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Series Solutions for Boundary Value Problems in Structural Engineering and Fluid Mechanics

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Abstract: In this study, Homotopy Analysis Method (HAM) is applied to solve linear and nonlinear boundary value problems with particular significance in structural engineering and fluid mechanics. These problems are used as mathematical models in viscoelastic and inelastic flows, deformation of beams and plate deflection theory. Comparison is made between the exact solutions and the results of the homotopy analysis method. The results reveal that this method is very effective and simple and that it yields the exact solutions. It was shown that this method can be used effectively for solving linear and nonlinear boundary value problems.

Key words: Homotopy analysis method . boundary value problems . structural engineering . fluid mechanics

INTRODUCTION

The HAM is developed in 1992 by Chinese researcher Shi Jun Liao in [1-8]. This method has been successfully applied to solve many types of nonlinear problems in science and engineering by many authors [9-15] and references therein. By the present method, numerical results can be obtained with using a few iterations. The HAM contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series for large values of t. Unlike, other numerical methods are given low degree of accuracy for large values of t. Therefore, the HAM handles linear and nonlinear problems without any assumption and restriction.

This paper discusses the analytical approximate solution for fourth-order equations with nonlinear boundary conditions. The general form of the equation for a fixed positive integer n, $n = 2$, is a differential equation of order $2n$:

$$y^{(2n)} + f(x) = 0 \tag{1}$$

subject to the boundary conditions

$$y^{(2j)}(a) = A_{2j}\,,\ y^{(2j)}(b) = B_{2j},\ \ j = 0(1)n-1 \tag{2}$$

where

$$-\infty < a \le x \le b < \infty,\ \ A_{2j}\,, B_{2j},\ \ j = 0(1)n-1$$

are finite constants.

It is assumed that y is sufficiently differentiable and that a unique solution of Eq.(1) exists. Problems of this kind are commonly encountered in plate-deflection theory and in fluid mechanics for modeling viscoelastic and inelastic flows [1-3]. Usmani [1, 2] discussed sixth order methods for the linear differential equation

$$y^{(4)} + P(x)y = q(x) \tag{3}$$

subject to the boundary conditions

$$y(a) = A_0\,,\ \ y'(a) = A_2,\ \ y(b) = B_0\,,\ \ y''(b) = B_2 \tag{4}$$

The method described in [1] leads to five-diagonal linear systems and involves p′, p″, q′, q″ at a and b, while the method described in [2] leads to nine diagonal linear systems.

Ma and Silva [4] adopted iterative solutions for Eq(1) representing beams on elastic foundations. Referring to the classical beam theory, they stated that if $u = u(x)$ denotes the configuration of the deformed beam, then the bending moment satisfies the relation M = -EIu″, where E is the Young modulus of elasticity and I is the inertial moment. Considering the deformation caused by a load $f = f(x)$, they deduced, from a free-body diagram, that f = -v′ and v = M′ = -EIu‴, where v denotes the shear force. For u representing an elastic beam of length L =1, which is clamped at its left

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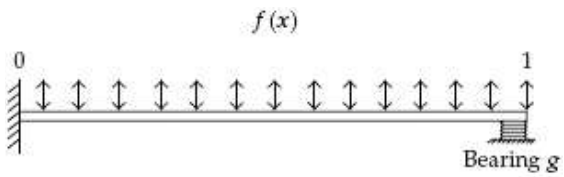

Fig. 1: Beam on elastic bearing

side $x = 0$ *and* resting on an elastic bearing at its right side $x = 1$ and adding a load f along its length to cause deformations (Fig. 1), Ma and Silva [4] arrived at the following boundary value problem assuming an $EI = 1$:

$$u^{(iv)} = f\left(x, u(x)\right), \quad 0 < x < 1 \tag{5}$$

the boundary conditions were taken as

$$u(0) = u'(0) = 1 \tag{6}$$

$$u'(1) = 0$$

and

$$u'''(1) = g\left(u(1)\right) \tag{7}$$

where $f \in C([0,1]xR)$ and $g \in C(R)$ are real functions. The physical interpretation of the boundary conditions is that $u'''(1)$ is the shear force at $x = 1$ *and* the second condition in (7) means that the vertical force is equal to $g(u(1))$, which denotes a relation, possibly nonlinear, between the vertical force and the displacement $u(1)$. Furthermore, since $u''(1) = 0$ indicates that there is no bending moment at $x = 1$, the beam is resting on the bearing g.

Solving (5) by means of iterative procedures, Ma and Silva [4] obtained solutions and argued that the accuracy of results depends highly upon the integration method used in the iterative process. Wang, Chen and Liao [5] investigated the large deformation of a cantilever beam under point load at the free tip by using the homotopy analysis method. Also Belendez *et al*. [6, 7] used classical numerical methods (Elliptic Functions, Runge-Kutta,etc) with the aid of Mathematica Software package for large and small deflections of a cantilever beam.

With the rapid development of nonlinear science, many different methods were proposed to solve differential equations, including boundary value problems (BVPs). In this paper, it is aimed to apply the homotopy analysis method proposed by Liao [8-15] to different forms of Eq.(1) subject to boundary conditions of physical significance. This method has been successfully applied to solve many types of nonlinear problems in science and engineering by many authors [16-22] and references therein. By the present method, numerical results can be obtained with using a few iterations. The HAM contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series for large values of t. Unlike, other numerical methods are given low degree of accuracy for large values of t. Therefore, the HAM handles linear and nonlinear problems without any assumption and restriction.

BASIC IDEA OF HOMOTOPY ANALYSIS METHOD

We apply the HAM [8-15] to linear and nonlinear boundary value problems with particular significance in structural engineering and fluid mechanics. We consider the following differential equation

$$N\left[u(x)\right] = 0 \tag{8}$$

where N is a nonlinear operator for this problem, x denotes an independent variables, u(x) is an unknown function.

In the frame of HAM [8-15], we can construct the following zeroth-order deformation:

$$(1-q)L\left(U(x;q) - u_0(x)\right) = q\hbar H(x) N\left(U(x,q)\right) \tag{9}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $H(x) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x)$ is an initial guess of u(x) and U(x;q) is an unknown function on the independent variables x and q.

Obviously, when q = 0 and q = 1, it holds

$$U(x;0) = u_0(x), \quad U(x;1) = u(x) \tag{10}$$

respectively. Using the parameter q, we expand U(x;q) in Taylor series as follows:

$$U(x;q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) q^m \tag{11}$$

where

$$u_m = \frac{1}{m!} \frac{\partial^m U(x;q)}{\partial^m q}\bigg|_{q=0} \tag{12}$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function H(x) are selected such that the series (11) is convergent at q = 1, then due to (10) we have

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) \tag{13}$$

Let us define the vector

$$\vec{u}_n(x) = \{u_0(x), u_1(x), \ldots, u_n(x)\} \tag{14}$$

Differentiating (9) *m* times with respect to the embedding parameter *q*, then setting q = 0 and finally dividing them by m!, we have the so-called *m*th-order deformation equation

$$L\left[u_m(x) - \chi_m u_{m-1}(x)\right] = \hbar H(x) R_m(\vec{u}_{m-1}) \tag{15}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N(U(x;q))}{\partial^{m-1} q}\bigg|_{q=0} \tag{16}$$

and

$$\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1 \end{cases} \tag{17}$$

Finally, for the purpose of computation, we will approximate the HAM solution (13) by the following truncated series:

$$\phi_m(x) = \sum_{k=0}^{m-1} u_k(x) \tag{18}$$

THE APPLICATIONS OF HAM

In this section, the HAM is applied to different forms of the fourth-order boundary value problem introduced in through Eq.(1).

Example 1: Consider the following linear boundary value problem:

$$u^{(4)}(x) = 4e^x + u(x) \quad , \quad 0 < x < 1 \tag{19}$$

subject to the boundary conditions

$$u(0) = 1, u'(0) = 2, u(1) = 2e, u'(1) = 3e \tag{20}$$

The exact solution for this problem is

$$u(x) = (1 + x)e^x \tag{21}$$

According to (9), the zeroth-order deformation can ve given by

$$(1-q)L\left(U(x;q) - u_0(x)\right) = q\hbar H(x)\begin{pmatrix} \dfrac{\partial^4 U(x;q)}{\partial x^4} - \\ U(x;q) - 4e^x \end{pmatrix} \tag{22}$$

Now it is assumed that an initial approximation has the form

$$u_0(x) = ax^3 + bx^2 + cx + d \tag{23a}$$

where *a, b, c* and *d* are unknown constants to be further determined.

We choose the auxiliary linear operator

$$L(U(x;q)) = \frac{\partial^4 U(x;q)}{\partial x^4} \tag{23b}$$

with the property

$$L(Ax^3 + Bx^2 + Cx + D) = 0$$

where *A,B, C* and *D* are integral constants.

The linear operator *L* normally consists of the homogeneous part of nonlinear operator *N* whereas parameter $\hbar$ and function *H(x)* are introduced in order to optimize the initial guess. In the present form of the article *H(x)* is set to 1 and we try to choose $\hbar$ in such a way that they get a convergent series. Under the Rule of Solution Expression [12] denoted by (11), the auxiliary function *H(x)* can be chosen as *H(x)* =1. In this way we obtain good approximations of such problems without having to go up to high order of approximation and without requiring a small parameter.

We also choose the auxiliary function to be

$$H(x) = 1$$

Hence, the *m*th-order deformation can be given by

$$L\left[u_m(x) - \chi_m u_{m-1}(x)\right] = \hbar H(x) R_m(\vec{u}_{m-1})$$

where

$$R_1(\vec{u}_0) = \frac{\partial^4 u_{m-1}(x;q)}{\partial x^4} - u_{m-1}(x;q) - 4e^x \tag{24}$$

$$R_m(\vec{u}_{m-1}) = \frac{\partial^4 u_{m-1}(x;q)}{\partial x^4} - u_{m-1}(x;q), \; m \ge 2 \tag{25}$$

Now the solution of the *m*th-order deformation equations (24-25) for m≥1 become

$$u_m(x) = \chi_m u_{m-1}(x) + \hbar L^{-1}\left[R_m(\vec{u}_{m-1})\right] \quad (26)$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_0(x) = ax^3 + bx^2 + cx + d$$

$$u_1(x) = \hbar\left(-\frac{1}{840}ax^7 - \frac{1}{360}bx^6 - \frac{1}{120}cx^5 - \frac{1}{24}dx^4 - 4e^x\right)$$

$$u_2(x) = u_1 + \hbar u_1 + \hbar^2\left(\begin{array}{c}\frac{1}{6652800}ax^{11} + \frac{1}{1814400}bx^{10} \\ + \frac{1}{362880}cx^9 + \frac{1}{40320}dx^8 + 4e^x\end{array}\right)$$

.

.

and so on.

Our solution series contain the auxiliary parameter $\hbar$. Similarly, we could choose proper values of $\hbar$ to ensure that the solution series converge. We can plot the $\hbar$-curve error (at x= 0.5), as shown in Fig. 2. It is seen that convergent results can be obtained when $-2 < \hbar \le 0$. Thus, we can choose an appropriate value for $\hbar$ in this range to get convergent solution of the our problem. We iterated auxiliary parameter $\hbar$ from-2 to 0 with 0.01 step length. We obtained tenth order approximation which contains *a,b,c,d*. Then, we applied boundary conditions to these tenth order approximations and obtained values of these four parameters for each different $\hbar$ value. We plotted absolute error graphics ($\hbar$-curve error) related to values of these four parameters.

We take tenth order approximation as:

$$U_2 = u_0 + u_1 + u_2 + \ldots + u_{10} \quad (27)$$

Incorporating the boundary conditions Eq.(20), into Eq.(27), the following coefficients can be obtained: (the auxiliary parameter $\hbar$ is taken as -1.6)

a = -1.532105007, b = -4.952945425,
c = -10.7136, d = -11.6136

Table 1 is taken as our analytic solution, in which the auxiliary parameter $\hbar$ is taken as -1.6. In order to verify numerically whether the proposed methodology leads to high accuracy, we evaluate the numerical solutions using only tenth order approximation and compare it with the exact analytical solution (21). Table 1 shows the absolute errors between exact solution and numerical solution of Eq. (19) with initial condition (20). Table 1 shows that the numerical approximate solution has a high degree of accuracy. As we know, the more terms added to the approximate solution, the more accurate it will be. Although we only considered tenth order approximation, it achieves a high level of accuracy.

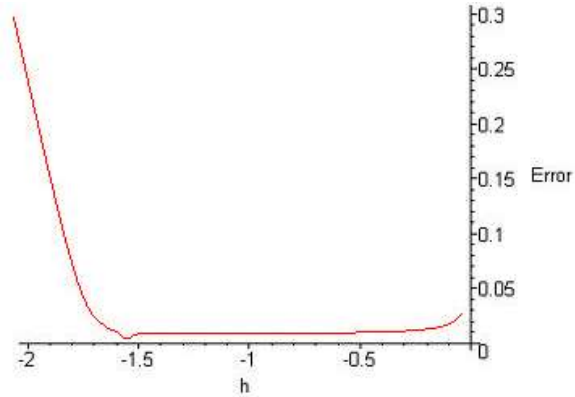

Fig. 2: $\hbar$-curve error for example 1

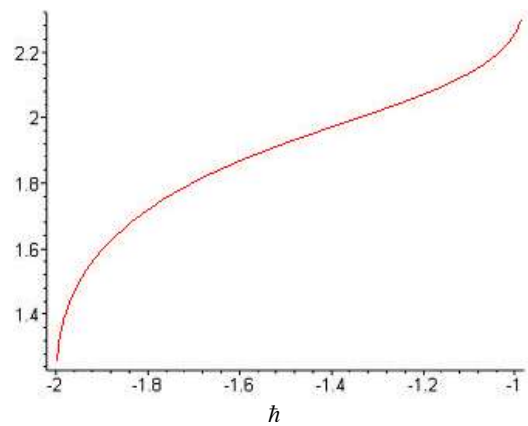

Fig. 3: $\hbar$-curve for example 1 in $u'(0)$

Table 1: Comparison of the tenth order approximate solution with exact solution

X	Exact solution	HAM-10th approximation	Absolute error
0	1,000000000	1,000000000	0,000000000
0,1	1,215688010	1,215683190	0,000004820
0,2	1,465683310	1,465689208	0,000005898
0,3	1,754816450	1,754810490	0,000005960
0,4	2,088554577	2,088550766	0,000003811
0,5	2,473081906	2,473081908	0,000000002
0,6	2,915390080	2,915395612	0,000005532
0,7	3,423379602	3,423379639	0,000000037
0,8	4,005973670	4,005973679	0,000000009
0,9	4,673245911	4,673240757	0,000005154
1	5,436563656	5,436563656	0,000000000

Convergence theorem: In this subsection, we prove that, if the solution series given by HAM is convergent, it must be an exact solution of the considered problem. If the series

$$u_0(x)+\sum_{m=1}^{\infty}u_m(x)$$

converges, where $u_m(x)$ is governed by the Eq. (26) under the definition (24-25), it must be an exact solution of Eq. (19) with initial conditions (20).

Proof: If the series is convergent, we can write

$$s=\sum_{m=0}^{\infty}u_m(x)$$

and it holds

$$\lim_{n\to\infty}u_n(x)=0$$

Then, using (15) and (23b), we have

$$\begin{aligned}&\sum_{m=1}^{\infty}\hbar H(x)R_m(\vec{u}_{m-1})\\&=\lim_{n\to\infty}\sum_{m=1}^{n}L[u_m(x)-\chi_m u_{m-1}(x)]\\&=L(\lim_{n\to\infty}\sum_{m=1}^{n}[u_m(x)-\chi_m u_{m-1}(x)])\\&=L(\lim_{n\to\infty}\sum_{m=1}^{n}u_n(x))\\&=0\end{aligned}$$

which gives, since $\hbar\neq 0$ and $H(x)\neq 0$,

$$\sum_{m=1}^{\infty}R_m(\vec{u}_{m-1})=0$$

Substituting (24-25) in the above expression,

$$\begin{aligned}&\sum_{m=1}^{\infty}R_m(\vec{u}_{m-1})=\\&\sum_{m=1}^{\infty}\frac{\partial^4 u_{m-1}(x;q)}{\partial x^4}-u_{m-1}(x;q)-(1-\chi_m)4e^x\\&=s^{(4)}-s-4e^x\\&=0\end{aligned}$$

This ends the proof.

Example 2: Consider the following linear boundary value problem:

$$u^{(4)}(x)=4e^x+u(x),\ 0<x<1 \tag{28}$$

subject to the boundary conditions

$$u(0)=1, u'(0)=0, u(1)=0, u'(1)=-e \tag{29}$$

The exact solution for this problem is

$$u(x)=(1-x)e^x \tag{30}$$

According to (9), the zeroth-order deformation can ve given by

$$(1-q)L\left(U(x;q)-u_0(x)\right)=q\hbar H(x)\begin{pmatrix}\frac{\partial^4 U(x;q)}{\partial x^4}-\frac{\partial^2 U(x;q)}{\partial x^2}\\-U(x;q)-e^x(x-3)\end{pmatrix} \tag{31}$$

Now it is assumed that an initial approximation has the form

$$u_0(x)=ax^3+bx^2+cx+d \tag{32}$$

where *a, b, c* and *d* are unknown constants to be further determined.
We choose the auxiliary linear operator

$$L(U(x;q))=\frac{\partial^4 U(x;q)}{\partial x^4}$$

with the property

$$L(Ax^3+Bx^2+Cx+D)=0$$

where *A,B, C* and *D* are integral constants. We also choose the auxiliary function to be

$$H(x)=1$$

Hence, the *m*th-order deformation can be given by

$$L[u_m(x)-\chi_m u_{m-1}(x)]=\hbar H(x)R_m(\vec{u}_{m-1})$$

where

$$\begin{aligned}R_1(\vec{u}_0)=&\frac{\partial^4 u_{m-1}(x;q)}{\partial x^4}-\frac{\partial^2 u_{m-1}(x,q)}{\partial x^2}\\&-u_{m-1}(x;q)-e^x(x-3)\end{aligned} \tag{33}$$

$$\begin{aligned}R_m(\vec{u}_{m-1})=&\frac{\partial^4 u_{m-1}(x;q)}{\partial x^4}-\frac{\partial^2 u_{m-1}(x,q)}{\partial x^2}\\&-u_{m-1}(x;q),\ m\geq 2\end{aligned} \tag{34}$$

Now the solution of the *m*th-order deformation equations (33-34) for m≥1 become

$$u_m(x)=\chi_m u_{m-1}(x)+\hbar L^{-1}\left[R_m(\vec{u}_{m-1})\right] \tag{35}$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_0(x) = ax^3 + bx^2 + cx + d$$

$$u_1(x) = \hbar(2x^4 - \frac{1}{5040}a^2x^{10} - \frac{1}{1512}bax^9 - \frac{1}{1680}(2ca + b^2)x^8 - \frac{1}{840}(2da + 2cb)x^7 - \frac{1}{360}(2db + c^2)x^6$$
$$- \frac{1}{60}dcx^5 - \frac{1}{24}d^2x^4 + \frac{1}{24024}x^{14} - \frac{1}{4290}x^{13} + \frac{1}{2970}x^{12} + \frac{1}{1980}x^{11} - \frac{1}{630}x^{10} + \frac{1}{420}x^8 - x^5)$$

….

and so on.

Our solution series contain the auxiliary parameter $\hbar$. Similarly, we could choose proper values of $\hbar$ to ensure that the solution series converge. We can plot the $\hbar$-curve error (at x= 0.5), as shown in Fig. 4. It is seen that convergent results can be obtained when $-1 \le \hbar \le 2$. Thus, we can choose an appropriate value for $\hbar$ in this range to get convergent solution of the our problem. We iterated auxiliary parameter $\hbar$ from-1 to 1 with 0.01 step length. The same procedure was used with Example 1.

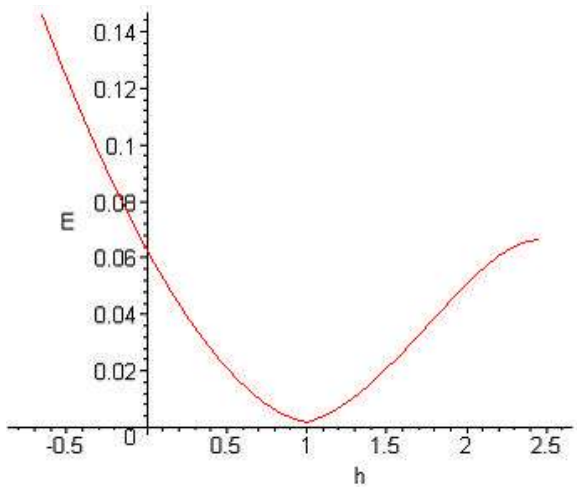

Fig. 4: $\hbar$-curve error for example 2

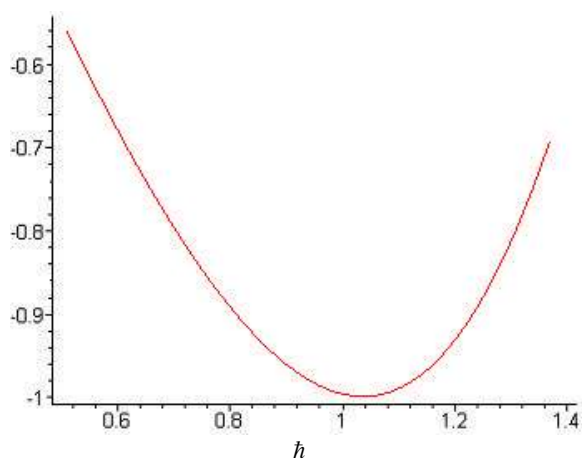

Fig. 5: $\hbar$-curve for example 2 in $u''(0)$

Table 2: Comparison of the tenth order approximate solution with exact solution

X	Exact solution	HAM-10th approximation	Absolute error
0	1,0000000000	1,0000000000	0,0000000000
0,1	0,9946538262	0,9946538262	0,0000000000
0,2	0,9771222064	0,9771222064	0,0000000000
0,3	0,9449011656	0,9449011656	0,0000000000
0,4	0,8950948188	0,8950948188	0,0000000000
0,5	0,8243606355	0,8243606355	0,0000000000
0,6	0,7288475200	0,7288475200	0,0000000000
0,7	0,6041258121	0,6041258121	0,0000000000
0,8	0,4451081856	0,4451081856	0,0000000000
0,9	0,2459603111	0,2459603111	0,0000000000
1	0,0000000000	0,0000000000	0,0000000000

We take tenth order approximation as

$$U_2 = u_0 + u_1 + u_2 + \ldots + u_{10} \tag{36}$$

Incorporating the boundary conditions Eq.(29), into Eq.(36), the following coefficients can be obtained: (the auxiliary parameter $\hbar$ is taken as 1.0)

$$a = b = c = d = 0$$

Table 2 is taken as our analytic solution, in which the auxiliary parameter $\hbar$ is taken as 1.0. In order to verify numerically whether the proposed methodology leads to high accuracy, we evaluate the numerical solutions using only tenth order approximation and compare it with the exact analytical solution (30). Table 2 shows the absolute errors between exact solution and numerical solution of Eq. (28) with initial condition (29). Table 2 shows that the numerical approximate solution has a high degree of accuracy. As we know, the more terms added to the approximate solution, the more accurate it will be. Although we only considered tenth order approximation, it achieves a high level of accuracy.

Example 3: Consider the following nonlinear boundary value problem:

$$u^{(4)}(x) = u^2(x) + g(x), \quad 0 < x < 1 \tag{37}$$

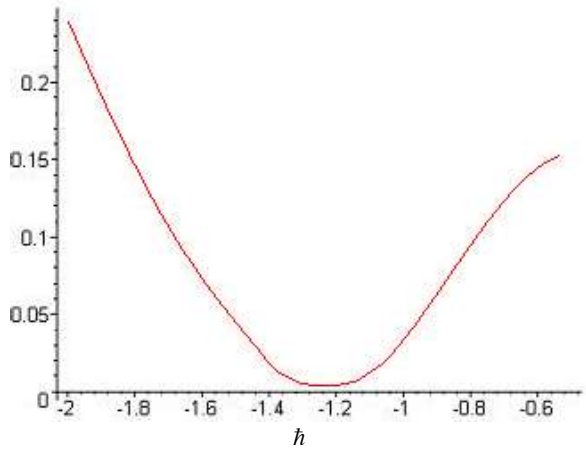

Fig. 6: $\hbar$-curve for example 3 in $u''(1)$

Table 3: Comparison of the tenth order approximate solution with exact solution

X	Exact solution	HAM-10th approximation	Absolute error
0	0,000000000	0,000000000	0,000000000
0,1	0,019810000	0,019811253	0,000001253
0,2	0,077120000	0,077110956	0,000000956
0,3	0,166230000	0,166231052	0,000001052
0,4	0,279040000	0,279041637	0,000001637
0,5	0,406250000	0,406252210	0,000002210
0,6	0,538500000	0,538501513	0,000001513
0,7	0,667870000	0,667871989	0,000001989
0,8	0,788480000	0,788480056	0,000000056
0,9	0,898290000	0,898290123	0,000000123
1	1,000000000	1,000000000	0,000000000

subject to the boundary conditions

$$u(0) = 0, u'(0) = 0, u(1) = 1, u'(1) = 1 \quad (38)$$

Where

$$g(x) = -x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48 \quad (39)$$

The exact solution for this problem is

$$u(x) = x^5 - 2x^4 + 2x^2 \quad (40)$$

We can use similar procedures which was used in Example 1-2 and obtain tenth order approximation. Table 3 shows the absolute errors between exact solution and numerical solution of Eq. (37) with initial condition (38).

CONCLUSION

This study showed that the homotopy analysis method is remarkably effective for solving boundary value problems. A fourth-order differential equation with particular engineering applications was solved using the HAM in order to prove its effectiveness. Different forms of the equation having boundary conditions of physical significance were considered. Comparison between the approximate and exact solutions showed that one iteration is enough to reach the exact solution. Therefore the HAM is able to solve partial differential equations using a minimum calculation process. This method is a very promoting method, which promises to find wide applications in engineering problems.

REFERENCES

1. Usmani, R.A., 1977. On the numerical integration of a boundary value problem involving a fourth order linear differential equation. BIT, 17 (2): 227-234.
2. Usmani, R.A., 1977. An O(h6) finite difference analogue for the solution of some differential equations occurring in plate deflection theory. Journal of the Institute of Mathematics and Its Applications, 20 (3): 331-333.
3. Momani, S.M., 1991. Some problems in non-Newtonian fluid mechanics. Ph.D. Thesis, Wales University, Wales, UK.
4. Ma, T.F. and J. da Silva, 2004. Iterative solutions for a beam equation with nonlinear boundary conditions of third order. Applied Mathematics and Computation, 159 (1): 11-18.
5. Wang, J., J.-K. Chen and S. Liao, 2008. An explicit solution of the large deformation of a cantilever beam under point load at the free tip. Journal of Computational and Applied Mathematics, 212: 320-330.
6. Belendez, T., C. Neipp and A. Belendez, 2002. Large and small deflections of a cantilever beam, European Journal of Physics, 23: 371-379.
7. Belendez, T., M. Perez-Polo, C. Neipp and A. Belendez, 2005. Numerical and experimental analysis of large deflections of cantilever beams under a combined load, Physica Scripta, 118: 61-65.
8. Liao, S.J., 1992. The proposed homotopy analysis technique for the solution of nonlinear problems. Ph.D Thesis, Shanghai Jiao Tong University.
9. Liao, S.J., 1995. An approximate solution technique which does not depend upon small parameters: A special example. Int. J. Nonlinear Mech., 30: 371-380.
10. Liao, S.J., 1997. An approximate solution technique which does not depend upon small parameters (II): An application in fluid mechanics. Int. J. Nonlinear Mech., 32: 815-822.

11. Liao, S.J., 1999. An explicit, totally analytic approximation of Blasius viscous flow problems. Int. J. Nonlinear Mech., 34 (4): 759-778.
12. Liao, S.J., 2003. Beyond perturbation: Introduction to the homotopy analysis method. Boca Raton: Chapman & Hall, CRC Press.
13. Liao, S.J., 2004. On the homotopy analysis method for nonlinear problems. Appl. Math. Comput., 147: 499-513.
14. Liao, S.J., 2002. Campo A. Analytic solutions of the temperature distribution in Blasius viscous flow problems. J. Fluid. Mech., 453: 411-425.
15. Liao, S.J., 2003. On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet. J. Fluid Mech., 488: 189-212.
16. Ayub, M., A. Rasheed and T. Hayat, 2003. Exact flow of a third grade fluid past a porous plate using homotopy analysis method. Int. J. Eng. Sci., 41: 2091-2103.
17. Hayat, T., M. Khan and S. Asghar, 2004. Homotopy analysis of MHD flows of an Oldroyd 8-constant fluid. Acta Mech., 168: 213-232.
18. Hayat, T., M. Khan and S. Asghar, 2004. Magnetohydrodynamic flow of an Oldroyd 6-constant fluid. Appl. Math. Comput., 155: 417-425.
19. Abbasbandy, S., 2007. Homotopy analysis method for heat radiation equations. Int. Comm. Heat Mass Transfer 34: 380-387.
20. Abbasbandy, S., 2007. The application of homotopy analysis method to solve a generalized Hirota-Satsuma coupled KdV equation. Phys. Lett. A, 361: 478-483.
21. Abbasbandy, S., 2007. Approximate solution for the nonlinear model of diffusion and reaction in porous catalysts by means of the homotopy analysis method. Chem. Eng. J., doi:10.1016/j.cej.2007.03.022.
22. Bataineh, A.S., M.S.M. Noorani and I. Hashim, Solving systems of ODEs by homotopy analysis method. Commun Nonlinear Sci. Numer. Sim, In Press, doi:10.1016/j.cnsns.2007.05.026.