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Semi Generalized Local Functions in Ideal Generalized Topological Spaces

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Abstract: In this paper we define semi generalized local function $A_\mu(I,\mu)$ by using μ-semi-open sets in an ideal generalized topological spaces (X,μ,I). Some properties and characterizations of semi generalized local functions are explored.

Key words: Ideal . ideal generalized topology . μ-semi-open set . semi-generalized open set . semi-generalized local function . generalized semi compatible space . semi lindeloff space

INTRODUCTION

Á. Császár initiated the study of Generalized Topology as a generalization of topology. His fundamental concepts have been studied by many topologists in the recent years. The notion of ideal topological spaces was first introduced by Kuratowski [5]. In this paper we introduce μ-semi-generalized open sets and infer some results according to this definition. We will define semi generalized local function and explore its associated properties. The notion of semi compatibility of a generalized topology μ with an ideal I is studied. It is given in [4] that for an ideal I_f(resp, I_c) of finite (resp. countable) subsets of $X, \tau \sim I_f$ (resp $\tau \sim I_c$) if and only if (X,τ) is a hereditary compact (resp. hereditary Lindeloff). We define and characterize generalized semi compatibility in ideal generalized topological space.

PRELIMINARY

A generalized topology μ [1], on *X* is nonempty collection of subsets of *X* which satisfies the following two axioms: $\phi \in \mu$ and μ is closed under arbitrary union. The elements of μ are called μ-open and their compliment is called μ-closed. The pair (X, μ) is called a generalized topological space. And sometimes it is represented as GTS on *X*. Let (X, μ) be a generalized topological space and $A \subseteq X$, then interior and the closure of *A* are represented as $Int_\mu(A)$ and $C_\mu(A)$. The ideals on non empty set X is a non-empty collection of subsets of X which satisfy the following properties:

(i) $A \in I$ and $B \subseteq A \Rightarrow B \in I$
(ii) $A \in I$ and $B \in I \Rightarrow A \cup B \in I$.

In 2005, Császár [1] defined local function in ideal generalized topological spaces and constructed a new generalized topology μ^*. Along with that he studied its several properties.

Definition 1.1: [1] Let A be a subsets of X, the set $A^* \subset X$ is defined by $x \in A^*$ if and only if $x \in M \in \mu$ implies that $M \cap A \in I$. If $M_\mu = \cup \{M: M \in \mu\}$ and $x \in M_\mu$ then by definition $x \in A^*$. If there is no ambiguity then we write A^* in place of $A^*(I,\mu)$. And $x \notin A^*$ implies that $M \cap A \in I$.

Definition 1.2: Let (X, μ) be a generalized topological space. Then $A \subseteq X$ is said to be μ-semi-open set if there exists a μ-open set $M \in \mu$, such that $M \subseteq A \subseteq cl(M)$. The compliment of μ-semi-open set is said to be μ-semi-closed. The collection of all μ-semi-open (resp. μ-semi-closed) sets in *X* containing *x* is denoted by $SO_\mu(X, x)$ (resp. $SC_\mu(X,x)$). The μ-semi-closure of A in (X, μ) is defined by the intersection of all μ-semi-closed sets containing A and is denoted by $Scl_\mu(A)$.

SEMI GENERALIZED LOCAL FUNCTION

Let (X,μ,I) be an ideal generalized topological space and $A \subseteq X$, then we define a set $A_\mu(I,\mu) \subset X$ by $x \in A_\mu(I,\mu)$ if and only if $x \in M \in SO_\mu(X,x)$ implies that $M \cap A \notin I$. If $M_\mu = \cup\{M: M \in SO_\mu(X)\}$ and $x \notin M_\mu$ then by definition $x \in A_\mu$. And $x \notin A_\mu$, implies that $M \cap A \in I$, when there is no ambiguity, A_μ is written in place of $A_\mu(I,\mu)$.

Remarks. 2.1

(1) $A_\mu(I,\mu) \subseteq A^*(I,\mu)$ for every subset *A* of *X*.

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(2) $A_\mu(I,\mu) = A^*(I,\mu)$ if $SO_\mu(X,\mu) = \mu$.

(3) The simplest ideals are $\{\phi\}$ and $P(X) = \{A : A \subseteq X\}$ we observe that $A_\mu(\{\phi\}) = Scl_\mu(A) \neq cl_\mu(A)$ and $A^*(P(X)) = \varphi$ gives $A_\mu(P(X)) = \varphi$ for every $A \subseteq X$.

(4) If $A \in I$ then $A_\mu = \phi$.

(5) Neither $A \subseteq A_\mu$ nor $A_\mu \subseteq A$ in general.

Proof

(1) Let $x \in A_\mu(I,\mu)$, then $A \cap U \notin I$ for every $U \in SO_\mu(X,x)$. Since every μ-open set is μ-semi-open, therefore $x \in A^*(I,\mu)$. Converse is not true in general. This is shown in Example 2.2.

(2) Proof is trivial.

(3) It is quite obvious.

(4) If $A \in I$, then by the definition of semi generalized local function, It's clear that $A_\mu = \phi$.

(5) By example 2.2, $A \not\subset A_\mu$ and by example 2.3 $(A_\mu)_* \not\subset A$.

Example 2.2: Let $X = \{a,b,c,d\}$ and

$$\mu = \{\phi, \{a\}, \{a,b\}\} \text{ and } I = \{\phi, \{a\}, \{b\}, \{a,b\}\}$$

and Let A = {a,d} then

$$SO_\mu(X) = \left\{ \begin{matrix} \phi, \{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \\ \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, X \end{matrix} \right\}$$

$A_\mu = \{d\}$ which shows that $A \not\subset A_\mu$.

Example 2.3: Similarly in the above example if we substitute the value for I = {ϕ} then for A = {a,d} then $A_\mu = \{a,b,c,d\}$. This shows that $A_\mu \not\subset A$.

Theorem 2.4: Let be an ideal generalized topological space and *A, B* are subsets of *X*. Then, for semi generalized local functions in ideal generalized topology (X, μ,I), the following properties hold:

(1) If $A \subset B \subset X \Rightarrow A_\mu \subset B_\mu$.

(2) $(A \cup B)_\mu = A_\mu \cup B_\mu$

(3) If $B \in I$, then $(A - B)_\mu \subset A_\mu \subset (A \cup B)_\mu$, equality holds only if for every $x \in X$, then there exists $N \in SO_\mu(X,x)$.

(4) $A_{\mu\mu} \subset A_\mu$.

(5) If $I \subseteq \vartheta \Rightarrow A_\mu(\vartheta,\mu) \subseteq A_\mu(I,\mu)$ where $\notin \vartheta$ is an ideal on X.

(6) $A_\mu = Scl_\mu(A_\mu) \subset Scl_\mu(A)$ and A_μ is μ-semi-open set.

(7) $A_\mu - B_\mu = (A - B)_\mu - B_\mu \subset (A - B)_\mu$.

Proof

(1) Suppose that $A \subset B$ and $x \notin B_\mu$. There exists $U \in SO_\mu(X,x)$ such that $U \cap B \in I$. Since $A \subset B$, $U \cap A \in I$ and $x \notin A_\mu$ This proves that $A_\mu \subset B_\mu$.

(2) First by (1) we can have, $A_\mu \cup B_\mu \subseteq (A \cup B)_\mu$.

Now suppose that $x \in (A \cup B)_\mu$ then for every $U \in SO_\mu(X,x)$ we can write that $(U \cap A) \cup (U \cap B) = U \cap (AUB) \notin I$. Therefore, $(U \cap A) \notin I$ or $(U \cap B) \notin I$, this implies that $x \in A_\mu$ or $x \in B_\mu$, that is $x \in A_\mu \cup B_\mu$. Therefore, we have $(A \cup B)_\mu \subseteq A_\mu \cup B_\mu$. Consequently, we obtain that $A_\mu \cup B_\mu = (A \cup B)_\mu$.

(3) Result is quite obvious.

(4) Let $x \in A_{\mu\mu}$, then for ever $U \in SO_\mu(X,x)$, $U \cap A_\mu \neq \phi$. Now let $y \in U \cap A_\mu$, then $U \in SO_\mu(X,y)$ and $y \in A$ and $A \cap U \notin I$. Hence for $U \cap A \notin I$ and $x \in A_\mu$. This shows that $A_{\mu\mu} \subset A_\mu$.

(5) $x \notin A_\mu(I)$, then there exists $U \in SO_\mu(X,x)$ such that $U \cap A \notin I$ or $U \cap A \in \vartheta$ because $I \subseteq \vartheta$. This gives $x \notin A_\mu(\vartheta)$ because $(A_\mu)_*(\vartheta) \subset (A_\mu)_*(I)$.

(6) We have $A_\mu \subseteq Scl_\mu(A_\mu)$ in general. Let $x \in Scl_\mu(A_\mu)$, then $U \cap A_\mu \neq \phi$ whenever $U \in SO_\mu(X,x)$. Therefore, there exists some $y \in A_\mu \cap U$ and $U \in SO_\mu(X,y)$.

Again, let $x \in Scl_\mu(A_\mu) = A_\mu$ implies $U \cap A \notin I$ fore very $U \in SO(X,x)$. This implies $U \cap A \neq \phi$ for every $U \in SO_\mu(X,x)$. Therefore, $x \in Scl_\mu(A_\mu)$, this proves that $A_\mu = Scl_\mu(A_\mu) \subset Scl_\mu(A)$.

(7) Since $A = (A \backslash B) \cup (B \cap A)$, then by (2) $A_\mu = (A \backslash B)_\mu \cup (B \cap A)_\mu$, and hence

$$\begin{aligned} A_\mu - B_\mu &= [(A \backslash B)_\mu \cup (B \cap A)_\mu] - B_\mu \\ &= [(A - B)_\mu - B_\mu] \cup [(B \cap A)_\mu - B_\mu] \\ &= [(A \backslash B)_\mu \backslash (B)_\mu] \cup \phi \subset (A \backslash B)_\mu \end{aligned}$$

GENERALIZED SEMI-COMPATIBLE TOPOLOGY WITH AN IDEAL

Definition 3.1: Let (X,μ,I) be an ideal generalized topological space. We say that the generalized topology μ is semi-compatible with the ideal I, denoted by μ~~I, if the following holds: for $A \subseteq X$, if for $x \in A$ there exists $U \in SO_\mu(X,x)$, such that $U \cap A \notin I$, then $A \in I$.

Definition 3.2: We say that (X,μ,I) is compatible with an ideal I if $A \subseteq X$ and for $x \in A$ there exist μ-open set *U* containing *x* such that $U \cap A \notin I$, then $A \notin I$.

Remark 3.2: A compatible generalized topological space μ is semi-compatible but the converse is not true in general.

Theorem 3.3: Let (X,μ,I) be an ideal generalized topological space, then the following are equivalent:

(1) μ~~I.
(2) If a subset *A* of *X* has a cover of μ-semi-open sets each of whose intersection with *A* is in *I*, then *A* is in *I*.
(3) For every $A \subseteq X$, $A \cap A_\mu = \phi \rightarrow A \in I$.
(4) For every $A \subseteq X$, $A \setminus A_\mu \in I$.
(5) For every $A \subseteq X$, if *A* contains non-empty subset B with $B \subseteq B_\mu$, then $A \in I$.

Proof:
(1) to (2) is quite obvious from the definition of A_μ.
(2) $\Rightarrow$ (3) Let $A \subset X$ and $x \in A$, then $x \notin A_\mu$ and there exists $U \in SO_\mu(X,x)$, such that $U \cap A \notin I$, therefore we have $A \subset \cup\{U : x \in A\}$ and $U \in SO_\mu(X,x)$ and by (2) $A \in I$.
(3) $\Rightarrow$ (4) For any $A \subset X$, $A \setminus A_\mu \subset A$ and $A \setminus A_\mu \cap A \setminus A_{\mu\mu} \subset A \setminus A_\mu \cap A_\mu = \phi$ By (3) $A / A_\mu \in I$.
(4)$\Rightarrow$(5), by (4) for every $A \subset X$, $A \setminus A_\mu \in I$. Let $A/A_\mu = J \in I$, then $A = J \cup (A \cup A_\mu)$ and by theorem 2.4(2), $A_\mu = J_\mu \cup (A \cup A_{\mu\mu})$. Therefore, we have $A \cap A_\mu = A \cap (A \cap A_{\mu\mu}) \subset A \cap A_{\mu\mu}$ and $A \cap A_\mu \subset A$. By the assumption $A \cap A_\mu = \phi$ and hence $A = A \setminus A_\mu \in I$.
(5)$\Rightarrow$(1) Let $A \subset X$ and assume that for every $x \in A$, there exists $U \in SO_\mu(X,x)$, such that $U \cap A \in I$. Then $A \cap A_\mu$, since $(A \setminus A_\mu) \cap (A \setminus A_{\mu\mu}) \subset (A \setminus A_\mu) \cap A_\mu = \phi$. $A \setminus A_\mu$ contains no non-empty subset B with $B \subset B_\mu = \phi$. By (5), $A \setminus (A_\mu)_* \in I$ and we have

$$A = A \cap (X \setminus A_\mu) = A \setminus A_\mu \in I.$$

Definition 3.5: (*X*, μ) is said to be semi-Lindeloff if every cover of *X* by μ-semi-open sets of *X* has a countable subcover. A space (*X*, μ) is said to be hereditary semi-Lindeloff property.

Definition 3.6: Let (X,μ,I) be an ideal generalized topology and X is a μ-open set then (X, μ) is said to be μ-semi-compact if every cover of X has a finite subcover.

Definition 3.7: Let (X,μ,I) be an ideal generalized topology and an ideal is called the σ-ideal [8] if it is countably additive, that is if $I_n \in I$ for each $n \in N$, then $\cup\{I_n : n \in N\} \in I$.

Theorem 3.8: Let (*X*, μ) be the hereditary semi-lindeloff space satisfying condition C_1 [6] and I be a σ-ideal on *X* then μ~~I.

Proof: Let $A \subset X$ and assume that for every $x \in A$ there exists a $U \in SO_\mu(X,x)$, such that $U \cap A \in I$. This imply that $U \cap Int_\mu(A) \in I$. Now $\{U \cap Int_\mu(A) \in I.; x \in A\}$ is a cover of $Int_\mu(A)$ by μ-semi-open sets of $Int_\mu(A)$.

Let (*X*, μ) is hereditary μ-semi-Lindeloff, therefore this cover has a countable sub cover $\{U_{x(n)} \cap Int_\mu(A); x \in A, n \in N\}$. Since I is a σ-ideal, therefore $Int_\mu(A) = \cup\{U_{x(n)} \cap int(A); x \in A, n \in N\}$ is in I. If A is μ-open subset of X, then the proof is complete otherwise proposition (1) $\Rightarrow A/Int_\mu(A)$ is finite. For every $x \in A \setminus int(A)$, there exists $U \in SO_\mu(X,x)$, such that $U \cap A \in I$; hence $U \cap (A \setminus Int_\mu(A)) \in I$. By the finite additivity of I, we have $A/Int_\mu(A) = \cup\{U_x \cap (A \setminus Int_\mu(A))\}$ is in I. This proves that $A = Int_\mu(A) \cup (A \setminus Int_\mu(A)) \in I$.

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