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Focal Curve of Biharmonic Curves in the Special Three-Dimensional ϕ-Ricci Symmetric Para-Sasakian Manifold P

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Abstract: In this paper, we study biharmonic curves in the special three-dimensional ϕ-Ricci symmetric para-Sasakian manifold P. Finally, we construct parametric equations of focal curve of biharmonic curves in the special three-dimensional ϕ-Ricci symmetric para-Sasakian manifold P.

Mathematics Subject Classifications: 53C41, 53A10
Key words: Biharmonic curve · Para-Sasakian manifold · Focal curve

INTRODUCTION

The differential geometry of space curves is a classical subject which usually relates geometrical intuition with analysis and topology. For any unit speed curve γ, the focal curve C_γ is defined as the centers of the osculating spheres of γ. Since the centerof any sphere tangent to at a point lies on the normal plane to γ at that point, the focal curve of γ may be parameterized using the Frenet frame $(\mathbf{t}(s),\mathbf{n}_1(s),\mathbf{n}_2(s))$ of γ as follows:

$$C_\gamma(s) = (\gamma + c_1\mathbf{n}_1+c_2\mathbf{n}_2)(s),$$

Where the coefficients c_1, c_2 are smooth functions that are called focal curvatures of γ.

The aim of this paper is to study biharmonic curves in the special three-dimensional ϕ-Ricci symmetric para-Sasakian manifold P.

A smooth map ϕ: $N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2}\left|\mathrm{T}(\phi)\right|^2 dv_h,$$

Where $\mathrm{T}(\phi) := \mathrm{tr}\nabla^\phi d\phi$ is the tension field of ϕ

The Euler--Lagrange equation of the bienergy is given by $\mathrm{T}_2(\phi) = 0$. Here the section $\mathrm{T}_2(\phi)$ is defined by

$$\mathrm{T}_2(\phi) = -\Delta_\phi \mathrm{T}(\phi) + \mathrm{tr}R(\mathrm{R}(\phi),d\phi)d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study biharmonic curves in the special three-dimensional ϕ-Ricci symmetric para-Sasakian manifold P. Finally, we construct parametric equations of focal curve of biharmonic curves in the special three-dimensional ϕ-Ricci symmetric para-Sasakian manifold P.

Preliminaries: An n-dimensional differentiable manifold M is said to admit an almost para-contact Riemannian structure (ϕ,ξ,η,g), where ϕ is a ξ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi\xi = 0, \eta(\xi) = 1, g(X,\xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = X - \eta(X)\xi, \tag{2.2}$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

for any vector fields X,Y on M.

In addition, if (ϕ,ξ,η,g) satisfy the equations

$$d\eta = 0,\ \nabla_X\xi = \phi X, \tag{2.4}$$

$$(\nabla_X\phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, X,Y \in \chi(M), \tag{2.5}$$

then M is called a para-Sasakian manifold or, briefly a P–Sasakian manifold. In particular, a P–Sasakian manifold M is called a special para-Sasakian manifold or briefly a SP–Sasakian manifold if M admits a 1-form η satisfying

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$$(\nabla_X \eta)Y = -g(X,Y) + \eta(X)\eta(Y). \tag{2.6}$$

It is known [16] that in a P–Sasakian manifold the following relations hold:

$$S(X,\xi) = -(n-1)\eta(X),$$

$$Q\xi = -(n-1)\xi,$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi,$$

$$R(\xi,X)\xi = X - \eta(X)\xi,$$

$$\eta(R(X,Y)Z) = \eta(Y)g(X,Z) - \eta(X)g(Y,Z),$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y),$$

for any vector fields X, Y, Z on M.

A para-Sasakian manifold is said to be Einstein if the Ricci tensor S is of the form

$$S(X,Y) = \lambda g(X,Y)$$

Where λ is a constant.

Special Three-Dimensional ϕ-Ricci Symmetric Para-Sasakian Manifold P

Definition 3.1: A para-Sasakian manifold M is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0,$$

for all vector fields X,Y,Z,W orthogonal to ξ. This notion was introduced by Takahashi [16], for a Sasakian manifold.

Definition 3.2: A para-Sasakian manifold M is said to be ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0,$$

for all vector fields X,Y,Z,W on M.

Definition 3.3: A para-Sasakian manifold M is said to be ϕ-Ricci symmetric if the Ricci operator satisfies

$$\phi^2((\nabla_X Q)(Y)) = 0,$$

for all vector fields X and Y on M and $S(X,Y) = g(QX,Y)$.

If X,Y are orthogonal to ξ, then the manifold is said to be locally ϕ-Ricci symmetric.

We consider the three-dimensional manifold

$$\mathrm{P} = \left\{ (x^1,x^2,x^3) \in \mathrm{R}^3 : (x^1,x^2,x^3) \neq (0,0,0) \right\},$$

Where (x^1,x^2,x^3) are the standard coordinates in R^3. We choose the vector fields

$$\mathbf{e}_1 = e^{x^1}\frac{\partial}{\partial x^2},\ \mathbf{e}_2 = e^{x^1}\left(\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3}\right),\ \mathbf{e}_3 = -\frac{\partial}{\partial x^1} \tag{3.1}$$

are linearly independent at each point of P. Let g be the Riemannian metric defined by

$$g(\mathbf{e}_1,\mathbf{e}_1) = g(\mathbf{e}_2,\mathbf{e}_2) = g(\mathbf{e}_3,\mathbf{e}_3) = 1, \tag{3.2}$$

$$g(\mathbf{e}_1,\mathbf{e}_2) = g(\mathbf{e}_2,\mathbf{e}_3) = g(\mathbf{e}_1,\mathbf{e}_3) = 0.$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z,\mathbf{e}_3) \mathrm{for any} Z \in \chi(\mathrm{P}).$$

Let be the (1,1) tensor field defined by

$$\phi(\mathbf{e}_1) = \mathbf{e}_2, \phi(\mathbf{e}_2) = \mathbf{e}_1, \phi(\mathbf{e}_3) = 0. \tag{3.3}$$

Then using the linearity of and g we have

$$\eta(e_3) = 1, \tag{3.4}$$

$$\phi^2(Z) = Z - \eta(Z)\mathbf{e}_3, \tag{3.5}$$

$$g(\phi Z,\phi W) = g(Z,W) - \eta(Z)\eta(W) \tag{3.6}$$

for any $Z,W \in \chi(\mathrm{P})$. Thus for $\mathbf{e}_3 = \xi$, (ϕ,ξ,η,g) defines an almost para-contact metric structure on P.

Let ∇ be the Levi-Civita connection with respect to g. Then, we have

$$\mathbf{e}_1,\mathbf{e}_2] = 0, [\mathbf{e}_1,\mathbf{e}_3] = \mathbf{e}_1, [\mathbf{e}_2,\mathbf{e}_3] = \mathbf{e}_2.$$

Taking $\mathbf{e}_3 = \xi$ and using the Koszul's formula, we obtain

$$\nabla_{\mathbf{e}_1}\mathbf{e}_1 = -\mathbf{e}_3, \nabla_{\mathbf{e}_1}\mathbf{e}_2 = 0, \quad \nabla_{\mathbf{e}_1}\mathbf{e}_3 = \mathbf{e}_1,$$

$$\nabla_{\mathbf{e}_2}\mathbf{e}_1 = 0, \quad \nabla_{\mathbf{e}_2}\mathbf{e}_2 = -\mathbf{e}_3, \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_2, \tag{3.7}$$

$$\nabla_{\mathbf{e}_3}\mathbf{e}_1 = 0, \quad \nabla_{\mathbf{e}_3}\mathbf{e}_2 = 0, \quad \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0.$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k,\ R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l),$$

Where the indices i,j,k and l take the values 1,2 and 3.

$R_{122} = -\mathbf{e}_1, R_{133} = -\mathbf{e}_1, R_{233} = -\mathbf{e}_2,$
and

$$R_{1212} = R_{1313} = R_{2323} = 1 \tag{3.8}$$

Biharmonic Curve in the Special Three-Dimensional ϕ-Ricci Symmetric Para-Sasakian Manifold P: Let us consider biharmonicity of curves in the special three-dimensional ϕ-Ricci symmetric para-Sasakian manifold P. Let $\{\mathbf{t},\mathbf{n}_1\mathbf{n}_2\}$, be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet--Serret equations:

$$\nabla_{\mathbf{t}}\mathbf{t} = \kappa\mathbf{n}_1,$$

$$\nabla_{\mathbf{t}}\mathbf{n}_1 = -\kappa\mathbf{t} + \tau\mathbf{n}_2 \tag{4.1}$$

$$\nabla_{\mathbf{t}}\mathbf{n}_2 = -\tau\mathbf{n}_1,$$

Where κ is the curvature of γ and τ its torsion and

$$g(\mathbf{t},\mathbf{t}) = 1, g(\mathbf{n}_1,\mathbf{n}_1) = 1, g(\mathbf{n}_2,\mathbf{n}_2) = 1, \tag{4.2}$$

$$g(\mathbf{t},\mathbf{n}_1) = g(\mathbf{t},\mathbf{n}_2) = g(\mathbf{n}_1,\mathbf{n}_2) = 0.$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\mathbf{t} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3,$$

$$\mathbf{n}_1 = n_1^1\mathbf{e}_1 + n_1^2\mathbf{e}_2 + n_1^3\mathbf{e}_3, \tag{4.3}$$

$$\mathbf{n}_2 = \mathbf{t} \times \mathbf{n}_1 = n_2^1\mathbf{e}_1 + n_2^2\mathbf{e}_2 + n_2^3\mathbf{e}_3.$$

Theorem 4.1: (see [12]) γ: $I \to$ P is a biharmonic curve if and only if

$$\kappa = \text{constant} \neq 0,$$
$$\kappa^2 + \tau^2 = 1, \tag{4.4}$$
$$\tau' = 0.$$

Proof: Using Frenet formulas (4.1), we have (4.4).

Theorem 4.2: [12] All of biharmonic curves in the special three-dimensional ϕ-Ricci symmetric para-Sasakian manifold P are helices.

Theorem 4.3: [12] Let γ: $I \to$P be a unit speed non-geodesic curve with constant curvature. Then, the parametricurve. Then, the parametric equations of γ are

$$x^1(s) = -s\cos\varphi + a_1,$$

$$x^2(s) = a_2 - \frac{\sin^3\varphi}{\kappa^2 - \sin^4\varphi} e^{-s\cos\varphi + a_1}([\Pi + \cos\varphi]\cos[\Pi s + a]$$

$$+[-\Pi + \cos\varphi]\sin[\Pi s + a]), \tag{4.5}$$

$$x^3(s) = C_3 - \frac{\sin^3\varphi}{\kappa^2 - \sin^4\varphi} e^{-s\cos\varphi + C_1}(-\cos\varphi\cos[\Pi s + a]$$

$$+[\Pi s + C]\sin[\Pi s + a]),$$

Where a, a_1, a_2, a_3 are constants of integration and
$\Pi = \dfrac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}$.

Focal Curve of Biharmonic Curves in the Special Three-dimensional ϕ–Ricci Symmetric Para-sasakian Manifold P: For a unit speed curve γ, the curve consisting of the centers of the osculating spheres of γ is called the parametrized focal curve of γ. The hyperplanes normal to γ at a point consist of the set of centers of all spheres tangent to γ at that point. Hence the center of the osculating spheres at that point lies in such a normal plane. Therefore, denoting the focal curve by C_γ we can write

$$C_\gamma(s) = (\gamma + c_1\mathbf{n}_1 + c_2\mathbf{n}_2)(s) \tag{5.1}$$

Where the coefficients c_1 : c_2 are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively. Further, the focal curvatures c_1, c_2 are defined by

$$c_1 = \frac{1}{\kappa}, c_2 = \frac{c_1'}{\tau}, \kappa \neq 0, \tau \neq 0. \tag{5.2}$$

Lemma 5.1: *Let $\gamma : I \to$ P be a unit speed biharmonic curve and C_γ its focal curve on* P. *Then,*

$$c_1 = \frac{1}{\kappa} = \text{constant} \text{and} c_2 = 0. \tag{5.3}$$

Proof: Using (3.3) and (5.2), we get (5.3).

Lemma 5.2: *Let $\gamma : I \to$ P be a unit speed biharmonic curve and C_γ its focal curve on* P *Then,*

$$C_\gamma(s) = (\gamma + c_1\mathbf{n}_1)(s) \tag{5.4}$$

Theorem 5.3: *Let $\gamma : I \to \mathbb{P}$ be a biharmonic curve parametrized by arc length. If C_γ is a focal curve of γ, then the parametric equations of C_γ are*

$$\begin{aligned}
\tilde{x}^1(s) &= -\frac{c_1 \sin^2\varphi}{2\kappa}s^2 + \left(\bar{a}_1 - \cos\varphi\right)s + \bar{a}_2 + a_1 \\
\tilde{x}^2(s) &= a_2 - \frac{\sin^3\varphi}{\kappa^2 - \sin^4\varphi} e^{-s\cos\varphi + C_1}(\Pi\cos[\Pi s + a] + [-\Pi + \cos\varphi]\sin[\Pi s + a]) \\
&+ \frac{c_1 \sin\varphi}{\kappa} e^{-\frac{\sin^2\varphi}{2}s^2 + \bar{a}_1 s + \bar{a}_2}\left(\Pi\sin[\Pi s + a] + \cos\varphi\cos[\Pi s + a]\right) \\
&+ \frac{c_1 \sin\varphi}{\kappa} e^{-\frac{\sin^2\varphi}{2}s^2 + \bar{a}_1 s + \bar{a}_2}\left(-\Pi\cos[\Pi s + a] + \cos\varphi\sin[\Pi s + a]\right), \\
\tilde{x}^3(s) &= a_3 - \frac{\sin^3\varphi}{\kappa^2 - \sin^4\varphi} e^{-s\cos\varphi + a_1}(-\cos\varphi\cos[\Pi s + a] + [\Pi s + a]\sin[\Pi s + a]) \\
&- \frac{c_1 \sin\varphi}{\kappa} e^{-\frac{\sin^2\varphi}{2}s^2 + \bar{a}_1 s + \bar{a}_2}\left(-\Pi\cos[\Pi s + a] + \cos\varphi\sin[\Pi s + a]\right),
\end{aligned} \tag{5.5}$$

Where $a, \bar{a}_1, \bar{a}_2, a_1, a_2, a_3$ *are constants of integration and* $\Pi = \frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}$.

Proof: Let C_γ is a focal curve of γ Recalling [12], we have

$$\mathbf{T} = \sin\varphi\cos[\Pi s + a]\mathbf{e}_1 + \sin\varphi\sin[\Pi s + a]\mathbf{e}_2 + \cos\varphi\mathbf{e}_3, \tag{5.6}$$

Where $\Pi = \frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}$ and a is a constant of integration.

On the other hand, using first equation of (4.3) we get

$$\nabla_{\mathbf{t}}\mathbf{t} = \left(t_1' + t_1 t_3\right)\mathbf{e}_1 + \left(t_2' + t_2 t_3\right)\mathbf{e}_2 + \left(t_3' - \left(t_1^2 - t_2^2\right)\right)\mathbf{e}_3. \tag{5.7}$$

From (4.1) and (5.6), we get

$$\begin{aligned}
\nabla_{\mathbf{t}}\mathbf{t} &= \sin\varphi\left(-\Pi\sin[\Pi s + a] + \cos\varphi\cos[\Pi s + a]\right)\mathbf{e}_1 + \\
&\sin\varphi\left(\Pi\cos[\Pi s + a] + \cos\varphi\sin[\Pi s + a]\right)\mathbf{e}_2 - \sin^2\varphi\mathbf{e}_3,
\end{aligned} \tag{5.8}$$

Where $\Pi = \frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}$.

Taking into account Frenet formulas (4.1), we derive that

$$\begin{aligned}
\mathbf{n}_1 &= \frac{1}{\kappa}\nabla_{\mathbf{t}}\mathbf{t} = \frac{1}{\kappa}[\left(\Pi\sin\varphi\sin[\Pi s + a] + \cos\varphi\sin\varphi\cos[\Pi s + a]\right)\mathbf{e}_1 \\
&+ \left(-\Pi\sin\varphi\cos[\Pi s + a] + \cos\varphi\sin\varphi\sin[\Pi s + a]\right)\mathbf{e}_2 - \sin^2\varphi\mathbf{e}_3].
\end{aligned} \tag{5.9}$$

Substituting (3.1) in (5.9), we arrive at

$$\mathbf{n}_1 = \frac{1}{\kappa}(-\frac{\sin^2\varphi}{2}s^2 + \bar{a}_1 s + \bar{a}_2,$$
$$e^{-\frac{\sin^2\varphi}{2}s^2+\bar{a}_1 s+\bar{a}_2}\left(\Pi\sin\varphi\sin[\Pi s + a] + \cos\varphi\sin\varphi\cos[\Pi s + a]\right)+$$
$$e^{-\frac{\sin^2\varphi}{2}s^2+\bar{a}_1 s+\bar{a}_2}\left(-\Pi\sin\varphi\cos[\Pi s + a] + \cos\varphi\sin\varphi\sin[\Pi s + a]\right),- \quad (5.10)$$
$$e^{-\frac{\sin^2\varphi}{2}s^2+\bar{a}_1 s+\bar{a}_2}\left(-\Pi\sin\varphi\cos[\Pi s + a] + \cos\varphi\sin\varphi\sin[\Pi s + a]\right)),$$

Where $\bar{a}_1, \bar{a}_2$ are constants of integration.
Next, we substitute (5.10) and (4.5) into (5.4), we get (5.5). The proof is completed.

Theorem 5.3: *Let $\gamma : I \to$ P be a biharmonic curve parametrized by arc length. If C_γ is a focal curve of γ, then the parametric equations of C_γ in terms of τ are*

$$\tilde{x}^1(s) = -\frac{c_1\sin^2\varphi}{2\sqrt{1-\tau^2}}s^2 + \left(\bar{a}_1 - \cos\varphi\right)s + \bar{a}_2 + a_1$$
$$\tilde{x}^2(s) = a_2 - \frac{\sin^3\varphi}{1-\tau^2-\sin^4\varphi}e^{-s\cos\varphi+a_1}(\Omega\cos[\Omega s + a] + [-\Omega + \cos\varphi]\sin[\Omega s + a]) + \quad (5.11)$$
$$\frac{c_1\sin\varphi}{\sqrt{1-\tau^2}}e^{-\frac{\sin^2\varphi}{2}s^2+\bar{a}_1 s+\bar{a}_2}\left(\Omega\sin[\Omega s + a] + \cos\varphi\cos[\Omega s + a]\right)+$$
$$\frac{c_1\sin\varphi}{\sqrt{1-\tau^2}}e^{-\frac{\sin^2\varphi}{2}s^2+\bar{a}_1 s+\bar{a}_2}\left(-\Omega\cos[\Omega s + a] + \cos\varphi\sin[\Omega s + a]\right),$$
$$\tilde{x}^3(s) = a_3 - \frac{\sin^3\varphi}{1-\tau^2-\sin^4\varphi}e^{-s\cos\varphi+a_1}(-\cos\varphi\cos[\Omega s + a] + [\Omega s + a]\sin[\Omega s + C]) -$$
$$-\frac{c_1\sin\varphi}{\sqrt{1-\tau^2}}e^{-\frac{\sin^2\varphi}{2}s^2+\bar{a}_1 s+\bar{a}_2}\left(-\Omega\cos[\Omega s + a] + \cos\varphi\sin[\Omega s + a]\right),$$

Where $a, \bar{a}_1, \bar{a}_2, a_1, a_2, a_3$ are constants of integration and $\Omega = \frac{\sqrt{\cos^2\varphi - \tau^2}}{\sin\varphi}$.

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