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A New Solution Approach for the Integer Fair Division Problem

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Abstract: In this paper, a solution method for classical fair division problem utilizing cut-and-choose protocol has been examined and a new method has been suggested in order to eliminate the deficiencies of that method. Furthermore, the problem has been generalized and mathematical models have been written according to oddness or evenness of the number of persons and solution techniques have been given.

2010 mathematics subject classification: 05C30

Key words: Fair division . cut-and-choose protocol . graph theory . organization theory

INTRODUCTION

The fair division problem is generally known as receiving a fair share for each one of the n players. Then the amount of the share for each of the player is equals to 1/n [1]. Nevertheless in integer fair division problem each of the share is integer-valued. The problem is used in the share of uncountable objects.

CLASSICAL CUT-AND-CHOOSE PROTOCOL AND ITS MODIFICATION

In the solution of the fair division problem, being fair of division means that each individual accepts his share with no objection, without being concerned about the exact equality of shares per capita. For example, in the solution of the problem of fair division for two persons with cut-and-choose protocol, first person cuts the object into two halves; second person chooses the half he wants [2]. In this case, at the end of choosing, second person must perform the cutting process almost excellently in order to get *most probably* the half of the shared object, whereas the second person will get the half of the object in the worst case, anyways. Handicap of the cut-and-choose protocol appears here at this point, the first person is disadvantageous because the protocol is not symmetric. Since protocol processes are unidirectional, compensation of the move of the first person is not possible and also no one wants to be the first person in a division like this. It is because the second person does not get the cutting responsibility. Let us assume the situation in which the second person takes responsibility. In other words, after the first division, if the protocol can be applied inversely once again, that is, if the second person cuts the object into two halves again and the first person performs choosing from these halves, then each party will have performed both cutting and choosing processes once and participated into division on an equal basis. Therefore, they will not be able to object to the result. Thus, the handicap of the cut-and-choose protocol is eliminated. However, at this point, since both persons have the half of the shared object theoretically after the first division, now another problem on how the protocol will be applied for the second time occurs. The reason for that problem is that while the protocol is processed inversely, the second person cuts the object despite the number of pieces, the first person who is the one choosing this time adds the chosen piece on the existing half object (approximately) and the second person looses from his half object (approximately). Since the second person will be losing from his share, the object will not been shared equally and the second person will be aggrieved. This problem arises from cutting the object into two halves at the first stage of the protocol and the solution of the problem is given in the following paragraph.

When the cutting process begins with the assumption of the one unit object, it is seen that an integer solution does not exist, because despite the number of pieces of which the object is divided, at the end of the division, the shares of the persons will be a fractional number. On the other hand, it is important to decide the number of the pieces in which the object will be divided initially in order to ensure the shares integer numbers. At this point, when the problem is considered backwards, it must be determined that into how many pieces the object of two units must be cut by the first person in order to ensure that each person will have an object of one unit as a result of division. Also, in this

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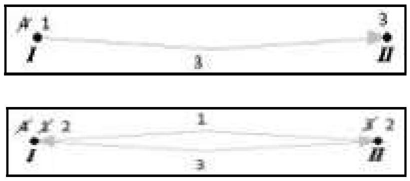


Fig. 1:

assumption, it is clear that there is no integer solution, because if the first person cuts the object into two pieces, after performing his choice the second person will be aggrieved, as it was explained in the previous paragraph, despite the number of pieces in which he cuts the object. Moreover, if the first person cuts the object into a number greater than 2, an integer solution would not be possible.

In that case, in order to reach the integer solution, under the assumption of both persons will have an object of two units as a result of division so that the division is started with object of four units, at the first stage of the division, first person cuts the object into four pieces and second person chooses 3 out of these four pieces. At the second stage, the second person cuts the object into three pieces and the first person chooses one out of these three pieces. Eventually, two units of the object remain for both persons, in other words, the first person receives a share such as:

$$\frac{4}{4}-\frac{3}{4}+\frac{1}{4}=\frac{2}{4}$$

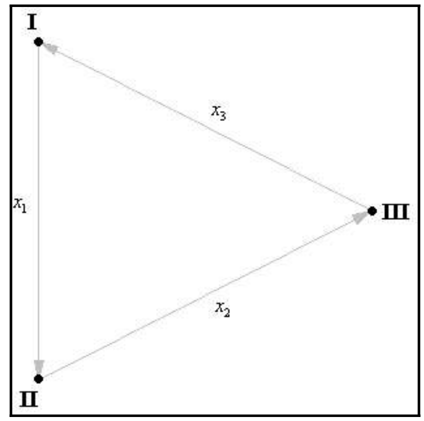
and the second person receives a share such as:

$$0+\frac{3}{4}-\frac{1}{4}=\frac{2}{4}$$

Total amount of object moved between two persons is as 3+1 = 4 objects and smaller integer solution does not exist (Fig. 1).

GENERALIZATION OF FAIR DIVISION PROBLEM AND ITS SOLUTION

Here a question may come to mind such as "If the number of persons who will participate into dividing is more than two persons, can cut-and-choose protocol still be applied?" may come to mind. When we model the problem with graph theory representing the persons which participate in the division and their cut processes as vertices of the graph and representing the choose processes between persons as the details of the graph in the problem, it is observed that the graph must be a full graph considering the fact that each person will participate into cut-and-choose process with all other



$$\begin{aligned}
&\min X = x_1 + x_2 + x_3 && \\
&3 - x_1 + x_3 = 1 && \mathrm{I} \\
&x_1 - x_2 = 1 && \mathrm{II} \\
&x_2 - x_3 = 1 && \mathrm{III} \\
&x_1, x_2\, x_3 \in \mathbb{Z}^+ &&
\end{aligned}$$

Fig. 2:

persons. As it was mentioned in the previous paragraphs, the number of cuts each person does and the number of choices every person makes must be equal for a fair division. It means that the degree of each vertex must be even. Therefore, the problem is solved easily with simple full graphs when n=3,5,7, is an odd number, because the fact that n is an odd number ensures the evenness of the degree of vertices n-1 in full graphs. However, when n = 2, 4, 6 is even, the problem can not be solved with simple full graphs and full graph that includes multiple edges are needed. Due to these reasons solution method differs according to the even and odd values of n and should be analyzed separately.

Number of persons as n which is odd: We can find which ratios will be used while cutting of first, second and third persons for n=3 by drawing the graph of the problem and writing mathematical model of the problem (Fig. 2). First of all, let us assume that there were 3 units of object in total at the beginning, assuming that everyone has a piece of one unit at the end of the division. Our aim is to move minimum unit objects with minimum number of cycles without vertex repetitions while applying the cut-and-choose protocol and is to show that outgoing (cut) and incoming (choose) quantities of each person as three consecutive constraints must be equal to one. And since the moved quantities will be at least 1 unit, decision variables are chosen as positive integer. When this model is run to solve the problem, solution vector is found as $x_1 = 3$,

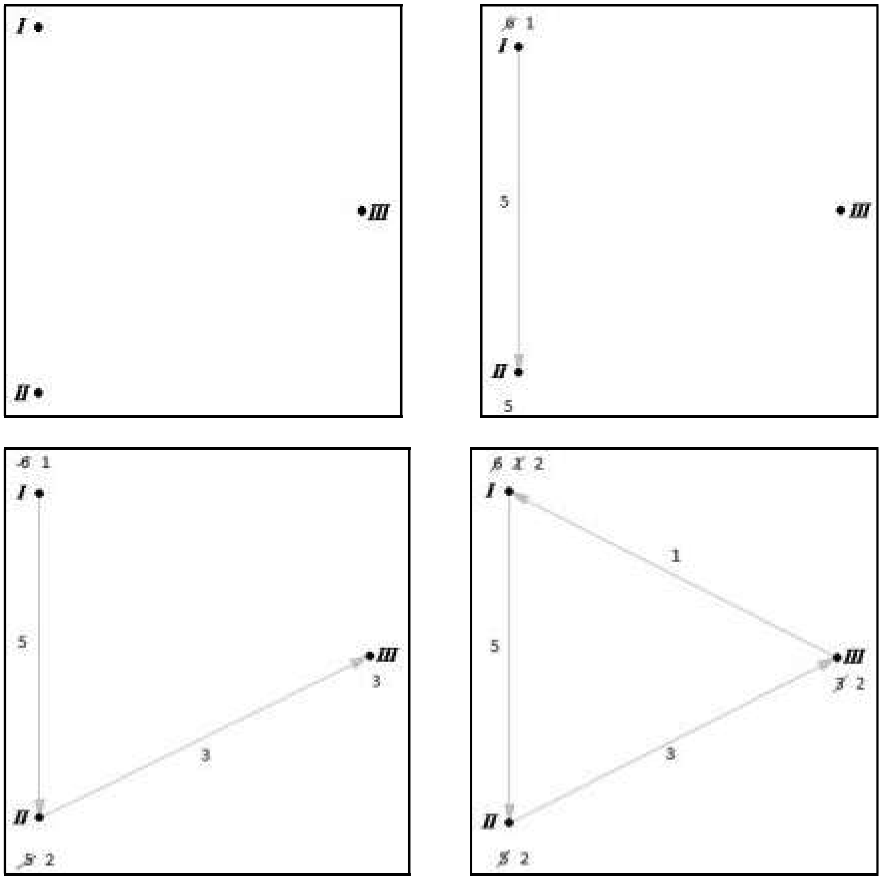


Fig. 3

$x_2 = 2$, $x_3 = 1$, but this solution is against the nature of the problem, because the first person has cut the object into 3 pieces from the beginning and the second person has chosen all of these three pieces. Therefore at this point cutting of first person does not make a sense.

This time we have arranged the model in order to start with six units of objects at the beginning by assuming that two units of objects have remained for each of the persons after division and run the model once again, we have found the following solution vector:

$$\min X = x_1 + x_2 + x_3$$

$$6 - x_1 + x_3 = 2 \qquad \mathrm{I}$$

$$x_1 - x_2 = 2 \qquad \mathrm{II}$$

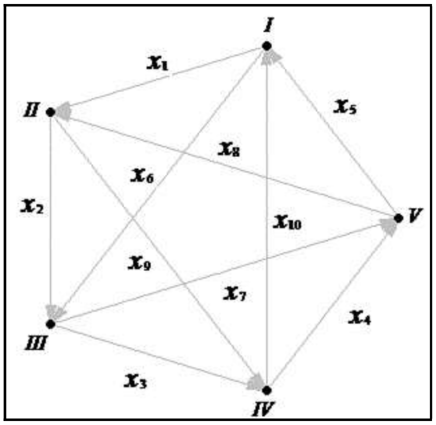
$$x_2 - x_3 = 2 \qquad \mathrm{III}$$

$$x_1, x_2\, x_3 \in \mathbb{Z}^+$$

$$x_1 = 5 \quad x_2 = 3 \quad x_3 = 1$$

This solution vector is adequate, because the first person cuts the total initial object into six pieces and the second person chooses five of these six units. Thereafter, the second person cuts his total object into five pieces and the third person chooses three of these and second person completes the problem with approximately two units of object. Finally, third person cuts his total object into three pieces and first person chooses one of these three pieces. The problem is completed with approximately two units of the object for third and first persons respectively (Fig. 3). Total amount of object moved between three persons is as 5+3+1 = 9 objects and smaller integer solution does not exist.

On the other hand, model written for n=3 is quite simple and a generalization of this model for great values of n is seen clearly. Therefore, when we try to solve the problem by writing the following mathematical model for n = 5 and by assuming that one unit of object will be left for each person as a result of the division, we see that the model does not have a solution (Fig. 4). Here, we have $x_{j-1} - x_j \geq 1 \quad j = \overline{2,10}$ constraints which are different from the previous model enable the consecutive choose processes in descending order to guarantee the cut processes become meaningful.



$$
\begin{array}{ll}
\min X = \sum_{i=1}^{10} x_i & \\
5 - x_1 + x_5 - x_6 + x_{10} = 1 & \mathrm{I} \\
x_1 - x_2 + x_8 - x_9 = 1 & \mathrm{II} \\
x_2 - x_3 + x_6 - x_7 = 1 & \mathrm{III} \\
x_3 - x_4 + x_9 - x_{10} = 1 & \mathrm{IV} \\
x_4 - x_5 + x_7 - x_8 = 1 & \mathrm{V} \\
x_{j-1} - x_j \geq 1 \quad j = \overline{2,10} & \\
x_i \in \mathbb{Z}^+ \quad i = \overline{1,10} &
\end{array}
$$

Fig. 4:

$$
\begin{array}{lll}
\min X = \sum_{i=1}^{10} x_i & & x_1 = 10 \\
 & & x_2 = 9 \\
10 - x_1 + x_5 - x_6 + x_{10} = 2 & \mathrm{I} & x_3 = 8 \\
x_1 - x_2 + x_8 - x_9 = 2 & \mathrm{II} & x_4 = 7 \\
x_2 - x_3 + x_6 - x_7 = 2 & \mathrm{III} & x_5 = 6 \\
x_3 - x_4 + x_9 - x_{10} = 2 & \mathrm{IV} & x_6 = 5 \\
x_4 - x_5 + x_7 - x_8 = 2 & \mathrm{V} & x_7 = 4 \\
x_{j-1} - x_j \geq 1 \quad j = \overline{2,10} & & x_8 = 3 \\
 & & x_9 = 2 \\
x_i \in \mathbb{Z}^+ \quad i = \overline{1,10} & & x_{10} = 1
\end{array}
$$

Fig. 5:

This time we have arranged the model by assuming that two units of objects have remained for each of the persons after division and run the model once again, we have found the following solution vector (Fig. 5). However, this solution is against the nature of the problem, because the first person has cut the ten units of the object into ten and the second person has chosen ten pieces out of them, therefore, at this point, cutting of the first person does not have a meaning.

As the next step, we have arranged the model by assuming that three unit objects will be remained for each of the persons after division and run the model once again, we can find the following solution vector

$$
\begin{array}{lll}
 & & x_1 = 14 \\
\min X = \sum_{i=1}^{10} x_i & & x_2 = 12 \\
15 - x_1 + x_5 - x_6 + x_{10} = 3 & \mathrm{I} & x_3 = 10 \\
x_1 - x_2 + x_8 - x_9 = 3 & \mathrm{II} & x_4 = 8 \\
x_2 - x_3 + x_6 - x_7 = 3 & \mathrm{III} & x_5 = 6 \\
x_3 - x_4 + x_9 - x_{10} = 3 & \mathrm{IV} & x_6 = 5 \\
x_4 - x_5 + x_7 - x_8 = 3 & \mathrm{V} & x_7 = 4 \\
 & & x_8 = 3 \\
x_{j-1} - x_j \geq 1 \quad j = \overline{2,10} & & x_9 = 2 \\
x_i \in \mathbb{Z}^+ \quad i = \overline{1,10} & & x_{10} = 1
\end{array}
$$

Fig. 6:

$$
\begin{array}{lll}
\min X = \sum_{i=1}^{10} x_i & & \\
5 - x_1 + x_5 = 1 & \mathrm{I} & x_1 = 5 \\
x_1 - x_2 = 1 & \mathrm{II} & x_2 = 4 \\
x_2 - x_3 = 1 & \mathrm{III} & x_3 = 3 \\
x_3 - x_4 = 1 & \mathrm{IV} & x_4 = 2 \\
x_4 - x_5 = 1 & \mathrm{V} & x_5 = 1 \\
5 - x_6 + x_{10} = 1 & \mathrm{I} & x_6 = 5 \\
x_8 - x_9 = 1 & \mathrm{II} & x_7 = 4 \\
x_6 - x_7 = 1 & \mathrm{III} & x_8 = 3 \\
x_9 - x_{10} = 1 & \mathrm{IV} & x_9 = 2 \\
x_7 - x_8 = 1 & \mathrm{V} & x_{10} = 1 \\
x_i \in \mathbb{Z}^+ \quad i = \overline{1,10} & &
\end{array}
$$

Fig. 7:

which is valid and consistent with the characteristic of the model (Fig. 6).

Remarkable point of the models for n=5 problem is that there must be two directional cycle in order for enable each person to perform both cutting and choosing processes. Because the vertex degrees of the full graph with five vertices are four, half of this value must be used for cutting process and the other half must be used for choosing process for each vertex. Here after the cycle that starts at x_1 and finishes at x_5, with the second cycle that starts at x_6 and finishes at x_{10}, it is provided that each person is included in cut-and-choose protocol by visiting all vertices twice. When we run the new mathematical model with this approach, we find the following solution vector (Fig. 7). Moreover solution stages of the problem for n = 5 can be seen from Fig. 8. The directional cycles that are preferred here are (I-II-III-IV-V-I) and (I-III-V-II-IV-I). The formula below can be used in order to find the vertex numbers in these cycles.

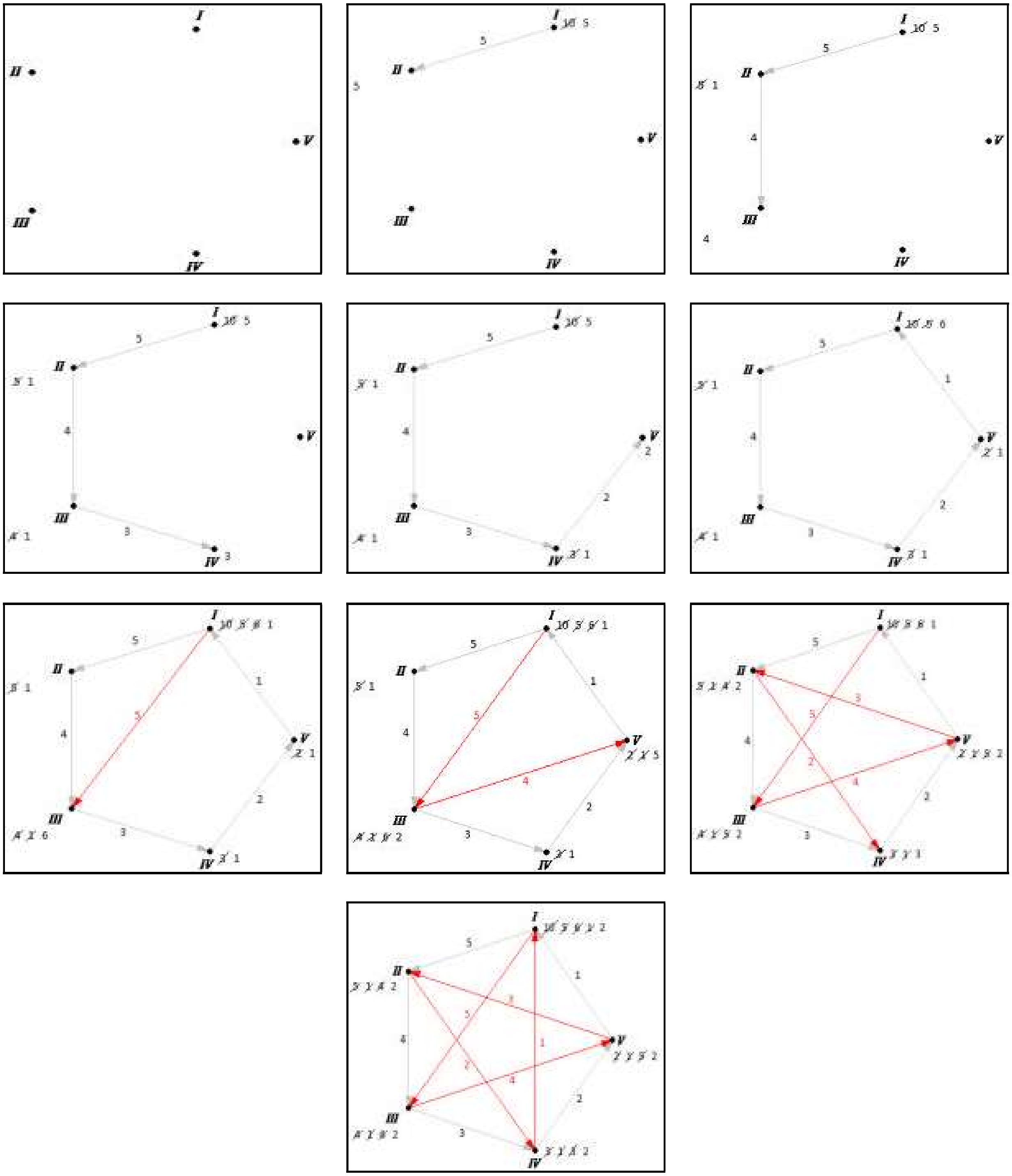


Fig. 8:

$$\text{j. tur için}\begin{cases} v_i = I & i = 1, \quad i = n+1 \\ v_{i+1} = (v_i + j) \bmod n & 2 \le i \le n \quad 1 \le j \le T \end{cases}$$

We can find the total number of cycle, or in other words, the number of the units of the object will be left for each person after division with T = (n-1)/2 formula. The only exception of this rule is n=3 and the reason of this exception has been explained above. For example for n=7, the result is T = 3 and these three cycles become (I-II-III-IV-V-VI-VII), (I-III-V-VII-II-IV-VI-I), (I-IV-VII-III-VI-II-V-I), according to the formula above.

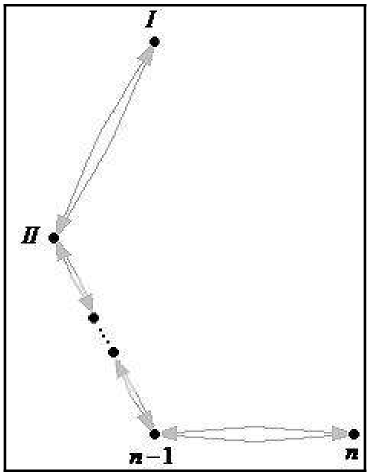


Fig. 9

Number of persons as n which is even: When n is even, n-1's which are vertex degrees of the full graph become odd; therefore there is no directional cycle where the cut and choose values are equal. In order to solve this problem, let us assume that there are multiple edges formed from two edges which are in the reverse direction between all vertex pairs in full graph with n vertices. With this move, the vertex degrees will be even and we can find a directional cycle number of n-1. The n-1 values are coming from the number of cut and choose couples that are correlated to each vertex. There must be 3 directional cycles for n=4 according to n-1 formula. Also the number of vertices in each cycle can be found from

$$\frac{n-2}{2}+2=\frac{n+2}{2}$$

formula. The value n in the formula comes from the number of edge with n vertices, thereby with n edges, in order to obtain minimum cycle value. When the vertex degrees which the cycle starts and the multiple edge is firstly used, are two, the value of internal vertices is four. The reason for starting the cycle from n value in the formula and subtracting two (which is the number of the multiple edges being firstly used) and dividing into two is to find the number of interim vertices. The vertex number is found finally by adding two vertices which are subtracted at the beginning. For example, I and n are extremal vertices and II,…,n-1 are interim vertices in Fig. 9.

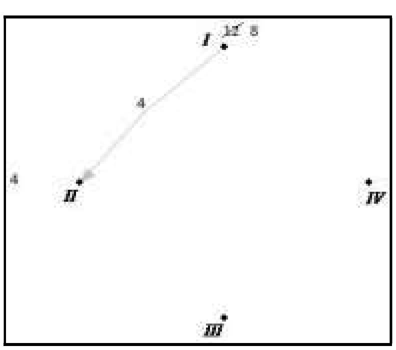
In order to find the vertex numbers in cycles, the following technique can be used. The cells at the first row having grey background color show the vertex indices of the cycle obtained from (n+2)/2 formula. The second gray row shows the value of the related vertex. At the first stage, the first column is filled with n-1 unit one vertex index the second column is filled from 2 to n. All the other remaining columns are filled with a formula j' = j-2 which is applied to the previous column of the mentioned column. If j'>n, then j' is updated with j' = (j' mod n) +1.

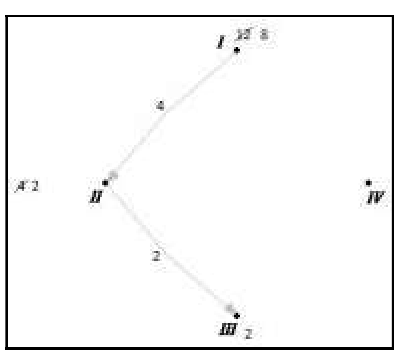
Mathematical model and solution v zector for n = 4 is as shown in Fig. 9. As it seen, total number of moved units is 24 and smaller integer solution does not exist.

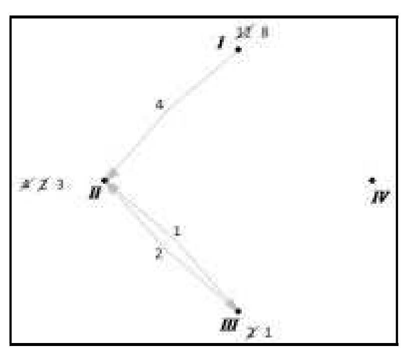
n=4		
1	2	3
2	4	2
1	2	3
1	3	4
1	4	2

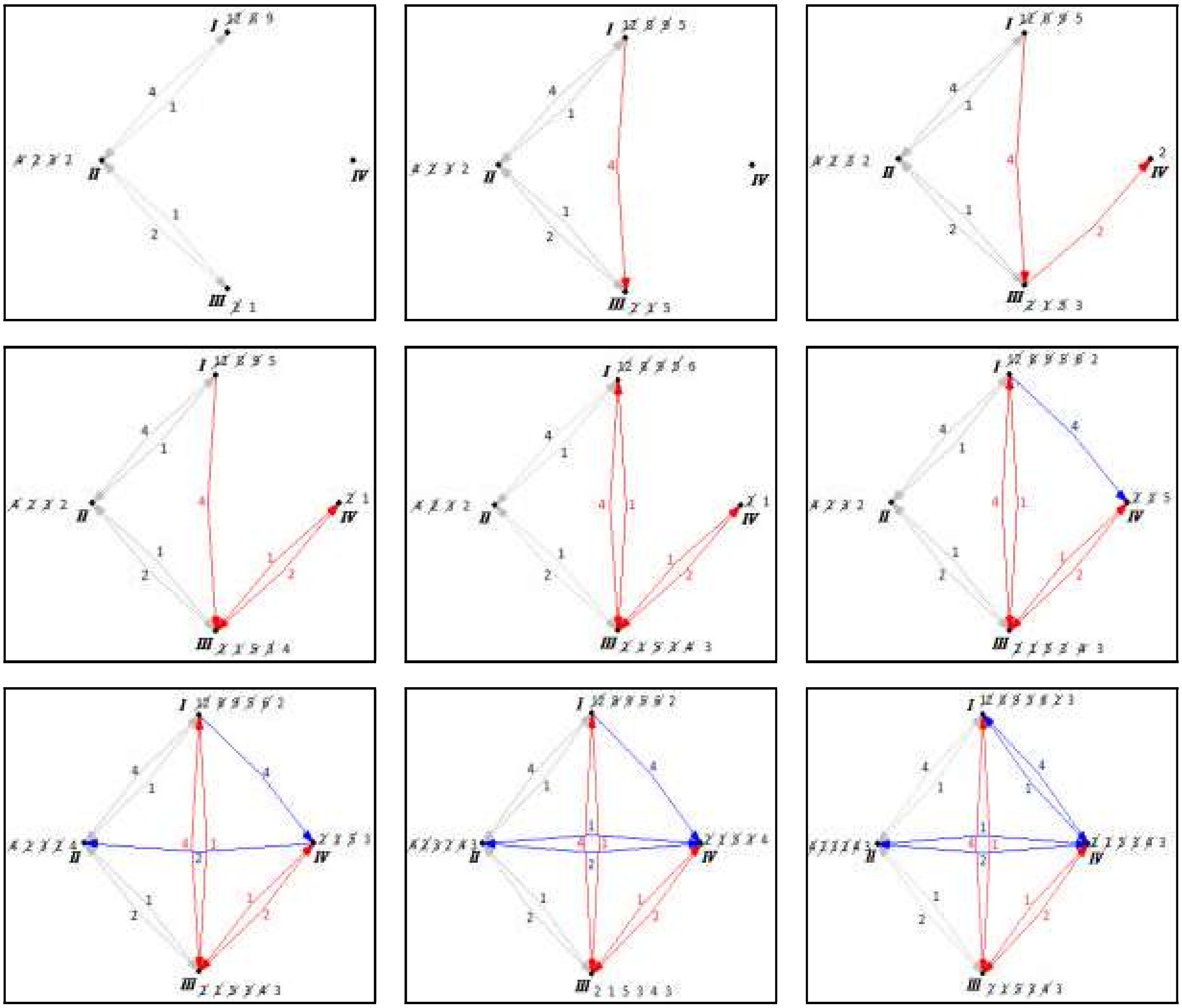
n=6			
1	2	3	4
2	4	4	2
1	2	3	5
1	3	4	6
1	4	5	2
1	5	6	3
1	6	2	4

n=8				
1	2	3	4	5
2	4	4	4	2
1	2	3	5	8
1	3	4	6	2
1	4	5	7	3
1	5	6	8	4
1	6	7	2	5
1	7	8	3	6
1	8	2	4	7









$$\min X = \sum_{i=1}^{12} x_i$$

$$\begin{array}{ll}
4 - x_1 + x_4 = 1 & \text{I} \\
x_1 - x_2 + x_3 - x_4 = 2 & \text{II} \\
x_2 - x_3 = 1 & \text{IV} \\
4 - x_5 + x_8 = 1 & \text{I} \\
x_5 - x_6 + x_7 - x_8 = 2 & \text{III} \\
x_6 - x_7 = 1 & \text{II} \\
4 - x_9 + x_{12} = 1 & \text{I} \\
x_9 - x_{10} + x_{11} - x_{12} = 2 & \text{II} \\
x_{10} - x_{11} = 1 & \text{IV} \\
x_i \in \mathbb{Z}^+ \qquad i = \overline{1,12} &
\end{array}$$

$$\begin{array}{l}
x_1 = 4 \\
x_2 = 2 \\
x_3 = 1 \\
x_4 = 1 \\
x_5 = 4 \\
x_6 = 2 \\
x_7 = 1 \\
x_8 = 1 \\
x_9 = 4 \\
x_{10} = 2 \\
x_{11} = 1 \\
x_{12} = 1
\end{array}$$

Fig. 10:

CONCLUSION

A distinctive solution method which is utilizing cut-and-choose protocol used in the solution of the classical fair division problem has been suggested in order to eliminate the deficiencies of the existing method. A program has been written based on this solution method and computational experiments have been done on various examples.

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