World Applied Sciences Journal 13 (11): 2288-2295, 2011
ISSN 1818-4952

Financial Modeling by Ordinary and Stochastic Differential Equations

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Abstract: In this paper, applications of ordinary and stochastic differential equations (ODEs and SEDs) in the finance will be described. First, the bond valuation and its sensitivity to interest rate change is defined as an ordinary differential equation. Then, a noise term is added to ordinary differential equations in order to use them as a powerful mathematical tool for risky assets. Consequently, the stochastic differential equation applications are analyzed through this procedure. Finally, jump terms, stochastic volatility and Markov switching models are used to fulfill the stochastic differential equation. In each section, MATLAB is used to show applications of ODEs and SEDs in financial modeling.

Key words: Black and Scholes Model . ODE . SDE . Ito lemma . Brownian motion . mathematical finance . MATLAB

INTRODUCTION

Ordinary differential equations are used for moving (changing) phenomena and by solution of ODEs; it is possible to understand behavior of them [1]. Because of variety of these phenomena, we need different models that some of them are stochastic and need stochastic differential equations to explain their behaviors.

To understand importance of differential equations, some financial quantities will be modeled by ordinary or stochastic differential equations.

BOND PRICE CHANGES: ORDINARY DIFFERENTIAL EQUATIONS

Imagine that you are an investor with defined debts in the future and because of this you have bond portfolio. On the other hand, you want to minimize your trading costs and of course your portfolio sensitivity to interest rate changes. For better understanding of concept the following section is devoted to define some special words which will be applicable in later sections.

Securities: There are a large number of securities in which an investor may be interested. Despite the virtually infinite variety of securities, we may start by classifying the fundamental securities as bonds, stocks and derivatives. This paper is not intended to explain all kinds of securities; however, it will illustrate their mathematical role and application of MATLAB software to analysis bonds and stocks.

Fixed income securities: Fixed income securities are securities that issuer of them obligate to pay predetermined amount of money to the owner of the securities [3]. Bonds are one of the instruments that firms and public administrations may use to fund their activities. They are debt instruments which, unlike stocks, do not imply any ownership of a firm on the part of the buyer.

Basically, the buyer of a bond lends some money to the issuer, over some time span ending at bond maturity. At maturity the issuer will pay the bond owner an amount of money corresponding to the face value, also called the par value, of the bond. In the simplest bonds, coupons are fixed and expressed as a percentage of face value; coupons are usually paid annually or semi-annually.

There is another class of bonds, which just promise the payment of face value at maturity. They are called zero-coupon bonds and are typically characterized by shorter maturities [2].

Present value of bonds: Profit rate of zero coupon bonds with face value of *F*, one year maturity and price of *P* equals to

$$r = \frac{F - P}{P}$$

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Then we have

$$P = \frac{F}{1+r}$$

This equation is a valuation formula and it is not more than a simple discount formula. Consequently periodic cash flows $C_1, C_2, \ldots, C_n$ in n years results in following equation for the present value of bond,

$$PV = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_n + F}{(1+r)^2}$$
$$= \sum_{k=1}^{n} \frac{C_k}{(1+r)^k} + \frac{F}{(1+r)^n}$$

where C_k is coupon paid at year number k. If there are m payments per year at regular time intervals, we have

$$PV = \sum_{k=1}^{mn} \frac{C_k^+}{\left(1+\frac{r}{m}\right)^k} + \frac{F}{\left(1+\frac{r}{m}\right)^{mn}}$$
$$= \sum_{k=1}^{n} \frac{C_k^+}{\left(1+\frac{r}{m}\right)^k} + \frac{F}{\left(1+\frac{r}{m}\right)^{mn}}$$

or

$$PV = \sum_{k=1}^{mn} \frac{C_k}{\left(1+\frac{r}{m}\right)^k}$$

where, $C_k = C_k^+$ and $C_n = C_n^+ + F$ [2].

Interest rate sensitivity of bond prices: As we explained, by change of interest rate, r, bond prices, P, will be changed so we can say bond prices are a function of interest rate:

$$p = p(r)$$

Now let's define duration of the stream as

$$D = \frac{PV(t_0)t_0 + PV(t_1)t_1 + \cdots + PV(t_n)t_n}{PV}$$

where PV is the present value of the whole stream and $PV(t_i)$ is the present value of cash flow C_i occurring at time t_i, $i = 0,1,\ldots, n$. In some sense, the duration looks like a weighted average of cash flow times, where the weights are the present values of the cash flows [2]. Remember that for zero coupon bonds, D is equal to T, where T is time to maturity.

Considering a generic bond and employing the yield as the discount rate for computing the present values develop the following equation

$$t_0 = 0 \quad , \quad t_1 = \frac{1}{m} \quad , \quad t_k = \frac{k}{m}$$
$$P(t_k) = \frac{C_k}{\left(1+\frac{r}{m}\right)^k}$$

we get Macaulay duration:

$$D = \frac{\sum_{k=0}^{n} PV(t_k)t_k}{PV} = \frac{\sum_{k=1}^{n} \frac{k}{m} \frac{C_k}{\left(1+\frac{r}{m}\right)^m}}{\sum_{k=1}^{n} \frac{C_k}{\left(1+\frac{r}{m}\right)^m}}$$

then, the derivative of the price with respect to r is compute:

$$\frac{dp}{dr} = \frac{d}{dr}\left(\sum_{k=1}^{n} \frac{C_k}{\left(1+\frac{r}{m}\right)^k}\right)$$
$$= \sum_{k=1}^{n} C_k \frac{d}{dr}\left(\frac{1}{\left(1+\frac{r}{m}\right)^k}\right)$$
$$= -\sum_{k=1}^{n} C_k \frac{k}{m} \frac{1}{\left(1+\frac{r}{m}\right)^k} \frac{1}{\left(1+\frac{r}{m}\right)}$$

Defining D_m as $D_m = \frac{D}{1+\frac{r}{m}}$ results

$$\frac{dp}{dr} = -D_m p$$

which is an ordinary differential equation. MATLAB can compute D_m using cfdur function as follow:

```
>>[Duration,Mod_Duration]=cfdur(ChashFlow,Yield)
```

This returns both Macauley and modified duration. Now, a numeric estimation for above equation can be estimated by Taylor series expansion

$$p(r_{i+1}) = p(r_i) + (r_{i+1} - r_i)p'(r_i) + \frac{(r_{i+1} - r_i)^2}{2!}p''(r_i) + \cdots$$

Using linear approximation results,

$$p(r_{i+1}) \simeq p(r_i) + \delta\lambda_i p'(r_i)$$
$$p'(r_i) - \frac{dp_i}{dr_i}$$

which implies a first order price sensitivity model

$$\frac{\delta p_i}{\delta r_i} - -D_m p_i$$

or

$$\delta p = \quad D_m p_i \delta r$$

In order to obtain a higher order equation higher order derivatives needed to be calculated with respect to *r,*

$$\frac{d^2p}{dr^2} = \sum_{k=1}^{n} C_k \frac{k(k+1)}{m^2\left(1+\frac{r}{m}\right)^m\left(1+\frac{r}{m}\right)^2}$$

Assuming,

$$C = \frac{1}{p\left(1+\frac{r}{m}\right)^2}\sum_{k=1}^{n}\frac{K(k+1)}{m^2}\frac{C_k}{\left(1+\frac{r}{m}\right)^k}$$

Then,

$$\frac{d^2p}{dr^2} = Cp$$

where, *C* is called convexity [2] by cfconv is MATLAB function can be used to calculate this quantity

```
>>Conv=cfconv(CashFlow,Yield)
```

Second order numerical approximation can be computed by Taylor series expansion:

$$p(r_{i+1}) \cong p(r_i) + (r_{i+1} - r_i)p'(r_i) + \frac{(r_{i+1} - r_i)^2}{2!}p''(r_i)$$

By means of first order numerical approximation and above equation we have

$$\delta p \cong -D_m p \delta r + \frac{Pc}{2}(\delta r)^2$$

which is a second order approximation by duration and convexity.

STOCK PRICE CHANGES: STOCHASTIC DIFFERENTIAL EQUATIONS

Considering environmental effects as a stochastic parameter in the equation, the result model becomes more realistic. Hence, modeling of stock price changes must be done based on realistic assumptions explaining dynamics of stock price behavior. For instance with a 1$ stock, we have to be able to predict its price distribution in tomorrow, one week later or more. In practice, stock price behavior is similar to a stochastic variable, in general, stochastic process (Fig. 1).

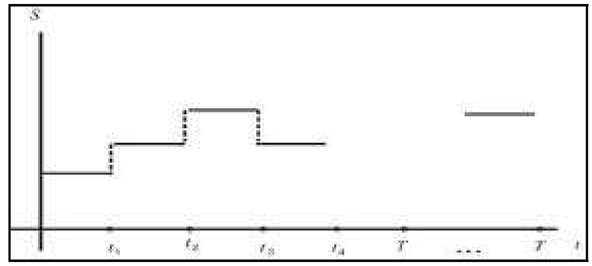

Fig. 1 Sample stock price behavior

Stock price behavior modeling: Suppose S(t) is stock price at time *t*. An initial guess consider to be a price changes which are proportional to stock price:

$$dS(t) \sim S(t)dt$$

or in differential equations form:

$$\frac{dS}{dt} = a(t)S(t)$$

with initial condition

$$S(.) = S$$

where *a(t)* is rate of stock price changes at time *t*. As explained previously, because of unknown factors, *a(t)* can be considered as a stochastic parameter:

$$a(t) = r(t) + \text{"noise"}$$

Since, the exact behavior of the noise term is indefinite and perhaps only its probability distribution is known, the rate of stock price changes is defined as below

$$\frac{dS}{dt} = (r(t) + \text{"noise"})S(t)$$

In general

$$\frac{dS}{dt} = \mu(t, xS_t) + \sigma(t, S_t).\text{"noise"}$$

where $S_t = S(t)$ and μ and σ are known functions. Let us first concentrate on the case when the noise is one dimensional. It is reasonable to look for some stochastic process W_t to represent the noise term, so that [8]

$$\frac{dS}{dt} = \mu(t,S_t) + \sigma(t,S_t)W_t$$

Now we can assume that W_t has at least approximately, these properties:

- For each $t_1 \neq t_2$ we have W_{t_2} and W_{t_1} are independent.
- The joint distribution of $\{W_{t_{k+t}}, \ldots, W_{t_{2+t}}\}$ does not depend on *t*.
- For all *t*, $E(W_t) = 0$

However, it turns out that no "reasonable" stochastic process satisfying I and II and such a W_t can't have continuous paths However, it is possible to represent W_t as generalized stochastic process called white noise process.

Now consider discrete version of this equation for $0 = t_. < t_1 < \ldots < t_m = T$ [11]

$$S_{k+1} \quad S_k = \mu(t_k,S_k)\Delta t_k + \sigma(t_k,S_k)W_k\Delta t_k$$

where

$$S_k - S_{t_k}, W_k - W_{t_k}, \Delta_{t_k} - t_{k+1} - t_k$$

then replace

$$W_k\Delta t_k = V_{k+1} \quad V_k$$

where $\{V_t\}_{t\geq}$ is some suitable stochastic process that have stationary independent increments with mean of zero. The only process with continuous paths is the Brownian motion B_t which obtained in previous equations

$$S_k = S_0 + \sum_{j=0}^{k-1} \mu(t_j,S_j).\Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j,S_j).\Delta B_j$$

Now, here is the question: Is it possible to prove that the limit of the right hand side of above equation when $\Delta t_j \to 0$ exits? If so we can have [10]

$$S_t = S_. + \int_.^t \mu(\tau,S_\tau)d\tau + \int_.^t \sigma(\tau,S_\tau)dB_\tau$$

That is an integral equation with stochastic terms or stochastic integral equation. In differential form we have stochastic differential equation that has so many applications in finance.

$$dS_t = \mu(t,S_t)dt + \sigma(t,S_t)dB_t$$

where, μ is drift and σ is volatility [6].

GENERALIZING STOCK PRICE CHANGES MODEL

There is a wide variety of generalized models that have a better performance rather than simple stochastic differential equations like geometric Brownian motion. Real data like stock returns turn out that financial market data have some special properties like:

- Assets can jump in value
- Logarithmic returns have skewness, kurtosis and asymmetrical distribution
- In this section we will explain some new models with abovementioned properties.

Jump diffusion models: Jump diffusion models are one of the new models that have better calibration results than simple models. On the other hand, these models can explain return skewness, volatility smile, market crashes.

The extra building block is required in jump diffusion model for an asset price is the Poisson process. Often in these models we cannot find a closed form solution and it's necessary to use numerical methods to solve SDEs with jump. Poisson process can be incorporated into a model for an asset as follow,

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t + d\left(\sum_{i=1}^{N(t)} Y_i\right)$$

where, Y_i is jump size with uniform iid distribution and N(t) is counter operator that in special cases is a Poisson process:

$$P(N(t) - n) - e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

In this equation Y_i is a stochastic process but in a special case of $Y_i = r$ (*r* is a constant), the resulted model is Merton jump diffusion model [12].

Markov switching models: Assume that stock price behavior in risk neutral environment follow this equation

$$\frac{dS_t}{S_{t-}} = (r - \lambda\mu_j)dt + \sigma_{\varepsilon_t d}dB_t + d\left(\sum_{i=1}^{N(t)} (V_i - 1)\right)$$

where, V_i is jump size and S_{t-} is stock price just before time *t*. Various jumps in different regimes in different stages of economy is observed in this model. Also we can add different behaviors in drift term. Generally these models don't have a closed form solution [7].

Stochastic volatility models: There is a plenty of evidence that stock return is not normally distributed; they have higher peaks and fatter tails than predicted by a normal distribution. This has been cited as evidence for non-constant and may be stochastic volatility. In stochastic volatility models, volatility of returns follows a stochastic process like below

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t)$$

where, $\{\sigma_t,\ 0 \le t \le T\}$ is a positive stochastic process like mean reverting process, Heston process and so on [10]. The following sections will explain some of the most famous stochastic volatility models.

Heston model: Heston Model assumes that volatility follows below stochastic process

$$dS(t) = S(t)\left(\mu dt + \sqrt{v(t)}dW_s(t)\right)$$

where

$$dv(t) = k\left(\theta - v(t)\right)dt + \varepsilon\sqrt{v(t)}dW_v(t)$$

and W_s and W_v are risk neutral Brownian motions. K,θ and ε are constant parameters so that θ is long term variance level and k is speed of mean reversion parameter [4, 10].

This model supposes that volatility follows Cox-Ingersoll-Ross diffusion model and weiner processes of stock returns and volatility are correlated. We can add a jump term to this model or other modifications to develop more accurate models.

Hull and white stochastic volatility model: Hull & White considered both general and specific volatility modeling. The most important result of their analysis is that when the stock and the volatility are uncorrelated and the risk neutral dynamics of the volatility are unaffected by the stock. One of the (risk-neutral) stochastic volatility models considered by Hull & White was [5, 10]

$$d(\sigma^2) = a(b \quad \sigma^2)dt + c\sigma^2 dX_2$$

GARCH diffusion model: Generalized Autoregressive Conditional Heteroscedasticity (GARCH) is a model for an asset and its associated volatility. The simplest form of this model is GARCH (1, 1) which has the form of

$$\sigma_{t+1}^2 = \omega + \beta\sigma_t^2 + \alpha\epsilon_t^2$$

where the $\in_t$ are the asset price returns after removing the drift. That is,

$$S_{t+1} = S_t(1 + \mu + \sigma_t\epsilon_t)$$

Because, historically, GARCH was developed in an econometrical and not a financial environment, the notation is different from, but related to, which are used by the current study. It can be shown that this simplest GARCH model becomes the same as the stochastic volatility model

$$d\upsilon = (a \quad b\upsilon)dt + c\upsilon dX_2$$

as the time step tends to zero and $\upsilon = \sigma^2$ [10].

3/2 model: In this model stochastic volatility follows below equation

$$d\upsilon = (a\upsilon - b\upsilon^2)dt + c\upsilon^{3/2}dX_2$$

where, mode $\upsilon = \sigma^2 l$. Like Heston model, it has a closed form solution [10].

Ornstein uhlenbeck process model: This model has the form of

$$dy = (a \quad by)dt + c dX_2$$

where, $y = \log\upsilon$ and $\upsilon = \sigma^2$ [10].

NUMERICAL RESULTS

MATLAB financial toolbox contains various functions for analysis of cash flows, sensitivity of bonds value to interest rate change, ODEs and of course SDEs.

Fixed income securities analysis in MATLAB: An example: Consider for instance the cash flow stream corresponding to a bond maturing in five years, with face value 100 and an 8% coupon rate. We can show its cash flows pattern in MATLAB by following function

```
>>cash_flow=[0 8 8 8 8 108]
```

It is obvious that present value of this stream if we discount it by an interest rate 8% is equal to face value, 100$. Now if interest rate increases to 8.8% present value will decrease to 96.8 $(equals to 3.12$ decrease).

```
>>bond_price1=pvvar(cash_flow,0.088)

bond_price1=96.8721
```

So we have 3.5$ decrease in bond value. Now we may compute the modified duration and the convexity

```
>>[duration modified_duration]=cfdur(cash_flow,0.08)

duration=5.3121

modified_duration =4.9186
>> convexity=cfconv(cash_flow,0.08)

convexity=30.1551
>>-modified_duration * bond_price1*0.01

ans =-3.8118
>>-modified_duration*bond_price1*0.008+ …
0.5*convexity*bond_price1*(0.008)^(2)

ans =-3.7183
```

As you can see, 3.7 is a good approximation for 3.5$ which is real decrease of bond value.

Stock price behavior modeling in MATLAB: One of the most important characteristics of Brownian motion is possibility of simulating this stochastic process by computer programs that is so useful to solve stochastic differential equations numerically [9]. A typical script (M-File) for simulating Brownian motion is given and its result in the following figure [6]

```
n=500;
m=5;
t=linspace(0,1,n);
dt=t(2)-t(1);
Bt=cumsum([zeros(1,5);sqrt(dt)*randn(n-1,m)]);
plot(t,Bt)
```

In this section, the Geometric Brownian Motion is employed to model Tehran stock exchange total index by build in gbm function of MATLAB.

Since GBM model just accept two parameters, mean and standard deviation of logarithmic returns (μ and σ). Following commands are required to obtaining GBM model parameters:

```
>> returns=price2ret(prices);
>> expReturn=mean(returns);
>> sigma=std(returns);
```

Then, by means of gbm command we can create Geometric Brownian Motion model as bellow

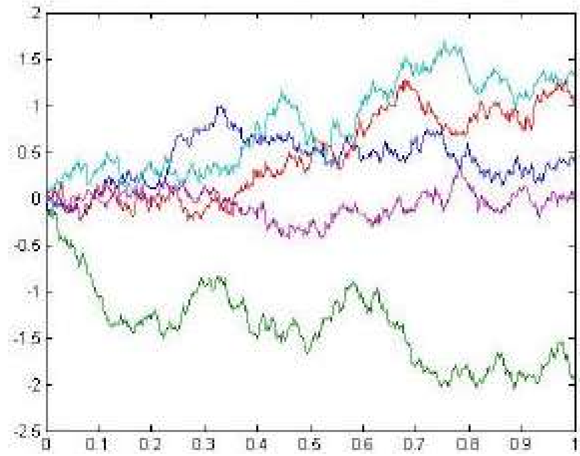

Fig. 3: Five typical sample path of Brownian motion

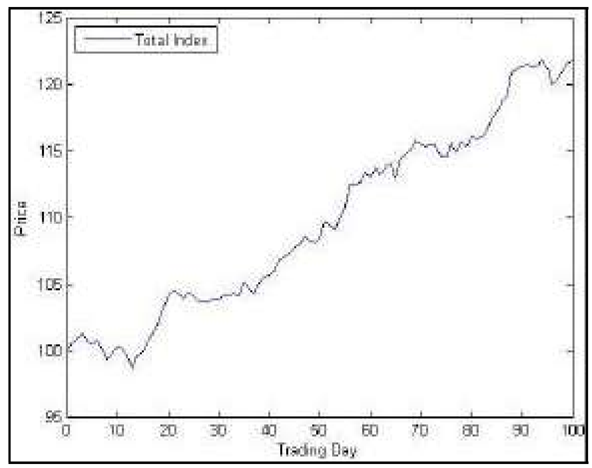

Fig. 3: Sample path of total index by GBM model

Table 1:

Function name	SDE
SDE	base SDE models
SDEMRD	mean reverting drift SDEs
GBM	Geometric Brownian Motion
HWV	Hull, White and Vasicek models
CIR	Cox, Ingersoll and Ross square root diffusion models
Heston	Heston stochastic volatility models

```
>> GBM=gbm(expReturn,sigma)
```

using graphical commands (i.e. plot) it is possible to plot a sample path for Tehran exchange total index (Fig. 3).

Also, since we can find a closed form solution for GBM model, it is possible to compare it to Euler approximation as a numerical method (Fig. 4).

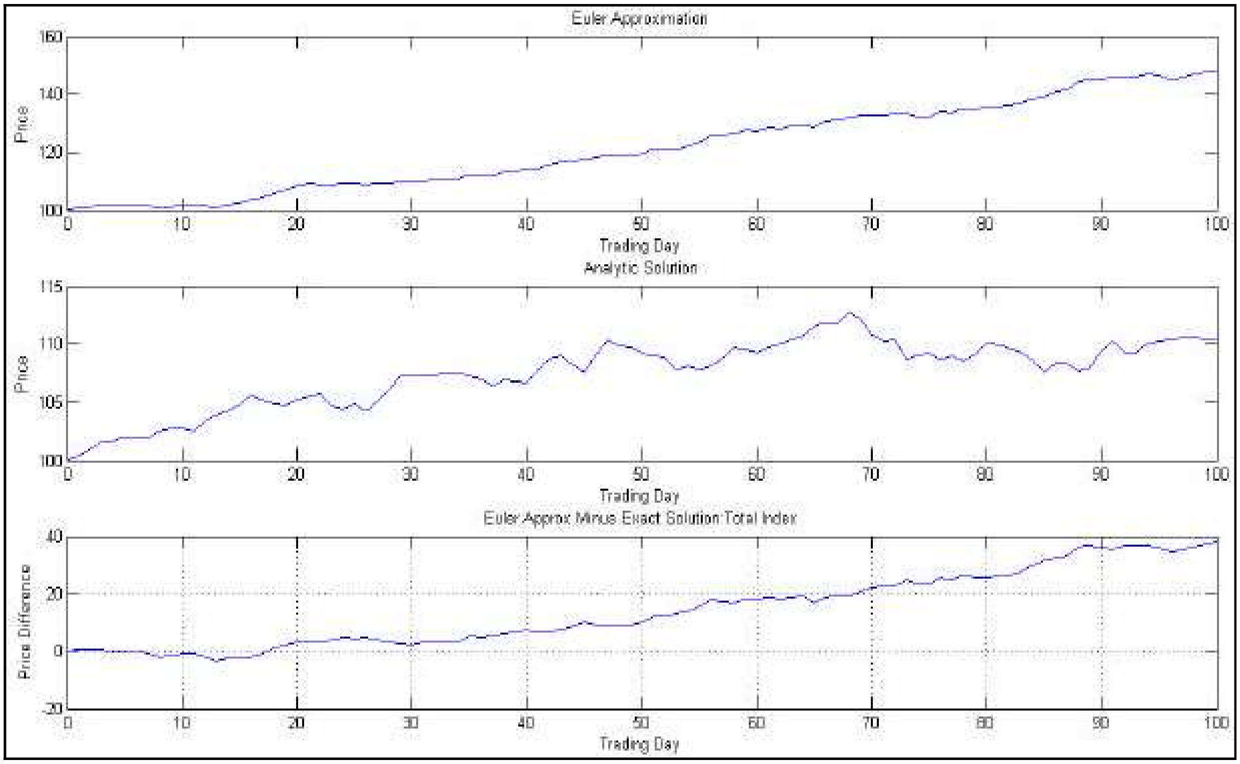

Fig. 4: Comparing closed form solution of GBM and euler approximation

Figure 4 presented a sample path of Euler approximation, sample path of closed form solution and their difference.

It is necessary to say that, MATLAB has useful and comprehensive functions to create SDE models. Table 1 introduces some of them:

CONCLUSION AND SUGGESTION FOR FUTURE STUDIES

All together, it is possible to say the most important SDEs that current paper studied are categorized in three major classes:
SDES without stochastic volatility and jump

$$dX_t = \mu(t, X_t, \theta)dt + \sigma(t, X_t, \theta)dW_t, \qquad 0 \le t \le T$$

where μ and σ are constant.

- SDES with stochastic volatility and without jump

$$\begin{pmatrix} dX_t \\ d\alpha_t \end{pmatrix} = \begin{pmatrix} \mu_x(X_t, \alpha_t, \theta) \\ \mu_\alpha(\alpha_t, \theta) \end{pmatrix} dt + \begin{pmatrix} \sigma_x(\alpha_t, \theta) & 0 \\ 0 & \sigma_\alpha(\alpha_t, \theta) \end{pmatrix} \begin{pmatrix} dB_t \\ dW_t \end{pmatrix}$$

- SDES with stochastic volatility and jump

$$\begin{pmatrix} dX_t \\ d\alpha_t \end{pmatrix} = \begin{pmatrix} \mu_x(X_t, \alpha_t, \theta) \\ \mu_\alpha(\alpha_t, \theta) \end{pmatrix} dt + \begin{pmatrix} \sigma_x(\alpha_t, \theta) & 0 \\ 0 & \sigma_\alpha(\alpha_t, \theta) \end{pmatrix} \begin{pmatrix} dB_t \\ dW_t \end{pmatrix} + \begin{pmatrix} d\left(\sum_{i=1}^{N(t)} Y_j\right) \\ d\left(\sum_{i=1}^{N(t)} Z_j\right) \end{pmatrix}$$

where $d\left(\sum_{i=1}^{N(t)} Y_j\right)$ is jump term of X and $d\left(\sum_{i=1}^{N(t)} Z_j\right)$ is jump term of α_t..

Despite of wide applications of stochastic differential models with stochastic volatility, they have two major problems in practice. First, selection of proper model and estimation of its parameters (i.e., volatility of volatility) is subjected to controversy.

Second, SDE with constant volatility provides one source of randomness; however, stochastic volatility models require two source of randomness which makes it very difficult to hedge our portfolio, because in financial markets, volatility does not trade! If we use two options to hedge our portfolio, there would be one equation and two unknown. This could be a good field of study for mathematical finance students.

On the other hand, there is no closed form solution for most of the complicated SDEs and we have to use numerical methods to solve them which can be second suggestion for future studies.

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