Tau Approximate Solution of Differential Algebraic Equations

F. Khaksar Haghani S. Karimi Vanani J. Sedighi Hafshejani

Department of Mathematics, Islamic Azad University, Shahrekord Branch, P.O. Box: 166, Shahrekord, Iran

Abstract: In this paper, an extension of the operational Tau method (OTM) for solving differential algebraic equations (DAEs) is presented. The proposed method converts the desired DAE to a set of algebraic equations using orthogonal polynomials as basis functions. Therefore, we present some notations and basic definitions of the Hessenberg forms of the differential algebraic equations. In addition, we state some concepts, properties and advantages of the OTM and its application for solving DAEs. Some illustrative numerical experiments are given to illustrate the capability and validity of the proposed method.

Key words: Spectral methods • Operational tau method • Differential algebraic equations • Hessenberg forms of differential algebraic equations

INTRODUCTION

Differential algebraic equations (DAEs) play a dominant role in many branches of science and engineering. DAEs have received considerable interest in recent years and have been extensively investigated and applied for many real problems which are modeled in various areas. For instances, circuit analysis, computer-aided design, power systems, simulation of mechanical systems and more general optimal control problems; thus, they have attracted the attention of numerical analysts [1-5].

The index of a DAE is a measure of the degree of singularity of the system and it is widely regarded as an indication of certain difficulties for numerical solution of ODE systems [1]. The following statements and discussion of DAEs have been described by Celik [6]. The most regular form of a DAE is as follows:

\[ F(t, y, y') = 0, \]  

(1)

\[ \frac{\partial F}{\partial y'} \] may be singular. The rank and structure of this Jacobian matrix depends, in general, on the solution \( y(t) \) and for simplicity, we shall always assume that it is independent of \( t \). The important case of a semi-explicit DAE, or an ODE with constraints, is given in the following equations:

\[ x' = f(t, x, z), \]

\[ 0 = g(t, x, z). \]  

(2)

This is a special case of equation (1). The index is 1 if \( \frac{\partial F}{\partial y'} \) is non-singular, because then one differentiation of equation (2) yields \( z' \) in principle. For the semi-explicit index-1 DAE, we can differentiate between differential variables \( x(t) \) and algebraic variables \( z(t) \). The algebraic variables may be less smooth than the differential variables by one derivative. In the general case, each component of \( y \) may contain a combination of differential and algebraic components, which make the numerical solution of such high-index problems much more difficult. The semi-explicit form is decoupled in this sense. On the other hand, any DAE can be written in the semi-explicit form (2), but with the index increased by 1 upon defining \( y' = z' \), which gives the following equations:

\[ y' = z, \]

\[ 0 = F(t, y, z). \]  

(3)

It is evident that re-writing by itself does not make the problem easier to solve. The converse transformation is also possible. Given a semi-explicit index-2 DAE system with \( z = w' \), the system is easily represented as follows:

\[ x' = f(t, x, w'), \]

\[ 0 = g(t, x, w'). \]  

(4)
Where this system is an index-1 DAE and yields exactly the same solution for $x$ as (2) above. The classes of fully-implicit index-1 DAEs of the form (1) and semi-explicit index-2 DAEs of the form (2) are therefore equivalent. We define the index of a DAE. For general DAE systems (1), the index along a solution $y(t)$ is the minimum number of differentiations of $y$ and $t$ that the system requires to solve for $y'$ uniquely in terms of $y$ and $t$ (i.e., to define an ODE for $y$). Thus, the index is defined in terms of the over-determined system as follows:

$$F(t, y, y') = 0, \quad \frac{\partial}{\partial t} F(t, y, y', y'') = 0, \quad \vdots \quad \frac{\partial^m}{\partial t^m} F(t, y, y', \ldots, y^{(m+1)}) = 0, \quad (5)$$

Where $m$ is assumed to be the smallest integer such that $y'$ in (5) can be determined in terms of $y$ and $t$. It should be noted that in practice, differentiating the system as in (5) is rarely done in a computation. However, such a definition is useful in understanding the underlying mathematical structure of the DAE system and therefore, in selecting an appropriate numerical method [5, 6].

In recent years, considerable efforts have been made to solve systems of DAEs. There are some numerical methods that solve DAEs; the important ones are the Padé approximate series [7], collocation method [8], the method which use both BDF [1, 4, 5, 7], implicit Runge-Kutta methods [1, 7, 8], Multiquadric approximation scheme [9], homotopy perturbation method [10] and Adomian decomposition method [11]. In this paper, we are interested in solving DAEs using OTM as a fast method with an easy resolvent algorithm.

Spectral methods provide a computational approach which achieved substantial popularity in the last three decades. Tau method is one of the most important spectral methods which is extensively applied for numerical solution of many problems. This method was invented by Lanczos [12] for solving ordinary differential equations (ODEs) and then the expansion of the method were done for many different problems such as partial differential equations (PDEs) [13-15], integral equations (IEs) [16], integro-differential equations (IDEs) [17] and etc. [18-21].

The organization of this paper is as follows. Section 2 gives some notations and basic definitions of the DAE forms. In Section 3, basic concepts of the OTM are given. Some orthogonal polynomials using in the OTM is presented in Section 4. To show the efficiency of the method, two illustrative numerical experiments are presented in Section 5. Finally, Section 6 consists of some obtained conclusions.

**Special Differential Algebraic Equation Forms:**

Many practical problems with higher-indices that arise in applied sciences can be considered to be a combination of more restrictive structures of ODEs coupled with constraints. In such systems, the algebraic and differential variables are explicitly identified for higher-index DAEs as well and the algebraic variables may all be eliminated using the same number of differentiations. These are called Hessenberg forms of the DAE and are given as follows [1].

**Hessenberg Index-1:**

$$x' = f(t, x, z), \quad 0 = g(t, x, z). \quad (6)$$

In the above equations, the Jacobian matrix function $\frac{\partial g}{\partial z}$ is assumed to be non-singular for all $t$. This system is also often referred to as a semi-explicit index-1 system. Semi-explicit index-1 DAEs are closely related to implicit ODEs. Using the implicit function theorem ([22, pp. 36-37]), we can actually solve for variable $z$ in constraint (6). Substituting the result in differential equation (6) yields an ODE in variable $x$.

**Hessenberg Index-2:**

$$x' = f(t, x, z), \quad 0 = g(t, x), \quad (7)$$

Where the product of Jacobians $\frac{\partial g}{\partial x}$ is non-singular for all $t$. Note the absence of the algebraic variable $z$ from constraint (7). This system is a pure index-2 DAE and all algebraic variables play the role of index-2 variables.

**Hessenberg Index-3:**

$$x' = f(t, x, y, z), \quad y' = g(t, x, y), \quad 0 = h(t, y), \quad (8)$$

Where the product of three matrix functions $\frac{\partial h}{\partial y} \frac{\partial g}{\partial y} \frac{\partial f}{\partial y}$ is non-singular. The index of a Hessenberg DAE is found, as in the general case, by differentiation. However, only algebraic constraints must be differentiated [8].
Operational Tau Method: In this section, we state some preliminaries and notations using in this work.

For any integrable functions \( \phi(t) \) and \( \phi'(t) \) on \([a,b]\), we define the scalar product \(<,>\) by

\[
<\psi(t),\phi(t)>_w = \int_a^b \psi(t)\phi(t) w(t) dt,
\]

Where \( \|\psi(t)\|_w = \int_a^b |\psi(t)|^2 w(t) dt \) and \( w(x) \) is a weight function. Let \( L_2[a,b] \) be the space of all functions \( f: [a,b] \to \mathbb{R} \), with \( \|f\|_2^2 < \infty \).

The main idea of the method is to seek a polynomial to approximate \( u(t) \in L_2[a,b] \). Let \( \{\phi_i|_{i=0}^\infty = F X\} \) be a set of arbitrary orthogonal polynomial bases defined by a lower triangular matrix \( \Phi \) and \( X = [1, t, t^2, \ldots]^T \).

**Lemma 1:** Suppose that \( u(t) \) is a polynomial as

\[
u(t) = \sum_{i=0}^\infty u_i^2 - uX_i, \]

then we have:

\[
D^r u(t) = \frac{d^r}{dt^r} u(t) = uM^r X_r, \quad r = 0, 1, 2, \ldots, \tag{9}
\]

\[
t^s u(t) = uN^s X_s, \quad s = 0, 1, 2, \ldots, \tag{10}
\]

and

\[
\int_a^x u(t) dt = uP X_t - uP X_a, \tag{11}
\]

Where \( u = [u_0, u_1, \ldots, u_m]^T, X = [1, t, t^2, \ldots]^T \) and \( M, N \) and \( P \) are infinite matrices with only nonzero elements

\[
M_{i+1,j} = i+1, N_{j+1,i} = 1, P_{j+1,i} = \frac{1}{i+1}, i = 0, 1, 2, \ldots.
\]

**Proof:** See [22].

Let us consider

\[
u(t) = \sum_{i=0}^\infty u_i^2 \phi_i(t) = u F X_i, \tag{12}
\]

to be an orthogonal series expansion of the exact solution, where \( u = [u_i]_{i=0}^\infty \) is a vector of unknown coefficients, \( \Phi X_i \) is an orthogonal basis for polynomials in \( \mathbb{R} \).

In the Tau method, the aim is to convert the linear and nonlinear terms to an algebraic system using some operational matrices.

In order to show the application of OTM, let us consider the well known form of DAEs as follows:

\[
A(t) \dot{U}(t) + B(t) U(t) = F(t), \quad \frac{\partial U}{\partial x} = U(a) = U_a, \tag{13}
\]

Where

\[
U(t) = [u_0(t), u_1(t), \ldots, u_m(t)]^T, \quad u_k(t) \in C, k = 0, 1, \ldots, m,
\]

is the state vector, \( A(t) \) and \( B(t) \) are \((m+1)\)-dimensional matrices which their entries are complex functions of \( t \).

Also

\[
U_a = [u_0(a), u_1(a), \ldots, u_m(a)]^T, \quad u_k(a) \in C, k = 0, 1, \ldots, m,
\]

\[
F(t) = [f_0(t), f_1(t), \ldots, f_m(t)]^T, \quad f_k(t) \in C, k = 0, 1, \ldots, m.
\]

Using equation (12), we can suppose that each element of the vector \( U(t) \) is as

\[
u_k(t) = u_k^T F X_t, \quad u_k = [u_k^0, u_k^1, \ldots], k = 0, 1, \ldots, m. \tag{17}
\]

Therefore, from equation (9) it is obvious that

\[
\dot{u}_k(t) = u_k^T F X_t, \quad u_k = [u_k^0, u_k^1, \ldots], k = 0, 1, \ldots, m. \tag{18}
\]

In next step, we desire to approximate each elements of matrices \( A(t) \) and \( B(t) \) in operational forms. Since, each elements of \( A(t) \) and \( B(t) \) are smooth functions therefore we can approximate them as follows:

\[
A_y(t) = \sum_{k=0}^n a_{yk} t^k, \quad B_y(t) = \sum_{k=0}^n b_{yk} t^k, \quad i, j = 0, 1, \ldots, m. \tag{19}
\]

Substituting above equations in equation (13) and using equations (10) and (18), we obtain:
Therefore
\[ \mathbf{A}(t) \mathbf{U}(t) = \mathbf{A}X_t, \quad \hat{\mathbf{A}} = \sum_{j=0}^{m} \sum_{k=0}^{n} d_{j} u_{j} F \mathbf{MN}^{k}, \quad i = 0,1,\ldots,m. \] (21)

In the same way, we have:
\[ \mathbf{B}(t) \mathbf{U}(t) = \hat{\mathbf{B}}X_t, \quad \hat{\mathbf{B}} = \sum_{j=0}^{m} \sum_{k=0}^{n} h_{j} u_{j} F \mathbf{N}^{k}, \quad i = 0,1,\ldots,m. \] (22)

Also, \( \mathbf{F}(t) \) can be considered as
\[ \mathbf{F}(t) = \bar{\mathbf{F}} \mathbf{X}_t, \quad \bar{f}_{ij} = f_{ij}, \quad i, j = 0,1,\ldots,m. \] (23)

Thus using equations (21) to (23), equation (13) is replaced by the following algebraic system,
\[ \hat{\mathbf{A}} \mathbf{F}^{-1} \mathbf{F} \mathbf{X}_t + \bar{\mathbf{B}} \mathbf{F}^{-1} \mathbf{F} \mathbf{X}_t = \bar{\mathbf{F}} \mathbf{X}_t \] (24)

So, the residual matrix \( \mathbf{R}(t) \) of equation (13), can be written as:
\[ \mathbf{R}(t) = [\hat{\mathbf{A}} \mathbf{F}^{-1} + \bar{\mathbf{B}} \mathbf{F}^{-1} - \bar{\mathbf{F}}] \mathbf{F} \mathbf{X}_t = \bar{\mathbf{R}} \mathbf{X}_t. \] (25)

Where
\[ \bar{\mathbf{R}} = [\hat{\mathbf{A}} \mathbf{F}^{-1} + \bar{\mathbf{B}} \mathbf{F}^{-1} - \bar{\mathbf{F}}]. \]

Now, we set the residual matrix \( \bar{\mathbf{R}} = 0 \) or we use the following inner products,
\[ \langle \mathbf{R}_i(t), \phi_k(t) \rangle_n = 0, \quad i, k = 0,1,\ldots. \] (26)

From supplementary conditions of equation (13), i.e \( \mathbf{U}(a) = \mathbf{U}_0 \) we obtain \( m + 1 \) equations. Therefore it is sufficient to choose \( n(m + 1) \) equations of the first equations in equation (26). Hence, a system of algebraic equations with \( (m + 1)(n + 1) \) equations and \( (m + 1)(n + 1) \) unknown is obtained. Solving this system yields the unknown vector \( \mathbf{U}(x) \). This is the so-called operational Tau method which is applicable for finite, infinite, regular and irregular domains.

Some Shifted Orthogonal Polynomials: In Sections 2 and 3, we have considered OTM based on arbitrary orthogonal polynomials. Orthogonal functions can be used to obtain a good approximation for transcendental functions. Since shifted Chebyshev and Legendre polynomials are more applicable orthogonal functions for a wide range of problems therefore we consider them, briefly.
Shifted Chebyshev Polynomials: The Chebyshev polynomials are defined on $[-1,1]$ as:

$$\begin{cases}
T_0(x) = 1, T_1(x) = x, \\
T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x) & i = 1, 2, 3, \ldots
\end{cases}$$

(27)

or

$$[T_0(x), T_1(x), T_2(x), \ldots]^T = TX, \text{ where } T = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and shifted Chebyshev polynomials are defined as:

$$\begin{cases}
T_0^*(x) = 1, T_1^*(x) = \frac{2x-(b+a)}{b-a}, & x \in [a,b], \\
T_{i+1}^*(x) = 2\left(\frac{2x-(b+a)}{b-a}\right)T_i^*(x) - T_{i-1}^*(x) & i = 1, 2, 3, \ldots
\end{cases}$$

(28)

Now, we consider the following lemma.

Lemma 3: Suppose that $T$ and $T'$ are coefficient matrices of Chebyshev polynomials \{\(T(x) \mid x \in [-1,1], I = 0,1,2,\ldots\)\} and shifted Chebyshev polynomials \{\(T_i^*(x) \mid x \in [a,b], i = 0,1,2,\ldots\)\} respectively. Hence, we have:

$$T' = TQ,$$

Where

$$Q_{i,j} = \binom{i}{j}v^{i-j}w^j, \text{ where } i, j = 0,1,2,\ldots; 0 < i < j,$$

With $v = \frac{2}{b-a}$ and $w = \frac{a+b}{a-b}$.

Proof: Definition of $T$ states that:

$$\begin{bmatrix} T_0(x) \\ T_1(x) \\ T_2(x) \\ \vdots \end{bmatrix} = T \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \end{bmatrix} \Rightarrow \begin{bmatrix} T_0^*(x) \\ T_1^*(x) \\ T_2^*(x) \\ \vdots \end{bmatrix} = T \begin{bmatrix} 1 \\ \frac{2x-(b+a)}{b-a} \\ \frac{2x-(b+a)}{b-a} \frac{2x-(b+a)}{b-a} \vdots \end{bmatrix} = T \begin{bmatrix} 1 \\ \frac{v^x+w}{(v^x+w)^2} \\ 2v^xw \\ w^2 \\ \vdots \end{bmatrix}.$$

We know that $(v^x+w)^n = \sum_{i=0}^{n} \binom{n}{i}v^i w^{n-i} x^i$, thus

$$\begin{bmatrix} 
1 \\
v^x+w \\
(v^x+w)^2 \\
\vdots \end{bmatrix} = \begin{bmatrix} 
1 & 0 & 0 & \cdots \\
v & w & 0 & \cdots \\
v^2 & 2vw & w^2 & \cdots \\
v^3 & 3vw^2 & 3v^2w & w^3 & \cdots \\
\vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 
x \\
x^2 \\
x^3 \\
\vdots \end{bmatrix}.$$
If we let \( Q \) to be the last coefficient matrix, then
\[
\begin{bmatrix}
T_0^*(x), T_1^*(x), T_2^*(x), \ldots
\end{bmatrix}^T = TQX,
\]
where
\[
Q_{i,j} = \begin{cases} 
1 & \text{if } j = 0, 1, 2, \ldots, \text{ and } \text{even } j, \text{ or if } \text{odd } j \text{ and } \text{odd } i, \\
0 & \text{if } i < j,
\end{cases}
\]
So
\[
T^* = TQ.
\]
Therefore, the lemma is valid.

**Shifted Legendre Polynomials:** The Legendre polynomials on \([-1,1]\) are defined as:
\[
P_0(x) = 1, P_1(x) = x,
\]
\[
P_i(x) = (2 - \frac{1}{i})xP_{i-1}(x) - (1 - \frac{1}{i})P_{i-2}(x), \ i = 2, 3, 4, \ldots,
\]
and we define shifted Legendre polynomials as:
\[
P_0^*(x) = 1, P_1^*(x) = 2x - \frac{2x - (b + a)}{b - a} = \frac{x}{x}, \ x \in [a,b],
\]
\[
P_i^*(x) = (2 - \frac{1}{i})\frac{2x - (b + a)}{b - a}P_{i-1}(x) - (1 - \frac{1}{i})P_{i-2}(x), \ i = 2, 3, 4, \ldots.
\]
In a similar manner with lemma 1 we can prove \( P^* = PC \), where \( P \) and \( P^* \) are coefficient matrices of Legendre and shifted Legendre polynomials, respectively.

**Numerical Experiments:** In this section, two experiments are presented to demonstrate the capability of the proposed method. The computations associated with the experiments discussed above were performed in *Maple 14* on a PC with a CPU of 2.4 GHz.

**Experiment 1:** Consider the following DAE [9],
\[
A(t)\dot{U}(t) + B(t)U(t) = F(t), \ 0 \leq t \leq 1,
\]
\[
U(0) = U_0.
\]
Where
\[
A(t) = \begin{bmatrix} 1 & 0 & -\lambda t \\ 0 & 1 & 5 - \lambda \\ 0 & 0 & 0 \end{bmatrix}, \ B(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i^2 \sin t & 0 \end{bmatrix}, \ F(t) = \begin{bmatrix} \dot{e} - \lambda(t \dot{e} + e^{-t}) \\ -\dot{e}^{-t} - (\lambda - 5)(\dot{e}^t + e^{-i}) \\ \dot{t}^2 \dot{e} + \sin t \dot{e}^{-t} \end{bmatrix}, \ U_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]
The exact solution is
\[
U(t) = [u_0(t), u_1(t), u_2(t)]^T = [e^t, e^{-t}, e^t - e^{-t}]^T.
\]
We have solved this experiment using OTM with shifted Legendre polynomials and for \( \lambda = 15 \). The sequence of approximate solution is obtained as follows:
Therefore, we can conclude that

\[
U_n(t) = \begin{bmatrix}
1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} \\
1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} \\
2t + \frac{t^3}{3} + \frac{t^5}{60}
\end{bmatrix}.
\]

This has the closed form \( u(t) = [\dot{e}, e^{-t}, e^{-t}]^T \), which is the exact solution of the problem.

**Experiment 2**: Consider the following DAE [9],

\[
A(t)U(t) + B(t)U(t) = 0, \quad 0 \leq t \leq 1, \quad U(0) = U_0.
\]

Where

\[
A(t) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad B(t) = \begin{bmatrix}
1 & 0 & -t & 1 \\
1 & 0 & -t^2 & t \\
0 & -t^3 & t^2 & 0 \\
t & -1 & t & -1
\end{bmatrix}, \quad U_0(t) = \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

The exact solution is \( U(t) = [e^{-t}, te^{-t}, e^{-t}]^T \).

We have solved this experiment using OTM with shifted Chebyshev polynomials. The sequence of approximate solution is obtained as follows:

\[
U_0(t) = \begin{bmatrix}
1 \\
0 \\
1 \\
0
\end{bmatrix}, \quad U_1(t) = \begin{bmatrix}
1 - t \\
t \\
t + t \\
t
\end{bmatrix}, \quad U_2(t) = \begin{bmatrix}
1 - t + \frac{t^2}{2} \\
t - t^2 \\
t + \frac{t^2}{2} \\
t + t
\end{bmatrix}, \quad U_3(t) = \begin{bmatrix}
1 - t + \frac{t^2}{2} + \frac{t^3}{6} \\
t - t^2 + \frac{t^3}{2} \\
t + \frac{t^3}{2} \\
t + \frac{t^3}{2}
\end{bmatrix}.
\]
Therefore, we can conclude that

\[
U_n(t) = \begin{bmatrix}
1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} \\
t - t^2 + \frac{t^3}{2} + \frac{t^4}{6} + \frac{t^5}{24} \\
1 + t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} \\
t + t^2 + \frac{t^3}{2} + \frac{t^4}{6} + \frac{t^5}{24}
\end{bmatrix}
\]

This has the closed form \( U(t) = \left[ e^{-t}, e^{-t}, e^{t}, te^{t} \right]^T \), which is the exact solution of the problem.

**CONCLUSIONS**

In this paper the OTM was made applicable to DAEs. The main idea of the proposed method is to convert the problem to an algebraic system using the orthogonal polynomials in order to decrease the computations. All the numerical results obtained using the OTM described earlier show very good agreement with the exact solutions in the forms of convergent series with easily calculable terms. All of these advantages of the OTM for solving DAEs assert the method as a convenient, reliable and powerful tool.

**REFERENCES**


