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An Efficient Numerical Method for Solving Linear and Nonlinear Partial Differential Equations by Combining Homotopy Analysis and Transform Method

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Abstract: In this article, we propose a reliable combination of Homotopy analysis method (HAM) and Laplace decomposition method (LDM) to solve linear and nonlinear partial differential equations effectively with less computation. The proposed method is called homotopy analysis transform method (HATM). This study represents significant features of HATM and its capability of handling linear and nonlinear partial differential equations.

Key words: Homotopy analysis method . laplace decomposition method . linear and nonlinear partial differential equations . homotopy analysis transform method

INTRODUCTION

A nonlinear phenomenon appears in a wide variety of scientific applications such as plasma physics, solid state physics, fluid dynamics and chemical kinetics. A broad range of analytical and numerical methods have been used in the analysis of these scientific models. Mathematical modeling of many physical systems leads to nonlinear ordinary and partial differential equations in various fields of physics and engineering. An effective method is required to analyze the mathematical model which provides solutions conforming to physical reality. Common analytic procedures linearize the system and assume that nonlinearities are relatively insignificant. Such assumptions sometimes strongly affect the solution with respect to the real physics of the phenomenon. Thus seeking solutions of nonlinear ordinary and partial differential equations are still significant problem that needs new techniques to develop exact and approximate solutions. Various powerful mathematical techniques such as Adomian decomposition method [1, 2], Homotopy analysis method [3-5], differential transform method [6, 7], Homotopy perturbation method [8, 9], Variational iterative method [10, 11], Laplace decomposition method [12-15], modified Laplace decomposition method [16, 17] and homotopy perturbation transform method [18] to obtain exact and approximate analytical solutions.

We use the homotopy analysis method combined with the Laplace transform for solving linear and nonlinear partial differential equations in this paper. It is worth mentioning that the proposed method is an elegant combination of the homotopy analysis method and Laplace transform. The advantage of this proposed method is its capability of combining two powerful methods for obtaining rapid convergent series partial differential equations. To the best of author knowledge no such attempt has been made to combine homotopy analysis method and Laplace transform for solving partial differential equations. This paper considers the effectiveness of the homotopy analysis transform method in solving linear and nonlinear equations.

The paper is organized as follows. In Section 2, the basic concepts of HAM is presented. Section 3 contains basic idea of HATM. Section 4, contains applications of HATM. The conclusions are given in last Section.

BASIC IDEA OF HAM

In HAM, a system can be writtenas:

$$N[u(x,t)] = 0, i = 1,2,3.... \quad (1)$$

where N is a nonlinear operator, u(x,t) is unknown function of x and t, $u_0(x,t)$ is the initial guess, $\hbar \neq 0$ an auxiliary parameter and L is a auxiliary linear operator. Also, $q \in [0,1]$ is an embedding parameter. We can construct a Homotopy as follows

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = q\hbar N[\phi(x,t;q)] \quad (2)$$

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Eq. (1), is so-called zero-order deformation equation. When q = 0, the zero-order deformation equation become

$$\phi(x,t;0) = u_0(x,t) \tag{3}$$

when q = 1, since $\hbar \neq 0$, we get final solution expression as follows

$$\phi(x,t;1) = u(x,t) \tag{4}$$

The embedding parameter q increases from 0 to 1. Using Taylor's theorem, $\phi_i(x,t;q)$ can be expanded in a power series of q as follows

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m \tag{5}$$

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m}\bigg|_{q=0} \tag{6}$$

where $u_0(x,t)$ is a initial guess, L_i is a auxiliary linear operator and $\hbar \neq 0$ is the non-zero auxiliary parameter. The power series in equation (5) is converges at q = 1. Therefore, we obtain

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \tag{7}$$

According to the equation (6), the governing equation of $u_i(x,t)$ can be derived from the zero-order deformation Eq.(2). Using m times differentiating with respective to q from the zero-order deformation equation (2) and setting q=1, we have the so-called mth-order deformation equation as:

$$L[u_m(x,t) - \chi_m u_0(x,t)] = \hbar R_m\left(\tilde{u}_{m-1}(x,t)\right) \tag{8}$$

where

$$R_m(u_{m-1}(x,t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0} \tag{9}$$

and

$$\chi_m = \begin{cases} 1, & m > 1 \\ 0, & m \leq 1 \end{cases} \tag{10}$$

HOMOTOPY ANALYSIS TRANSFORM METHOD

We consider a general nonlinear, non-homogenous partial differential equation

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = f(x,t) \tag{11}$$

where L is a linear operator, N is a nonlinear operator and $f(x,t)$ is a source function. The initial conditions are also as

$$u(x,0) = g(x), u_t(x,0) = f(x) \tag{12}$$

Applying the Laplace transforms $\wp$, we obtain

$$\wp Lu(x,t) + \wp Ru(x,t) + \wp Nu(x,t) = \wp f(x,t) \tag{13}$$

Therefore, we have

$$s^2 \wp u(x,t) - su(x,0) - u_t(x,0) = \wp f(x,t) - \wp Ru(x,t) - \wp Nu(x,t),$$

Or

$$\wp u(x,t) = \frac{h(x)}{s} + \frac{f(x)}{s^2} + \frac{1}{s^2}\wp f(x,t) - \frac{1}{s^2}\wp Ru(x,t) - \frac{1}{s^2}\wp Nu(x,t) \tag{14}$$

Now we embed the HAM in Laplace transform method. Hence we may write any equation in the form

$$N[u(x,t)] = 0, i = 1,2,3 \tag{15}$$

$$N[\phi(x,t;q)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} + \frac{1}{s^2}\wp f(x,t) - \frac{1}{s^2}\wp R[\phi(x,t;q)] - \frac{1}{s^2}\wp N[\phi(x,t;q)] - \wp[\phi(x,t;q)] \tag{16}$$

where N is a nonlinear operator, u(x,t) is unknown function of x and t, $\hbar \neq 0$ an auxiliary parameter and L an auxiliary linear operator. Also, $q \in [0,1]$ is an embedding parameter. Thus, we can construct such a Homotopy such as

$$(1-q)\wp[\phi(x,t;q) - u_0(x,t)] = q\hbar N[\phi(x,t;q)] \tag{17}$$

which is a zero-order deformation equation. When q = 0, the zero-order deformation equation become

$$\phi(x,t;0) = u_0(x,t) \tag{18}$$

when q = 1, since $\hbar_i \neq 0$ we get an analytical approximate solution as follows

$$\phi(x,t;1) = u(x,t) \tag{19}$$

The embedding parameter q increases from 0 to 1. Using Taylor's theorem, $\phi_i(x,t;q)$ can be expanded in a power series of q as follows

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m \tag{20}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}\bigg|_{q=0} \tag{21}$$

If $u_{i,0}(x,t)$ guesses the initial condition, L_i is the auxiliary linear operator, $\hbar \neq 0$ is the non-Zero auxiliary parameter then the power series in equation (20) is converges at q = 1. Therefore, we have:

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \tag{22}$$

According to the Eq. (6), the governing equation of $u_i(x,t)$ can be derived from the zero-order deformation Eq. (2) Using m times differentiating with respective to q from the zero-order deformation equation (2) and setting q=1, we have the so-called m-th-order deformation equation as

$$\wp[u_m(x,t) - \chi_m u_0(x,t)] = \hbar R_m\left(\tilde{u}_{m-1}(x,t)\right) \tag{23}$$

Using the last equation the series solution is given by

$$u_m(x,t) = \chi_m u_0(x,t) + \hbar \wp^{-1}\left(R_m\left(\tilde{u}_{m-1}(x,t)\right)\right) \tag{24}$$

where

$$R_{i,m}(u_{i,m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0} \tag{25}$$

and

$$\chi_m = \begin{cases} 1, & m > 1 \\ 0, & m \leq 1 \end{cases} \tag{26}$$

APPLICATION

In order to elucidate the solution procedure of the homotopy Analysis transform method, we solve three examples in this sections which shows the effectiveness and generalizations of our proposed method.

Example 1: Consider the two-dimensional initial values problem

$$u_{tt} = \frac{1}{2}y^2 u_{xx} + \frac{1}{2}x^2 u_{yy}, x > 0, y < 1, t > 0 \tag{27}$$

with the initial conditions

$$u(x,y,0) = x^2 + y^2, u_t(x,y,0) = -(x^2 + y^2) \tag{28}$$

Applying Laplace transform, we get

$$\begin{aligned} &\wp u(x,t) - \frac{1}{s}u(x,0) - \frac{1}{s^2}u_t(x,0) \\ &- \frac{1}{s^2}\wp\left[\frac{1}{2}y^2 u_{xx} + \frac{1}{2}x^2 u_{yy}\right] = 0 \end{aligned} \tag{29}$$

Using given the initial condition Eq. (28) becomes

$$\begin{aligned} &\wp u(x,t) - \frac{1}{s^2}\wp\left[\frac{1}{2}y^2 u_{xx} + \frac{1}{2}x^2 u_{yy}\right] \\ &- \frac{x^2+y^2}{s} + \frac{x^2+y^2}{s^2} = 0 \end{aligned} \tag{30}$$

Form Eq. (30), we define a nonlinear operator as

$$\begin{aligned} N[\phi(x,t;q)] &= \wp\phi(x,t;q) - \frac{1}{s^2}\wp\left[y^2\frac{\partial^2\phi(x,t;q)}{\partial x^2} + x^2\frac{\partial^2\phi(x,t;q)}{\partial y^2}\right] \\ &- \frac{x^2+y^2}{s} + \frac{x^2+y^2}{s^2} = 0 \end{aligned} \tag{31}$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)\wp[\phi(x,t;q) - u_0(x,t)] = q\hbar N[\phi(x,t;q)] \tag{32}$$

With initial conditions, where $q \in [0,1]$ is an embedding parameter, $\hbar$ is a non-zero auxiliary function, $\wp$ is Laplace transformation operator, $u_0(x,t)$ is an initial guess of u(x,t) and $\phi(x,t;q)$ is unknown function. When q = 0 and q = 1 we have

$$\phi(x,t;0) = u_0(x,t), \phi(x,t;1) = u(x,t) \tag{33}$$

Expanding $\phi(x,t;q)$ in Taylor series with respect to q, we obtain

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m \tag{34}$$

where

$$u_m(x,t) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\phi(x,t;q)}{\partial q^{m-1}} \tag{35}$$

The above series is convergent at q = 1, then

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \tag{36}$$

The Eq. (36) must be one of the solutions of the original nonlinear equation (1), which is proved by Liao [3]. Now, we define the vector

$$\vec{u}_{m-1} = \{u_0(x,t), u_2(x,t), ...u_{m-1}(x,t)\} \tag{37}$$

The *m*th order deformation equation is

$$u_m = \chi_m u_{m-1} + \hbar \wp^{-1} R_m(\vec{u}_{m-1}) \tag{38}$$

where

$$R_m(\vec{u}_{m-1}) = \wp u_{m-1} - \frac{1}{s^2}\wp\left[y^2 \frac{\partial^2 u_{m-1}}{\partial x^2} + x^2 \frac{\partial^2 u_{m-1}}{\partial y^2}\right] + (1-\chi_m)\left(-\frac{x^2+y^2}{s} + \frac{x^2+y^2}{s^2}\right) \tag{39}$$

Using the Mathematica 5.2 package, we obtain the solution as

$$u_1(x,y,t) = -\frac{\hbar}{2}(t^2 - 2t)(x^2 + y^2) \tag{40}$$

$$u_2(x,y,t) = \frac{\hbar t}{24}\left(\hbar t^3 - 4\hbar t^2 - 12\hbar t - 12t + 24\hbar + 24)(x^2 + y^2\right) \tag{41}$$

The solution is given by

$$u(x,y,t) = \left(1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6!} - \frac{t^7}{7!} + ...\right)(x^2 + y^2) \tag{42}$$

$$u(x,y,t) = e^{-t}(x^2 + y^2) \tag{43}$$

where $\hbar = -1$

Example 2: Consider the two dimensional nonlinear inhomogeneous initial value problems

$$u_{tt} = 2x^2 + 2y^2 + \frac{15}{2}(xu_{xx}^2 + yu_{yy}^2), x > 0, y < 1, t > 0 \tag{44}$$

with the initial conditions

$$u(x,y,0) = 0, u_t(x,y,0) = 0 \tag{45}$$

Applying Laplace transforms, we get

$$\wp u(x,t) - \frac{2x^2 + 2y^2}{s} - \frac{15}{2s^2}\wp\left[xu_{xx}^2 + yu_{yy}^2\right] = 0 \tag{46}$$

We define a nonlinear operator as

$$N[\phi(x,t;q)] = \wp\phi(x,t;q) - \frac{1}{s^2}\wp\left[x\left(\frac{\partial\phi(x,t;q)}{\partial x}\right)^2 + y\left(\frac{\partial\phi(x,t;q)}{\partial y}\right)^2\right] - \frac{2x^2+2y^2}{s} \tag{47}$$

The *m*-th order deformation equation is

$$u_m = \chi_m u_{m-1} + \hbar \wp^{-1} R_m(\vec{u}_{m-1}) \tag{48}$$

where

$$R_m(\vec{u}_{m-1}) = \wp u_{m-1} - \frac{12}{5s^2}\wp\left[x\sum_{k=0}^{m-1}\frac{\partial u_k}{\partial x}\frac{\partial u_{m-1-k}}{\partial x} + y\sum_{k=0}^{m-1}\frac{\partial u_k}{\partial y}\frac{\partial u_{m-1-k}}{\partial y}\right] - (1-\chi_m)\left(\frac{2x^2+2y^2}{s}\right) \tag{49}$$

Using the Mathematica 5.2 package, we obtain the solution as

$$u_1(x,y,t) = -\hbar t^2(x^2 + y^2)$$

$$u_2(x,y,t) = -\hbar(\hbar + 1)t^2(x^2 + y^2) \tag{50}$$

$$u_3(x,y,t) = -\hbar t^2(1+2\hbar+\hbar^2+\hbar^2 t^4 y)(x^2+y^2)$$

$$\ldots\ldots$$

The solution is given by

$$u(x,y,t) = (x^2+y^2)\,t^2 + (x^2+y^2)\,t^6 + \ldots\ldots \qquad 51)$$

where $\hbar = -1$.

Example 3: Consider the inhomogeneous problem [19]

$$u_t + uu_x = 0 \qquad (52)$$

with the initial condition

$$u(x,0) = -x \qquad (53)$$

Applying Laplace transform, we have

$$\wp\, u(x,t) + \frac{1}{s}\wp\left[uu_x\right] - \frac{x}{s^2} = 0 \qquad (54)$$

We define a nonlinear operator as

$$N[\phi(x,t;q)] = \wp\,\phi(x,t;q) - \frac{1}{s^2}\wp\left[\phi(x,t;q)\frac{\partial\phi(x,t;q)}{\partial y}\right] - \frac{x}{s^2} \qquad (55)$$

The *m*-th order deformation equation is

$$u_m = \chi_m u_{m-1} + \hbar\,\wp^{-1} R_m(\vec{u}_{m-1}) \qquad (56)$$

where

$$R_m(\vec{u}_{m-1}) = \wp\, u_{m-1} - \frac{1}{s}\wp\left[\sum_{k=0}^{m-1} u_k \frac{\partial u_{m-1-k}}{\partial x}\right] - \left(1-\chi_m\right)\left(\frac{x}{s^2}\right) \qquad (57)$$

Using the Mathematica 5.2 package, to obtain the series solution

$$u_1(x,y,t) = -\hbar xt$$

$$u_2(x,y,t) = -\hbar xt(\hbar t - \hbar - 1)$$

$$u_3(x,y,t) = \hbar xt(\hbar t - \hbar - 1)^2 \qquad (58)$$

$$u_4(x,y,t) = -\hbar xt(\hbar t - \hbar - 1)^3$$

The solution is given by

$$u(x,t) = -x(1 + t + t^2 + t^3 + \ldots) \qquad (59)$$

$$u(x,t) = \frac{x}{t-1} \qquad (60)$$

where $\hbar = -1$, which is exact solution obtained as special case of homotopy perturbation transform method [19].

CONCLUSION

The main aim here is to provide the series solution of linear and nonlinear partial differential equations using Homotopy Analysis Transform Method (HATM). The solution obtained with the help of HATM is more general as compared to HPTM, ADM and VIM solution. We can easily recover all results of HPTM, ADM and VIM by assuming $\hbar = -1$. The analysis given here shows further confidence on HATM.

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