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On the Numerical Solution of Generalized Pantograph Equation

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Absrtact: In this study, a numerical algorithm for solving a generalization of a functional differential equation known as the pantograph equation is presented. Firstly, the proposed algorithm produces an approximate polynomial solution as a power series for the problem. Then, we transform the obtained power series into Padé series form to obtain an approximate polynomial of an arbitrary order for solving pantograph equation. The structure and advantages of using of the proposed method are presented. To show the validity and applicability of the numerical method some linear and nonlinear experiments are examined. The results reveal the high accuracy and efficiency of the proposed method.

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INTRODUCTION

Functional differential equations with proportional delays are usually referred as pantograph equations or generalized equations. The name pantograph originated from the study [1] by Ockendon and Taylor and used by them to study how to electric current collected by the pantograph of an electric locomotive. Generalized pantograph equation is a special case of delay differential equations that arise in quite different fields of pure and applied mathematics such as number theory, dynamical systems, probability, quantum mechanics and electro-dynamics, population dynamics, infectious diseases, physiological and pharmaceutical kinetics and chemical kinetics, the navigational control of ships and aircraft and control problems and electronic systems [2-18].

The generalized pantograph equation is given in the following form [9-18]:

$$u^{(m)}(x)=\sum_{j=0}^{J}\sum_{k=0}^{m-1}p_{jk}(x)u^{(k)}(\alpha_j x+\beta_j)+f(x) \qquad (1)$$

with the initial conditions:

$$u^{(i)}(0)=\lambda_i,\ i=0,1,\ldots,m-1 \qquad (2)$$

where $p_{jk}(x)$ and $f(x)$ are analytical functions; c_{ik}, λ_i, α_j and β_j are real or complex constants.

Several numerical schemes have been developed for solving pantograph equations of the retarded and advanced type. The most important one are collocation method [10], spline methods [11], Runge-Kutta methods [12], θ-methods [13], Adomian decomposition method (ADM) [14], Taylor method [16], variational iteration method (VIM) [17] and homotopy perturbation method (HPM) [18].

Out of the above methods we are interested to solve this equation by a proposed method, because this method of solution produces an approximate solution in a few terms and is easy to implement.

The organization of this paper is as follows. Section 2 is devoted to introduce the mathematical preliminaries of proposed method and then we transform the obtained approximate polynomial to Padé approximate series. In Section 3, some linear and nonlinear illustrative experiments are included to demonstrate the validity and applicability of the presented technique. A brief conclusion is given in Section 4.

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THE PROPOSED ALGORITHM

Consider the generalized pantograph equation given by equation (1). The structure of the proposed algorithm is as follows.

In the first step, by m times integrating from both sides of equation (1), we convert the differential equation (1) to a integral equation as follows:

$$u(x)-\sum_{i=0}^{m-1}\lambda_i x^i = \int_0^x\int_0^x\cdots\int_0^x\left(\sum_{j=0}^{J}\sum_{k=0}^{m-1}p_{jk}(t)u^{(k)}(\alpha_j t+\beta_j)+f(t)\right)dt^m \tag{3}$$

Now, assumed that

$$u_0=\sum_{i=0}^{m-1}\lambda_i x^i=\mu(x)$$

and the approximate solution is

$$\tilde{u}_1(x)=u_0+ex=\mu(x)+ex \tag{4}$$

where e is a coefficient which is obtained as follows. We substitute (4) in (3), then we have:

$$\mu(x)+ex=\int_0^x\int_0^x\cdots\int_0^x\left(\sum_{j=0}^{J}\sum_{k=0}^{m-1}p_{jk}(t)\begin{bmatrix}\mu(\alpha_j t+\beta_j)\\+e(\alpha_j t+\beta_j)\end{bmatrix}^{(k)}+f(t)\right)dt^m \tag{5}$$

Since, the functions $p_{jk}(t)$ and $f(t)$ are assumed to be analytical functions therefore we can approximate them by suitable polynomials. Hence, the left right hands of (5) convert to a polynomial. Comparing the both sides of this equation and neglecting the higher order terms, the unknown e is obtained. Let us suppose that $e = u_1$. Thus, we have the following approximation of order one,

$$\tilde{u}_1(x)=\mu(x)+u_1x$$

In next step, we assume that new approximate solution is

$$\tilde{u}_2(x)=\mu(x)+u_1x+ex^2$$

In the same way and neglecting of higher order terms (here $O(x^3)$), the value of e will be obtained. Repeating the above procedure for aforementioned terms and higher terms, we can get the arbitrary order power series of the solutions for equation (3) as

$$\tilde{u}_n(x)=\mu(x)+u_1x+u_2x^2+\ldots+u_nx^n \tag{6}$$

Also, the power series given by the above procedure can be transformed into Padé series, easily.

Generally, Suppose that the power series $\sum_{i=0}^{\infty}a_i x^i$, represents a function $f(x)$, so that

$$f(x)=\sum_{i=0}^{\infty}a_i x^i \tag{7}$$

A Padé approximate is a rational fraction

$$[L/M]=\frac{p_0+p_1x+\ldots+p_Lx^L}{q_0+q_1x+\ldots+q_Mx^M} \tag{8}$$

which has a Maclaurin expansion which agrees with (7) as far as possible. Notice that in (8) there are L+1 numerator coefficients and M+1 denominator coefficients. There is a more or less irrelevant common factor between them and for definiteness we take $q_0 = 1$. This choice turns out to be an essential part of the precise definition and (8) is our conventional notation with this choice for q_0. So there are L+1 independent numerator coefficients and M independent denominator coefficients, making L+M+1 unknown coefficients in all. These numbers suggest that normally the [L/M] ought to fit the power series (7) through the orders 1,x, $x^2,\ldots,x^{L+M}$ in the notation of formal power series,

$$\sum_{i=0}^{n}a_i x^i=\frac{p_0+p_1x+\ldots+p_Lx^L}{q_0+q_1x+\ldots+q_Mx^M}+O(x^{L+M+1}) \tag{9}$$

Multiply the both side of (9) by the denominator of right side in (9) and compare the coefficients of both sides in (9), we have:

$$a_l+\sum_{k=1}^{M}a_{l-k}q_k=p_l,\ l=0,1,\ldots,M$$

$$a_l+\sum_{k=1}^{L}a_{l-k}q_k=0,\ l=M+1,\ldots,M+L \tag{10}$$

Solving the linear equation in (10), we have q_k, $k = 1,\ldots,L$ and substituting q_k into (10), we obtain p_l for all $l = 0,\ldots, M$; such as [19]. Therefore, we have

constructed a [L/M] Padé approximation, which agrees with $\sum_{i=0}^{\infty} a_i x^i$ through order x^{L+M}. If $M \le L \le M+2$, where M and L are the degree of numerator and denominator in Padé series, respectively, then Padé series gives an A-stable formula for an ODE [20].

ILLUSTRATIVE NUMERICAL EXPERIMENTS

In this section, five experiments of generalized pantograph equations are given to illustrate the efficiency of the method. The computations associated with the experiments discussed above were performed in *Maple 14* on a PC with a CPU of 2.4 GHz.

Experiment 3.1: Consider the following pantograph equation [14, 17]

$$u'(x) = \frac{1}{2} e^{\frac{x}{2}} u\left(\frac{x}{2}\right) + \frac{1}{2} u(x), \quad 0 \le x \le 1$$
$$u(0) = 1$$

The exact solution $u(x) = e^x$.

We have solved this problem using the proposed method. The sequence of approximate solution is obtained as follows:

$$\begin{aligned}
&n = 0 : u_0(x) = 1\\
&n = 1 : u_1(x) = 1 + x\\
&n = 2 : u_2(x) = 1 + x + \frac{x^2}{2}\\
&n = 3 : u_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}\\
&n = 4 : u_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\\
&\vdots \quad \vdots
\end{aligned}$$

and so on.
Thus, we obtain:

$$u_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!}$$

This has the closed form $u(x) = e^x$, which is the exact solution of the problem.

Experiment 3.2: Consider the following nonlinear pantograph equation of first-order [14, 18]:

$$u'(x) = 1 - 2u^2\left(\frac{x}{2}\right), \quad 0 \le x \le 1$$
$$u(0) = 0$$

which has the exact solution $u(x) = \sin x$.

We have solved this problem using the proposed method. The sequence of approximate solution is obtained as follows:

$$\begin{aligned}
&n = 0 : u_0(x) = 0\\
&n = 1 : u_1(x) = x\\
&n = 2 : u_2(x) = x\\
&n = 3 : u_3(x) = x - \frac{x^3}{6}\\
&n = 4 : u_4(x) = x - \frac{x^3}{6}\\
&n = 5 : u_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}\\
&n = 6 : u_6(x) = x - \frac{x^3}{6} + \frac{x^5}{120}\\
&\vdots \quad \vdots
\end{aligned}$$

and so on.
Thus, we obtain:

$$u_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

This convergent series solution has the closed form $u(x) = \sin x$, which is the exact solution of the problem.

Experiment 3.3: Consider the following pantograph equation of second order [16, 17]:

$$u''(x) = \frac{3}{4} u(x) + u\left(\frac{x}{2}\right) - x^2 + 2, \quad 0 \le x \le 1$$
$$u(0) = u'(0) = 0$$

The exact solution is $u(x) = x^2$. We have solved this problem using the proposed method. The sequence of approximate solution is obtained as follows:

$$\begin{aligned}
&n = 0 : u_0(x) = 0\\
&n = 1 : u_1(x) = 0\\
&n = 2 : u_2(x) = x^2\\
&n = 3 : u_3(x) = x^2\\
&n = 4 : u_4(x) = x^2\\
&\vdots \quad \vdots
\end{aligned}$$

and so on.

Therefore, the exact solution of the problem $u(x) = x^2$ is obtained just in three terms.

Experiment 3.4: Consider the pantograph equation of third order [14, 16, 17]

$$u'''(x) = -u(x) - u(x-0.3) + e^{-x+0.3}, \quad 0 \le x \le 1$$
$$u(0)=1 \;\; u'(0)=-1, \;\; u''(0)=1$$

The exact solution of this problem is e^{-x}.

We have solved this problem using the proposed method. The sequence of approximate solution is obtained as follows:

$$n=0: u_0(x)=1$$
$$n=1: u_1(x)=1-x$$
$$n=2: u_2(x)=1-x+\frac{x^2}{2}$$
$$n=3: u_3(x)=1-x+\frac{x^2}{2}-\frac{x^3}{6}$$
$$\vdots \qquad \vdots$$

and so on.
Thus, we obtain:

$$u_n(x)=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots+(-1)^n\frac{x^n}{n!}$$

This has the closed form $u(x) = e^{-x}$, which is the exact solution of the problem.

Experiment 3.5: Consider the following nonlinear pantograph equation:

$$u'''(x) = u'^2\left(\frac{x}{2}\right)\cos^2 x - u^2(x)\sin^2\left(\frac{x}{2}\right) - u'(x), \quad 0 \le x \le 1$$
$$u(0)=1, \;\; u'(0)=0, \;\; u''(0)=-1$$

The exact solution u(x) = cosx.

We have solved this problem using the proposed method. The sequence of approximate solution is obtained as follows:

$$n=0: u_0(x)=1$$
$$n=1: u_1(x)=1$$
$$n=2: u_2(x)=1-\frac{x^2}{2}$$
$$n=3: u_3(x)=1-\frac{x^2}{2}$$
$$n=4: u_4(x)=1-\frac{x^2}{2}+\frac{x^4}{24}$$
$$n=5: u_5(x)=1-\frac{x^2}{2}+\frac{x^4}{24}$$
$$\vdots \qquad \vdots$$

and so on.
Thus, we obtain:

$$u_{2n+1}(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots+(-1)^n\frac{x^{2n}}{(2n)!}$$

This convergent series solution has the closed form u(x) = cosx, which is the exact solution of the problem.

CONCLUSION

In the present paper, an efficient algorithm was proposed for solving generalized pantograph equation. The main idea of the proposed algorithm is to convert the problem including linear and nonlinear terms to Padé series. Furthermore, this method yields the desired accuracy only in a few terms and in a series form of the exact solution. The method is also quite straightforward to write computer code. These facts motivate us to consider the presented algorithm to solve generalized pantograph equations as a valid and powerful tool.

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