Semi Exact Solutions for Bi-Harmonic Equations Using Homotopy Analysis Method

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Abstract: In this article, the homotopy analysis method (HAM) for obtaining semi analytical solutions of bi-harmonic equations is introduced. Series solutions of the problem under consideration are developed by means of HAM and the recurrence relations are given explicitly. The initial approximation can be freely chosen with possible unknown constants which can be determined by imposing the boundary and initial conditions. The numerical examples show the rapid convergence of the series constructed by this method to the exact solution. Moreover, this technique does not require any discretization, linearization or small perturbations. Test problems have been considered to ensure that HAM is accurate and efficient compared with the variational iteration method (VIM).

Key words: Analytical solution • Bi-Harmonic equations • Homotopy analysis method

INTRODUCTION

It is well known that most of the scientific phenomena are modeled by ordinary or partial differential equations. Analytical solutions of these equations may well describe the various phenomena in science and nature, such as vibrations, solitons and propagation with a finite speed. The homotopy analysis method is an analytical technique for solving nonlinear differential equations devised by Shi-Jun Liao in 1992 [1].

The proposed method has been successfully applied to solve many types of nonlinear problems in science and engineering by many authors [2-9] and references therein. We aim in this work to effectively employ HAM to establish the semi analytical solutions for the bi-harmonic equations [10,11]. By the presented method, numerical results can be obtained with using a few iterations [12]. Moreover, HAM contains the auxiliary parameter \( h \), which provides us with a simple way to adjust and control the convergence region of solution series [12]. Therefore, HAM handles linear and nonlinear problems without any assumption and restriction. On the other side, M. Duy and T. Cong [13] reported an indirect radial-basis-function collocation method for solving numerically the bi-harmonic boundary value problem. M. Dehghan and A. Mohebbi [14] proposed two compact finite difference approximations algorithms for solving numerically the three-dimensional bi-harmonic equation with Dirichlet boundary conditions. M. Duy and R. Tanner [15] suggested a new spectral collocation method for solving numerically the two-dimensional bi-harmonic boundary value problem. In this paper, we implement HAM to obtain the solution of the one-and two-dimensional bi-harmonic equations. Test problems have been considered to ensure that HAM is accurate and efficient compared with the previous ones.

The paper has been organized as follows. In section 2, the basic idea of homotopy analysis method is described. In section 3, applying HAM for bi-harmonic equations. Also, in section 4, the convergence of the exact solution is illustrated. Discussion and conclusions are presented in section 5.

Basic Idea of HAM: In this section, the basic idea of HAM is described. We consider the following differential equation:

\[
N[u(x,y)] = 0, 
\]

where \( N \) is a linear operator for this problem, \( x \) and \( y \) denote independent variables, \( u(x,y) \) is an unknown function. For simplicity, we ignore all boundary and initial conditions, which can be treated in the similar way.
Zeroth-Order Deformation Equation: Liao [1], constructs the so-called zeroth-order deformation equation:

\[ (1-q) \varepsilon \left( \psi(x,y; q) - u_0(x,y) \right) = q h N(\psi(x,y; q)) \quad (2) \]

where is an auxiliary linear operator, \( u_0(x,y) \) is an initial guess, \( h = 0 \) is an auxiliary parameter and \( q = \varepsilon [0,1] \) is the embedding parameter. Obviously, when \( q = 0 \) and \( q = 1 \), it holds respectively:

\[ \psi(x,y;0) = u_0(x,y), \quad \psi(x,y;1) = u(x,y). \quad (3) \]

Thus, as \( q \) increasing from 0 to 1, the solution \( \psi(x,y; q) \) various from \( u_0(x,y) \), to \( u(x,y) \).

Expanding \( \psi(x,y;0) \) in Taylor series with respect to the embedding parameter \( q \), one has:

\[ \psi(x,y; q) = u_0(x,y) + \sum_{m=1}^{\infty} u_m(x,y) q^m \quad (4) \]

where

\[ u_m(x,y) = \frac{1}{m!} \left( \frac{\partial^m \psi(X,Y,q)}{\partial q^m} \right)_{q=0} \quad (5) \]

Assume that the auxiliary linear operator, the initial guess and the auxiliary parameter \( h \) are selected such that the series (4) is convergent at \( q = 1 \), then at \( q = 1 \) and by (3), the series (4) becomes:

\[ u(x,y) = u_0(x,y) + \sum_{m=1}^{\infty} u_m(x,y). \quad (6) \]

The Mth-Order Deformation Equation:

Define the vector

\[ \tilde{u}_n(x,y) = [u_0(x,y), u_1(x,y), \ldots, u_n(x,y)] \quad (7) \]

Differentiating Eq. 2 \( m \) times with respect to the embedding parameter \( q \), then setting \( q = 0 \) and dividing them by \( m! \), finally using (5), we have the so-called \( m \)-th order deformation equations:

\[ \varepsilon [u_m(x,y) - d_m \tilde{u}_{m-1}(x,y)] = h R_m(\tilde{u}_{m-1}), \quad (8) \]

where

\[ R_m(\tilde{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\psi(x,y;q)]}{\partial q^{m-1}} \bigg|_{q=0} \quad (9) \]

and

\[ d_m = \begin{cases} 0, & m \leq 1; \\ 1, & m > 1. \end{cases} \quad (10) \]

Applications the Proposed Method: We will apply HAM to two physical problems to illustrate the strength of the method and to establish semi exact solutions for these problems. We made comparison with the VIM [10,16-18].

HAM for the One-Dimensional Bi-harmonic Equation

Consider the one dimensional bi-harmonic equation in the form [19]:

\[ \frac{d^4 u(x)}{dx^4} + 4 u(x) = 0, \quad -1 \leq x \leq 1, \]

\[ u(-1) = e^2, \quad u(1) = 1, \quad \frac{d^2 u(-1)}{dx^2} = -2e^2 \tan(1), \quad \frac{d^2 u(1)}{dx^2} = 2\tan(1). \quad (12) \]

The exact solution of this equation is given by:

\[ u(x) = \frac{e^{1-x} \cos(x)}{\cos(1)}. \quad (13) \]

We choose the linear operator:

\[ \varepsilon [\psi(x,q)] = \frac{\partial^4 \psi(x; q)}{\partial x^4}, \quad (14) \]

with the property:

\[ \varepsilon [c_1 + c_2 x + c_3 x^2 + c_4 x^3] = 0, \]

where \( c = 1,2,3,4 \) are constants. We now define a linear operator as:

\[ N[\psi(x,q)] = \frac{\partial^4 \psi(x; q)}{\partial x^4} + 4\psi(x,q). \quad (15) \]

Using above definition, we construct the zeroth-order deformation equation:

\[ (1-q) \varepsilon \left[ \psi(x; q) - u_0(x) \right] = q h N[\psi(x; q)] \quad (16) \]

For \( q = 0 \) and \( q = 1 \), we can write:

\[ \psi(x,0) = u_0(x), \quad \psi(x,1) = u(x). \quad (17) \]
Thus, we obtain the mth-order deformation equations Now, the solution of the mth-order deformation equations for \( m \geq 1 \) becomes:

\[
\varepsilon \left[ u_m(x) - d_m u_{m-1}(x) \right] = h \mathcal{R}_m(\tilde{u}_{m-1})
\]

where

\[
\mathcal{R}_m(\tilde{u}_{m-1}) = \frac{d^4 u_m(x)}{dx^4} + 4\psi(x, q).
\]

This in turn gives the first few components of the approximate solution:

\[
\begin{align*}
 u_0(x) &= a + bx + cx^2 + dx^3, \\
 u_1(x) &= h \left( \frac{a}{6}x^4 + \frac{b}{30}x^5 + \frac{c}{90}x^6 + \frac{d}{210}x^7 \right), \\
 u_2(x) &= h \left( \frac{a}{6}x^4 + \frac{b}{30}x^5 + \frac{c}{90}x^6 + \frac{d}{210}x^7 \right) + h^2 \left( \frac{a}{6}x^4 + \frac{b}{30}x^5 + \frac{c}{90}x^6 \right) + h^3 \left( \frac{a}{6}x^4 \right).
\end{align*}
\]

other components of the approximate solution can obtain in the same manner.

Using the conditions (12). Then the values of the four constants \( a, b, c \) and \( d \) are:

\[
a = 5.03104, \quad b = -503104, \quad c = 3.71732 \times 10^{-9}, \quad d = 1.67701.
\]

We can see that HAM solution converges to the exact solution. Numerical and exact solutions at different values of \( x \) and \( h = -1 \) are listed in Table 1. We note that there is a complete agreement between computed results by presented algorithm and the exact solution.

Table 2, gives comparison between the error of our proposed method (HAM) and the VIM [10]. From these results we can see that the presented approach is more efficient than the different methods, regarding HAM which takes three components only of the solution.

It is noted that our approximate solutions converge at \( h = 0.85 \) and \( h = 1.3 \) (Tables 1 and 2). The explicit, analytic expression given by Eq. 19 contains the auxiliary parameter \( h \), which gives the convergence region and rate of approximation for HAM. However, the errors can be further be reduced by calculating higher order approximations. This proves that HAM is very useful analytic method to get accurate analytic solutions to linear and strongly nonlinear problems [2-6].

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Table 1: Comparison between the error of HAM and VIM.

<table>
<thead>
<tr>
<th>x</th>
<th>-1.0</th>
<th>-0.6</th>
<th>-0.2</th>
<th>0.2</th>
<th>0.6</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>HAM</td>
<td>4.4409e-15</td>
<td>2.9636e-09</td>
<td>4.2147e-09</td>
<td>4.1563e-09</td>
<td>2.8473e-09</td>
<td>2.2205e-15</td>
</tr>
<tr>
<td>VIM</td>
<td>4.4408e-15</td>
<td>2.9636e-09</td>
<td>4.2146e-09</td>
<td>4.1563e-09</td>
<td>2.8473e-09</td>
<td>2.2205e-15</td>
</tr>
</tbody>
</table>

Table 2: The error of solution using HAM at different value of \( h = -0.85 \)

<table>
<thead>
<tr>
<th>x</th>
<th>-1.0</th>
<th>-0.6</th>
<th>-0.2</th>
<th>0.2</th>
<th>0.6</th>
<th>1.0</th>
</tr>
</thead>
</table>
HAM for the Two Dimensional Bi-Harmonic Equation: Consider the two dimensional bi-harmonic equation in the form:

$$\frac{\partial^4 u(x,y)}{\partial x^4} + 2 \frac{\partial^4 u(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 u(x,y)}{\partial y^4} = 0, \quad -1 \leq x, y \leq 1,$$

with the boundary conditions:

$$u(x,1) = \frac{\sin(\pi x)}{\pi}, \quad u(x,-1) = \frac{\sin(x)(\sinh(2\pi) + 2\pi \cosh(2\pi))}{\pi^2 (2\pi + \sinh(2\pi))},$$

$$\frac{\partial^2 u(x,1)}{\partial y^2} = \sin(\pi x), \quad \frac{\partial^2 u(x,-1)}{\partial y^2} = \frac{\sin(x)(\sinh(2\pi) + 2\pi \cosh(2\pi))}{(2\pi + \sinh(2\pi))}.$$  

(21)

Assuming the function $u(x,y)$ can be expressed in the following form:

$$u(x,y) = \sin(\pi x)v(y),$$

(23)

where $v(y)$ is a function of $y$ only. Then we get a bi-harmonic equation as in the last section but different. So, by the same method we solve it and then we get the solution of the two dimensional bi-harmonic equation.

Substituting (23) in (21), we get the following equation:

$$\frac{d^4 v(y)}{dy^4} - \frac{2\pi}{2} \frac{d^2 v(y)}{dy^2} + \pi^2 v(y) = 0, \quad -1 \leq y \leq 1,$$

with the boundary conditions:

$$v(1) = \frac{1}{\pi}, \quad v(-1) = \frac{(\sinh(2\pi) + 2\pi \cosh(2\pi))}{\pi^2 (2\pi + \sinh(2\pi))},$$

$$\frac{d^2 v(1)}{dy^2} = 1, \quad \frac{d^2 v(-1)}{dy^2} = \frac{(\sinh(2\pi) + 2\pi \cosh(2\pi))}{(2\pi + \sinh(2\pi))}.$$  

(24)

The exact solution of this equation is given by:

$$v(y) = A \cosh(\pi y) + B \sinh(\pi y),$$

(26)

where

$$A = \frac{2 \sinh(\pi) + 2\pi \cosh(\pi)}{2\pi^3 + \pi^2 \sinh(2\pi)}, \quad B = \frac{-2 \sinh(\pi)}{2\pi^2 + \pi \sinh(2\pi)}.$$  

So, the exact solution of (21) is given by:

$$u(x,y) = \sin(\pi x) A \cosh(\pi y) + B \sinh(\pi y),$$

(27)

We choose the linear operator:

$$\varepsilon[y;q] = \frac{\partial^4 \psi(y;\xi)}{\partial y^4},$$

(28)

with the property:

$$\varepsilon [c_1 + c_2 y^2 + c_3 y^4 + c_4 y^6] = 0,$$

where $c_i = 1,2,3,4$ are constants. We now define a linear operator as:
Using above definition, we construct the zeroth-order deformation equation:

\[
(1 - q) \varepsilon \left[ ? (y;q) - v_0(y) \right] = q h N[y(y;q)]
\]

For \( q = 0 \) and \( q = 1 \), we can write:

\[
\psi(y;0) = v_0(y), \quad \psi(y;1) = v(y).
\]

Thus, we obtain the mth-order deformation equations

\[
\varepsilon[v_m(y) - d_m v_{m-1}(y)] = h \rho_l (\varepsilon_m - 1),
\]

where

\[
\rho_l = \frac{1}{m!} \left. \frac{\partial^{m-1} N[y(y;q)]}{\partial q^{m-1}} \right|_{q=0}.
\]

Now, the solution of the mth-order deformation equations (32) for \( m \geq 1 \) become:

\[
v_m(y) = d_m v_{m-1}(y) + h \rho_l [\varepsilon_m (\varepsilon_m - 1)].
\]

This in turn gives the first few components:

\[
v_0(y) = a + by + cy^2 + dy^3,
\]

\[
v_1(y) = h \left( \frac{-cp^2 + ap^4}{6} y^4 + \frac{-dp^2 + bp^4}{10} y^5 + \frac{cp^4}{360} y^6 + \frac{dp^4}{840} y^7 \right),
\]

\[
v_2(y) = h \left( \frac{-cp^2 + ap^4}{6} y^4 + \frac{-dp^2 + bp^4}{10} y^5 + \frac{cp^4}{360} y^6 + \frac{dp^4}{840} y^7 \right)
\]

\[
+ h^2 \left( \frac{-cp^2 + ap^4}{6} y^4 + \frac{-dp^2 + bp^4}{10} y^5 + \frac{cp^4}{360} y^6 + \frac{dp^4}{840} y^7 \right)
\]

\[
+ \left( \frac{-cp^6}{5040} + \frac{ap^8}{40320} \right) y^8 + \left( \frac{-dp^6}{15120} + \frac{bp^6}{362880} \right) y^9
\]

\[
+ \left( \frac{-cp^8}{1814400} + \frac{ap^8}{6652800} \right) y^{10} + \left( \frac{-dp^8}{604800} + \frac{bp^8}{2419200} \right) y^{11},
\]

other components of the approximate solution can obtain in the same manner.

Using the conditions (25), we obtain the values of the four constants \( a, b, c \) and \( d \) are:

\[
a = 0.0277666, \quad b = -0.0799744, \quad c = 0.163421, \quad d = 0.138818.
\]

Fig. 1 shows that the approximate solution is consistent with the exact solution which sure the accuracy of HAM for the one-and two-dimensional bi-harmonic equations.
Convergence of the Exact Solution: Liao [1] showed that whatever a solution series converges it will be one of the solutions of considered problem. Liao [1,8,12] presented to be controlled by the auxiliary parameter the rate of convergence of \( h \) the approximate solutions obtained by HAM. HAM and the VIM [10] give the same solutions of one-and two-dimensional bi-harmonic equations with boundary conditions when \( h = -1 \).

CONCLUSIONS

In this Letter, HAM was used for obtaining the semi exact solutions of one-and two-dimensional bi-harmonic equations using the PC-based Mathematica package for illustrated examples. The numerical results showed that this method has very accuracy and reductions of the size of calculations compared with the VIM [16,18] and the homotopy perturbation method [20,21]. In addition, we see that the homotopy perturbation method and VIM are special cases of HAM for \( h=-1 \). It may be concluded that this methodology is very powerful and efficient technique in finding exact solutions for wide classes of problems. It is also worth noting to point out that the advantage of this methodology shows a fast convergence of the solutions by means of the auxiliary parameter, \( h=-1 \) see Fig. 1. HAM is very easy applied to both differential equations and linear or nonlinear differential systems. The approximate solutions were almost identical to analytic solutions of the nonlinear evolution equations.

REFERENCES