_3^2}{\rho} )</td>
</tr>
</tbody>
</table>

Table 2.2: Longitudinal and transverse wave speed in rotating transversely isotropic medium

<table>
<thead>
<tr>
<th>Direction of propagation along</th>
<th>Speed of longitudinal wave m/s</th>
<th>Speed of transverse wave m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>x_1-axis</td>
<td>Nil</td>
<td>( \sqrt{ \frac{c_{33}}{\rho} } )</td>
</tr>
<tr>
<td>x_2-axis</td>
<td>Nil</td>
<td>( \sqrt{ \frac{c_{33}}{\rho} } )</td>
</tr>
<tr>
<td>x_3-axis</td>
<td>( \sqrt{ \frac{c_{33}}{\rho} } )</td>
<td>( \nu_1 )</td>
</tr>
<tr>
<td>x_1x_2-plane</td>
<td>Nil</td>
<td>( \sqrt{ \frac{c_{33}}{\rho} } )</td>
</tr>
<tr>
<td>x_2x_3-plane</td>
<td>( \sqrt{ \frac{c_{33} n_1^2}{\rho} } )</td>
<td>Nil</td>
</tr>
<tr>
<td>x_1x_3-plane</td>
<td>( \sqrt{ \frac{c_{33} n_1^2 + (c_{11} + 2c_{44}) n_3^2 - \Omega^2}{\rho} } ), ( \sqrt{ \frac{c_{33} n_3^2 + (c_{11} + 2c_{44}) n_1^2}{\rho} } )</td>
<td>Nil</td>
</tr>
</tbody>
</table>

plane perpendicular to the axis of symmetry. The results of Table 2.1 are in good agreement with the results produced by Chadwick [2].

**Elastic wave propagation in a rotating transversely isotropic medium:** Transversely isotropic medium rotating with a uniform angular velocity \( \Omega \) is considered. By using equation (1.4) and by choosing same \( \Omega_i \), the equation of motion for the rotating medium becomes,

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( c_{11} n_1^2 + \frac{1}{2} (c_{11} - c_{12}) n_2^2 + c_{44} n_3^2 - \rho c^2 \right) k^2 - \rho \Omega^2 p_1 + \left( \frac{1}{2} (c_{11} + c_{12}) n_2^2 \right) k^2 + 2 \text{i} \kappa \rho \Omega \right) p_2 + \left( c_{11} + c_{44} \right) n_3^2 k^2 - \rho \Omega^2 \right) p_3 = 0 \\
\left\{ \begin{array}{l}
\left( \frac{1}{2} (c_{11} + c_{12}) n_2 n_3 \right) k^2 - 2 \text{i} \kappa \rho \Omega \right) p_1 + \left( \frac{1}{2} (c_{11} - c_{12}) n_2^2 + c_{44} n_3^2 - \rho c^2 \right) k^2 + \rho \Omega^2 \right) p_2 + \left( c_{11} + c_{44} \right) n_3^2 k^2 - \rho \Omega^2 \right) p_3 = 0 \\
\left\{ (c_{11} + c_{44}) n_1 n_3 \right) p_1 + \left( (c_{11} + c_{44}) n_2 n_3 \right) p_2 + \left( c_{44} n_2^2 + c_{44} n_3^2 + c_{11} n_3^2 - \rho c^2 \right) p_3 = 0
\end{array} \right.
\end{align*}
\] (2.3)

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Table 3.1: Speed of longitudinal and transverse waves in orthotropic medium

<table>
<thead>
<tr>
<th>Direction of propagation along</th>
<th>Speed of Longitudinal waves m/c</th>
<th>Speed of Transverse waves m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>x1-axis</td>
<td>$\sqrt{\frac{c_{11}}{\rho}}$</td>
<td>$\sqrt{\frac{c_{66}}{\rho}}$, $\sqrt{\frac{c_{55}}{\rho}}$</td>
</tr>
<tr>
<td>x2-axis</td>
<td>$\sqrt{\frac{c_{22}}{\rho}}$</td>
<td>$\sqrt{\frac{c_{66}}{\rho}}$, $\sqrt{\frac{c_{44}}{\rho}}$</td>
</tr>
<tr>
<td>x3-axis</td>
<td>$\sqrt{\frac{c_{33}}{\rho}}$</td>
<td>$\sqrt{\frac{c_{55}}{\rho}}$, $\sqrt{\frac{c_{44}}{\rho}}$</td>
</tr>
<tr>
<td>x1x2-plane</td>
<td>$v_1$ and $v_2$</td>
<td>$\sqrt{\frac{c_{55}n_1^2 + c_{66}n_2^2}{\rho}}$</td>
</tr>
<tr>
<td>x2x3-plane</td>
<td>$v_3$ and $v_4$</td>
<td>$\sqrt{\frac{c_{66}n_2^2 + c_{55}n_3^2}{\rho}}$</td>
</tr>
<tr>
<td>x1x3-plane</td>
<td>$v_5$ and $v_6$</td>
<td>$\sqrt{\frac{c_{66}n_1^2 + c_{55}n_3^2}{\rho}}$</td>
</tr>
</tbody>
</table>

The speed of the longitudinal and transverse waves along the co-ordinate axes and planes are shown in Table 2.2. Where,

$$v_1 = \sqrt{\frac{2c_{11} + 2\rho \frac{\Omega^2}{k^2} \pm \sqrt{\left(2c_{11} + 2\rho \frac{\Omega^2}{k^2}\right)^2 - 4\left(c_{66}^2 + \rho^2 \frac{\Omega^2}{k^2} - 2\rho \frac{\Omega^2}{k^2} c_{66}\right)}}}{2 \rho}$$

Rotational effect allows the longitudinal wave to move along the axis of symmetry/rotation and planes containing it whereas transverse waves don’t propagate in the planes containing axis of rotation.

**ORTHOTROPIC MEDIUM**

**Basic equations and solution of the problem for the non-rotating material:** It is known that the generalized Hooke’s law for orthotropic medium is [3]:

$$\sigma_{ij} = \begin{bmatrix} \epsilon_{11} \epsilon_{12} \epsilon_{22} \epsilon_{23} \epsilon_{33} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{22} & 0 & 0 & 0 \\ c_{22} & c_{23} & c_{33} & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 \\ 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \end{bmatrix} \tag{3.1}$$

The equation of motion in the absence of body forces in an elastic medium by using the constitutive equations for the orthotropic material is

$$\begin{align*}
(p c^2 - c_{11} n_1^2 - c_{66} n_2^2 - c_{55} n_3^2) p_1 - (c_{12} + c_{66}) n_1 n_2 p_2 - (c_{13} + c_{55}) n_1 n_3 p_3 &= 0 \\
(c_{12} + c_{66}) n_1 n_2 p_1 - (p c^2 - c_{55} n_2^2 - c_{33} n_3^2) p_2 + (c_{23} + c_{55}) n_2 n_3 p_3 &= 0 \\
(c_{13} + c_{55}) n_1 n_3 p_1 + (c_{23} + c_{55}) n_3 n_3 p_2 - (p c^2 - c_{55} n_3^2 - c_{33} n_3^2) p_3 &= 0
\end{align*} \tag{3.2}$$

Speed of longitudinal and transverse waves in the medium is presented in the Table 3.1.
Table 3.2: Rotational effects on the longitudinal and transverse waves in orthotropic medium

<table>
<thead>
<tr>
<th>Direction of propagation along</th>
<th>Speed of Longitudinal waves m/c</th>
<th>Speed of Transverse waves m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>x₁-axis</td>
<td>Nil</td>
<td>( \sqrt{\frac{c_{55}}{\rho}} )</td>
</tr>
<tr>
<td>x₂-axis</td>
<td>Nil</td>
<td>( \sqrt{\frac{c_{66}}{\rho}} )</td>
</tr>
<tr>
<td>x₃-axis</td>
<td>( \sqrt{\frac{c_{33}}{\rho}} )</td>
<td>( \sqrt{\frac{c_{44} + c_{55} + \Gamma^2}{2\rho} - \sqrt{\frac{(c_{44} - c_{55})^2 + 2(c_{66} + c_{55})\Gamma^2}{\rho}} )</td>
</tr>
<tr>
<td>x₁x₂-plane</td>
<td>Nil</td>
<td>( \sqrt{\frac{c_{66}n_1^2 + c_{44}n_2^2}{\rho}} )</td>
</tr>
<tr>
<td>x₁x₃-plane</td>
<td>Nil</td>
<td>Nil</td>
</tr>
<tr>
<td>x₂x₃-plane</td>
<td>Nil</td>
<td>Nil</td>
</tr>
</tbody>
</table>

where,

\[
v_i = \sqrt{\frac{c_{ii}n_i^2 + c_{66}n_2^2 + c_{22}n_3^2}{2\rho} - \frac{1}{2\rho} \left( \frac{(c_{11}n_1^2 + c_{45} + c_{35}n_3^2)^2}{4(c_{44} - c_{55})^2 + 2(c_{66} + c_{55})\Gamma^2} + 2c_{66}n_1^2n_2^2 + c_{22}c_{66}n_2^2 \right) - 4c_{66}c_{66}n_2^2n_2^2 + c_{22}c_{66}n_2^2}
\]

In a similar way \( v_3, v_4, v_5, \) and \( v_6 \) can be calculated.

From the Table 3.1 it is observed that along the axes one longitudinal and two transverse waves, whereas in coordinate planes one transverse and two longitudinal waves exist.

Rotational effects on the elastic wave propagation in orthotropic medium: The equation of motion in the absence of body forces in rotating orthotropic medium can be written as follows

\[
\begin{align*}
\left( \rho c^2 - c_{11}n_1^2 - c_{55}n_2^2 - c_{35}n_3^2 \right) p_1 - \frac{\Omega^2}{k} + \left( 2i\rho \frac{\Omega}{k} + (c_{11} + c_{66})n_1n_2 \right) p_2 - (e_{15} + e_{15})n_2n_3p_3 &= 0 \\
2i\rho \frac{\Omega}{k} - (c_{13} + c_{33})n_1n_3p_1 + \left( \rho c^2 - c_{11}n_1^2 - c_{33}n_2^2 - c_{44}n_3^2 \right) p_2 - (e_{25} + c_{44})n_2n_3p_3 &= 0 \\
(c_{13} + c_{33})n_1n_3p_1 + (c_{23} + c_{44})n_2n_3p_2 - \left( \rho c^2 - c_{11}n_1^2 - c_{33}n_2^2 - c_{44}n_3^2 \right) p_3 &= 0
\end{align*}
\]  

(3.3)

Speed of Longitudinal and transverse waves in rotating orthotropic medium are shown in the Table 3.2

It is noted that rotational effect restricts the longitudinal waves to move only along the axis of rotation, whereas the transverse waves don’t exist in planes containing axis of rotation.

**MONOCLINIC MEDIUM**

Elastic wave propagation in non-rotating monoclinic medium: Generalized Hook’s law for monoclinic material, where \( x_2 = 0 \) is the plane of symmetry, can be written as follows [4]:

\[
\begin{align*}
\left( \rho c^2 - c_{11}n_1^2 - c_{55}n_2^2 - c_{35}n_3^2 \right) p_1 - \frac{\Omega^2}{k} + \left( 2i\rho \frac{\Omega}{k} + (c_{11} + c_{66})n_1n_2 \right) p_2 - (e_{15} + e_{15})n_2n_3p_3 &= 0 \\
2i\rho \frac{\Omega}{k} - (c_{13} + c_{33})n_1n_3p_1 + \left( \rho c^2 - c_{11}n_1^2 - c_{33}n_2^2 - c_{44}n_3^2 \right) p_2 - (e_{25} + c_{44})n_2n_3p_3 &= 0 \\
(c_{13} + c_{33})n_1n_3p_1 + (c_{23} + c_{44})n_2n_3p_2 - \left( \rho c^2 - c_{11}n_1^2 - c_{33}n_2^2 - c_{44}n_3^2 \right) p_3 &= 0
\end{align*}
\]  

(3.3)
Table 4.1: Longitudinal and transverse wave speed in a stationary monoclinic medium

<table>
<thead>
<tr>
<th>Direction of propagation along</th>
<th>Speed of longitudinal wave m/s</th>
<th>Speed of transverse wave m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>x1-axis</td>
<td>Nil</td>
<td>$c_T = \sqrt{\frac{c_{L1}^2}{\rho}}$</td>
</tr>
<tr>
<td>x2-axis</td>
<td>$c_L = \sqrt{\frac{c_{L2}^2}{\rho}}$</td>
<td>$v_{11}$ and $v_{12}$</td>
</tr>
<tr>
<td>x3-axis</td>
<td>Nil</td>
<td>$c_T = \sqrt{\frac{c_{T1}^2}{\rho}}$</td>
</tr>
<tr>
<td>x1x2-plane</td>
<td>Nil</td>
<td>Nil</td>
</tr>
<tr>
<td>x2x3-plane</td>
<td>Nil</td>
<td>Nil</td>
</tr>
<tr>
<td>x1x3-plane</td>
<td>$v_{11}$ and $v_{12}$</td>
<td>$v_{13}$</td>
</tr>
</tbody>
</table>

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{21} \\
\sigma_{31}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & c_{44} & c_{46} & c_{66} \\
0 & 0 & 0 & c_{46} & c_{66} & c_{46} \\
0 & 0 & 0 & c_{66} & c_{46} & c_{46}
\end{bmatrix}
\] (4.1)

The equation of motion in the component form can be written as follows

\[
\begin{aligned}
\left\{ c_{11} n_1^2 + 2 c_{12} n_1 n_2 + c_{66} n_2^2 + c_{46} n_1^2 - \rho c_n^2 \right\} + & \left\{ (c_{12} + c_{66}) n_1 n_2 + (c_{46} + c_{66}) n_2 n_3 \right\} p_2 \\
+ & \left\{ (c_{12} + c_{66}) n_1 n_2 + (c_{46} + c_{66}) n_2 n_3 \right\} p_1 = 0
\end{aligned}
\] (4.2)

Table 4.1 shows the speed of the longitudinal and transverse waves for monoclinic medium. Where,

\[
v_{11} = c_{11} = \sqrt{\frac{c_{11} n_1^2 + (c_{12} + 2 c_{66}) n_1 n_2 + 3 c_{12} n_2 n_3 + c_{66} n_3^2}{\rho}}
\]
\[
v_{12} = c_{12} = \sqrt{\frac{c_{12} n_2^2 + (c_{11} + 2 c_{66}) n_1 n_2 + 3 c_{11} n_1 n_3 + c_{66} n_3^2}{\rho}}
\]
\[
v_{13} = c_{13} = \sqrt{\frac{(c_{44} + c_{66}) + (c_{44} - 2 c_{66}) n_1^2 + 4 c_{44}}{2 \rho}}
\]
\[
v_{21} = c_{21} = \sqrt{\frac{(c_{44} + c_{66}) - (c_{44} - 2 c_{66}) n_1^2 + 4 c_{44}}{2 \rho}}
\]
\[
v_{22} = c_{22} = \sqrt{\frac{c_{22} n_2^2 + (c_{23} + 2 c_{66}) n_2 n_3 + 3 c_{23} n_1 n_3 + c_{66} n_3^2}{\rho}}
\]
\[
v_{23} = c_{23} = \sqrt{\frac{(c_{44} + c_{66}) + (c_{44} - 2 c_{66}) n_1^2 + 4 c_{44}}{2 \rho}}
\]
\[
v_{31} = c_{31} = \sqrt{\frac{c_{31} n_1^2 + (c_{32} + 2 c_{66}) n_1 n_2 + 3 c_{32} n_2 n_3 + c_{66} n_3^2}{\rho}}
\]
\[
v_{32} = c_{32} = \sqrt{\frac{c_{32} n_2^2 + (c_{31} + 2 c_{66}) n_1 n_2 + 3 c_{31} n_1 n_3 + c_{66} n_3^2}{\rho}}
\]
\[
v_{33} = c_{33} = \sqrt{\frac{(c_{44} + c_{66}) - (c_{44} - 2 c_{66}) n_1^2 + 4 c_{44}}{2 \rho}}
\]

and.

Elastic wave propagation in rotating monoclinic medium: The equation of motion in the absence of body forces in the rotating medium can be written in component form as follows,

\[
\begin{aligned}
\left\{ c_{11} n_1^2 + 2 c_{12} n_1 n_2 + c_{66} n_2^2 + c_{46} n_1^2 - \rho c_n^2 \right\} + & \left\{ (c_{12} + c_{66}) n_1 n_2 + (c_{46} + c_{66}) n_2 n_3 \right\} p_2 \\
+ & \left\{ (c_{12} + c_{66}) n_1 n_2 + (c_{46} + c_{66}) n_2 n_3 \right\} p_1 = 0
\end{aligned}
\] (4.3)
Table 4.2: Longitudinal and transverse wave speed in a rotating monoclinic medium

<table>
<thead>
<tr>
<th>Direction of propagation along</th>
<th>Speed of longitudinal wave m/s</th>
<th>Speed of transverse wave m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$-axis</td>
<td>Nil</td>
<td>$c_{T1}$</td>
</tr>
<tr>
<td>$x_2$-axis</td>
<td>Nil</td>
<td>$\sqrt{\frac{\Omega^2}{\rho}}$</td>
</tr>
<tr>
<td>$x_3$-axis</td>
<td>Nil</td>
<td>$\sqrt{\frac{\Omega^2}{\rho}}$</td>
</tr>
<tr>
<td>$x_1x_2$-plane</td>
<td>Nil</td>
<td>Nil</td>
</tr>
<tr>
<td>$x_1x_3$-plane</td>
<td>Nil</td>
<td>Nil</td>
</tr>
<tr>
<td>$x_2x_3$-plane</td>
<td>Nil</td>
<td>Nil</td>
</tr>
</tbody>
</table>

$c_{T1} = \sqrt{\frac{\Omega^2}{\rho}}$ 
$c_{T2} = \sqrt{\frac{\Omega^2}{\rho}}$

Table 5.1: Longitudinal and transverse wave speed in anisotropic medium

<table>
<thead>
<tr>
<th>Direction of propagation along</th>
<th>Speed of longitudinal wave m/s</th>
<th>Speed of transverse wave m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$-axis</td>
<td>Nil</td>
<td>$v_{T1}$</td>
</tr>
<tr>
<td>$x_2$-axis</td>
<td>Nil</td>
<td>$v_{T2}$</td>
</tr>
<tr>
<td>$x_3$-axis</td>
<td>Nil</td>
<td>$v_{T3}$</td>
</tr>
<tr>
<td>$x_1x_2$-plane</td>
<td>$v_{L1}$</td>
<td>Nil</td>
</tr>
<tr>
<td>$x_1x_3$-plane</td>
<td>$v_{L2}$</td>
<td>Nil</td>
</tr>
<tr>
<td>$x_2x_3$-plane</td>
<td>$v_{L3}$</td>
<td>Nil</td>
</tr>
</tbody>
</table>

Table 4.2 shows the speed of the longitudinal and transverse waves in rotating monoclinic medium.

It is clear from Table 4.2 that longitudinal waves exist, neither along the coordinate axis nor in the coordinate planes. Two transverse waves propagate when $\frac{\Omega^2}{k^2} \leq \frac{c_{ph}}{\rho}$ otherwise only one transverse wave propagates with constant speed $\sqrt{\frac{c_{ph}}{\rho}}$. It is also noted that transverse wave neither exist along the axis of rotation nor along the axis of symmetry.

**ANISOTROPIC MEDIUM**

Elastic wave propagation in general anisotropic medium: The constitutive equations for anisotropic medium are as follows,

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{31} \\
\sigma_{12}
\end{bmatrix}
= \begin{bmatrix}
\epsilon_{11} & \epsilon_{12} & \epsilon_{13} & \epsilon_{14} & \epsilon_{15} & \epsilon_{16} \\
\epsilon_{21} & \epsilon_{22} & \epsilon_{23} & \epsilon_{24} & \epsilon_{25} & \epsilon_{26} \\
\epsilon_{31} & \epsilon_{32} & \epsilon_{33} & \epsilon_{34} & \epsilon_{35} & \epsilon_{36} \\
\epsilon_{41} & \epsilon_{42} & \epsilon_{43} & \epsilon_{44} & \epsilon_{45} & \epsilon_{46} \\
\epsilon_{51} & \epsilon_{52} & \epsilon_{53} & \epsilon_{54} & \epsilon_{55} & \epsilon_{56} \\
\epsilon_{61} & \epsilon_{62} & \epsilon_{63} & \epsilon_{64} & \epsilon_{65} & \epsilon_{66}
\end{bmatrix}
\]

(5.1)

The equation of motion in the component form can be written as,

\[
\begin{align*}
\{c_{ij}n_i^2 + c_{m}n_i^2 + c_{m}n_i^2 + 2c_{11}n_in_j + 2c_{12}n_in_j + 2c_{13}n_in_j - \rho c^2\}p_i + \\
\{c_{12}n_i^2 + c_{46}n_i^2 + c_{46}n_i^2 + (c_{14} + c_{46})n_in_j + (c_{16} + c_{46})n_in_j + c_{16}n_j + c_{m}n_j + c_{m}n_j\}p_2 + \\
\{c_{3}n_i^2 + c_{14}n_i^2 + c_{14}n_i^2 + (c_{14} + c_{16})n_in_j + (c_{16} + c_{14})n_in_j + c_{16}n_j + c_{m}n_j + c_{m}n_j\}p_3 + 0, \\
\{c_{12}n_i^2 + c_{46}n_i^2 + c_{46}n_i^2 + (c_{14} + c_{46})n_in_j + (c_{16} + c_{46})n_in_j + c_{16}n_j + c_{m}n_j + c_{m}n_j\}p_4 + \\
\{c_{16}n_i^2 + c_{22}n_i^2 + c_{22}n_i^2 + 2c_{46}n_in_j + 2c_{46}n_in_j + c_{22}n_j + c_{m}n_j - \rho c^2\}p_5 + \\
\{c_{14}n_i^2 + c_{14}n_i^2 + c_{14}n_i^2 + (c_{14} + c_{46})n_in_j + (c_{16} + c_{46})n_in_j + c_{16}n_j + c_{m}n_j + c_{m}n_j\}p_6 + 0, \\
\{c_{3}n_i^2 + c_{46}n_i^2 + c_{46}n_i^2 + (c_{14} + c_{46})n_in_j + (c_{16} + c_{46})n_in_j + c_{16}n_j + c_{m}n_j + c_{m}n_j\}p_7 + \\
\{c_{12}n_i^2 + c_{12}n_i^2 + c_{12}n_i^2 + 2c_{46}n_in_j + 2c_{46}n_in_j + c_{22}n_j + c_{m}n_j - \rho c^2\}p_8 + \\
\{c_{3}n_i^2 + c_{46}n_i^2 + c_{46}n_i^2 + (c_{14} + c_{46})n_in_j + (c_{16} + c_{46})n_in_j + c_{16}n_j + c_{m}n_j + c_{m}n_j\}p_9 + 0.
\end{align*}
\]

(5.2)
Table 5.1 shows the speed of longitudinal and transverse waves in general anisotropic medium. Where,

\[
v_{L1} = \left\{ \frac{(c_{11} + c_{16}) n_1^2 + (c_{12} + c_{16}) n_2^2 + 2(c_{15} + c_{16}) n_3 n_3}{2p} \right\}^{\frac{1}{2}} - 4 \frac{c_{11} n_1^2 + c_{15} n_1^2 + 2c_{16} n_2 n_3}{n_3} \pm \frac{\left( c_{12} + c_{16} \right) n_1^2 + \left( c_{13} + c_{16} \right) n_2^2 + 2(c_{14} + c_{16}) n_3 n_3}{2p} - 4 \frac{c_{12} n_1^2 + c_{16} n_1^2 + 2c_{13} n_2 n_3}{n_3} \pm \left( c_{13} + c_{16} \right) n_2^2 + \left( c_{14} + c_{16} \right) n_3 n_3 \right\}^{\frac{1}{2}}
\]

exist if \( \frac{p_2}{p_1} = \frac{c_{11} n_1^2 + c_{12} n_2^2 + (c_{13} + c_{16}) n_3 n_3}{c_{11} n_1^2 + c_{14} n_1^2 + (c_{15} + c_{16}) n_3 n_3} \) holds, otherwise not.

\[
v_{T1} = \left\{ \frac{(c_{11} + c_{16}) n_1^2 + (c_{12} + c_{16}) n_2^2 + 2(c_{15} + c_{16}) n_3 n_3}{2p} \right\}^{\frac{1}{2}} - 4 \frac{c_{11} n_1^2 + c_{15} n_1^2 + 2c_{16} n_2 n_3}{n_3} \pm \frac{\left( c_{12} + c_{16} \right) n_1^2 + \left( c_{13} + c_{16} \right) n_2^2 + 2(c_{14} + c_{16}) n_3 n_3}{2p} - 4 \frac{c_{12} n_1^2 + c_{16} n_1^2 + 2c_{13} n_2 n_3}{n_3} \pm \left( c_{13} + c_{16} \right) n_2^2 + \left( c_{14} + c_{16} \right) n_3 n_3 \right\}^{\frac{1}{2}}
\]

exist if \( \frac{p_2}{p_3} = \frac{c_{11} n_1^2 + c_{12} n_2^2 + (c_{13} + c_{16}) n_3 n_3}{c_{11} n_1^2 + c_{14} n_1^2 + (c_{15} + c_{16}) n_3 n_3} \) holds, otherwise not.

\[
v_{T2} = \left\{ \frac{(c_{11} + c_{16}) n_1^2 + (c_{12} + c_{16}) n_2^2 + 2(c_{15} + c_{16}) n_3 n_3}{2p} \right\}^{\frac{1}{2}} - 4 \frac{c_{11} n_1^2 + c_{15} n_1^2 + 2c_{16} n_2 n_3}{n_3} \pm \frac{\left( c_{12} + c_{16} \right) n_1^2 + \left( c_{13} + c_{16} \right) n_2^2 + 2(c_{14} + c_{16}) n_3 n_3}{2p} - 4 \frac{c_{12} n_1^2 + c_{16} n_1^2 + 2c_{13} n_2 n_3}{n_3} \pm \left( c_{13} + c_{16} \right) n_2^2 + \left( c_{14} + c_{16} \right) n_3 n_3 \right\}^{\frac{1}{2}}
\]

exist if \( \frac{p_2}{p_3} = \frac{c_{11} n_1^2 + c_{12} n_2^2 + (c_{13} + c_{16}) n_3 n_3}{c_{11} n_1^2 + c_{14} n_1^2 + (c_{15} + c_{16}) n_3 n_3} \) holds, otherwise not.

\[
v_{T3} = \left\{ \frac{(c_{16} + c_{13}) n_1^2 + 2c_{16} n_2 n_3}{2p} \right\}^{\frac{1}{2}} - 4 \frac{c_{16} n_1^2 + c_{13} n_1^2 + 2c_{16} n_2 n_3}{n_3} \pm \left( c_{13} + c_{16} \right) n_2^2 + \left( c_{14} + c_{16} \right) n_3 n_3 \right\}^{\frac{1}{2}}
\]

exist if \( \frac{p_2}{p_3} = \frac{c_{13} n_3^2 + c_{16} n_3^2 + (c_{14} + c_{16}) n_2 n_2}{c_{13} n_3^2 + c_{16} n_3^2 + (c_{14} + c_{16}) n_2 n_2} \) holds, otherwise not.

\[
v_{T4} = \left\{ \frac{(c_{16} + c_{13}) n_1^2 + 2c_{16} n_2 n_3}{2p} \right\}^{\frac{1}{2}} - 4 \frac{c_{16} n_1^2 + c_{13} n_1^2 + 2c_{16} n_2 n_3}{n_3} \pm \left( c_{13} + c_{16} \right) n_2^2 + \left( c_{14} + c_{16} \right) n_3 n_3 \right\}^{\frac{1}{2}}
\]

exist if \( \frac{p_2}{p_3} = \frac{c_{13} n_3^2 + c_{16} n_3^2 + (c_{14} + c_{16}) n_2 n_2}{c_{13} n_3^2 + c_{16} n_3^2 + (c_{14} + c_{16}) n_2 n_2} \) holds, otherwise not.
Table 5.2: Longitudinal and transverse wave speed in rotating general anisotropic medium

<table>
<thead>
<tr>
<th>Direction of propagation along</th>
<th>Speed of longitudinal wave m/s</th>
<th>Speed of transverse wave m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>x1-axis</td>
<td>Nil</td>
<td>Nil</td>
</tr>
<tr>
<td>x2-axis</td>
<td>Nil</td>
<td>Nil</td>
</tr>
<tr>
<td>x3-axis</td>
<td>Nil</td>
<td>Nil</td>
</tr>
<tr>
<td>x1x2-plane</td>
<td>Nil</td>
<td>Nil</td>
</tr>
<tr>
<td>x2x3-plane</td>
<td>Nil</td>
<td>Nil</td>
</tr>
<tr>
<td>x1x3-plane</td>
<td>Nil</td>
<td>Nil</td>
</tr>
</tbody>
</table>

Elastic wave propagation in rotating general anisotropic medium: The equation of motion in the absence of body forces in a rotating elastic medium can be written in component form as follows,

\[
\begin{align*}
\left( s_1 - \rho \omega^2 \right) k^2 - \rho \Omega^2 &= 0 \\
\left( s_2 - \rho \omega^2 \right) k^2 &= 0 \\
\left( s_3 - \rho \omega^2 \right) k^2 &= 0 \\
\{ s_p \} p_1 + \{ s_t \} p_2 + \{ s_t - \rho \omega^2 \} p_3 &= 0.
\end{align*}
\]

(5.3)

where

\[
\begin{align*}
s_1 &= c_{11} n_1^2 + c_{12} n_2^2 + c_{13} n_3^2 + 2 c_{15} n_1 n_3 + 2 c_{16} n_2 n_3 + 2 c_{66} n_1 \cdot n_2 \\
s_2 &= c_{22} n_2^2 + c_{12} n_1^2 + c_{23} n_3^2 + 2 c_{25} n_1 n_3 + 2 c_{55} n_2 n_3 + 2 c_{56} n_1 \cdot n_2 \\
s_3 &= c_{33} n_3^2 + c_{23} n_2^2 + c_{13} n_1^2 + 2 c_{35} n_1 n_3 + 2 c_{55} n_2 n_3 + 2 c_{56} n_1 \cdot n_2 \\
s_4 &= (c_{12} + c_{66}) n_2 n_3 + (c_{25} + c_{14}) n_1 n_3 + (c_{46} + c_{56}) n_1 n_2 + (c_{56} + c_{14}) n_2 n_3 + (c_{46} + c_{66}) n_1 n_2 + (c_{56} + c_{66}) n_1 n_3 + (c_{56} + c_{66}) n_2 n_3 \\
s_5 &= (c_{13} + c_{35}) n_1 n_3 + (c_{14} + c_{66}) n_1 n_2 + (c_{46} + c_{35}) n_2 n_3 + (c_{56} + c_{35}) n_2 n_3 + (c_{56} + c_{66}) n_1 n_2 + (c_{56} + c_{66}) n_1 n_3 + (c_{56} + c_{66}) n_2 n_3.
\end{align*}
\]

It is observed that due to the rotation there do not propagate any longitudinal and transverse wave in general anisotropic material, neither along the coordinate axes nor in the coordinate plane.

It is an interesting that in rotating general anisotropic medium neither longitudinal nor transverse wave propagate along co-ordinate axes and planes.

CONCLUSION

In isotropic medium longitudinal and transverse waves propagates in all directions simultaneously whereas in rotating medium the longitudinal wave propagate along the axis of rotation and perpendicular to the plane of the axis of rotation but the transverse wave propagate everywhere except in the planes containing the axes of rotation. It is further observed that the results of Table 1.1 are in agreement with Schoenberg and Censor [1].

In transversely isotropic mediums longitudinal wave propagates only along the axis of rotation and plane containing the axis of rotation whereas transverse wave does not propagates in the plane containing the axis of rotation. Rotational effect also reduces the number of waves along the coordinate planes and axes. It is further observed that the speed of the longitudinal and transverse waves depends upon the frequency of rotation of the medium as shown in Table 2.2.

In orthotropic medium it is observed that there is only longitudinal wave propagate along the axis of rotation whereas transverse wave does not propagates in the plane containing the axis of rotation. Rotational effect also effected the speed of the wave propagate along the axis of rotation.

In monoclinic medium it is observed that the rotational effect does not allow the longitudinal wave to propagate along the co-ordinate axes and planes whereas transverse wave propagate along an axis.

In general anisotropic medium rotational effect does not allow the longitudinal and transverse waves to propagate along the co-ordinate axes and planes.

REFERENCES