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The Nodal Points for Uniqueness of Inverse Problem in Boundary Value Problem with Aftereffect

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Abstract: In this paper, the inverse nodal is studied for the second order differential operators on a finite interval. The oscillation of the eigenfunctions corresponding to the large modulus eigenvalues is established and an asymptotic of the nodal points is obtained. However, the uniqueness theorem is proved.

Key words: Nodal points · Inverse problem · Asymptotic form · Eigenvalues

INTRODUCTION

Inverse nodal problems consist in constructing operators from given nodes (zeros) of their eigenfunctions. Mclaughlin seems to be the first to consider this sort of inverse problem (see [1]). Later on, some remarkable results were obtained. For example, X.F. Yang got the uniqueness for general boundary conditions using the same method as McLaughlin (see [2]; C.K. Law and Ching-Fu yang (see [3]) have reconstructed the potential function and its derivatives from nodal data. Besides, the readers can refer to [4, 5]. We consider boundary value problem with "aftereffect" L=L(q,M) of the form

$$ly(x):=-y''(x)+q(x)y(x)+\int_0^x M(x-t)y(t)dt=\lambda y(x)=\rho^2 y(x), \quad (1)$$

$$y(0)=y(\pi)=0. \quad (2)$$

Here λ is the spectral parameter and $q(x)\in L_2(0,\pi)$ is a real function. Let $\{\lambda_n\}_{n\geq 1}$ be the eigenvalues of a boundary value problem L. The presence of an aftereffect in a mathematical model produces qualitative changes in the study of the inverse problem. The uniqueness theorem for boundary value problem with aftereffect by the transformation operator method was studied in [6]. It says that the function M(x) is uniquely determined from the given q(x) and the spectrum $\{\lambda_n\}_{n\geq 1}$. We describe this method, Because some techniques will be used. In this paper, using of the nodal points we show uniqueness of M(x). In orther word, the function M(x) is uniquely determined from a dense set of nodal points and given q(x). The paper is organized as follows. Section 2 deals with uniqueness theorem by the transformation operator method. In section 3, we obtain the eigenfunctions corresponding to the large modulus eigenvalues and an asymptotic of the nodal points. Furthermore, we will give a uniqueness theorem.

Uniqueness Theorem by the Transformation Operator Method: In this section, we study the uniqueness theorem by the transformation operator method. Put

$$M_0(x)=(p-x)M(x),\ M_1(x)=\int_0^x M(t)dt, Q(x)=M_0(x)-M_1(x).$$

We shall assume that $Q(x)\in L_2(0,\pi)$, $M_k(x)\in L(0,\pi)$, k = 01.

Let $S(x,\lambda)$ be the solution of (1) under the initial conditions $S(0,\lambda)=0$, $S'(0,\lambda)=1$. Denote $\Delta(\lambda)=S(\pi,\lambda)$. The eigenvalues $\{\lambda_n\}_{n\geq 1}$ of the boundary value problem are real and coincide with the zeros of the function $\Delta(\lambda)$ and like in the proof of Theorem 1.1.3 in [1], we get for $n\to\infty$

$$\rho_n=\sqrt{\lambda_n}=n+\frac{A_1}{n}+\frac{\kappa_n}{n},\ \{\kappa_n\}\in l_2,\ A_1=\frac{1}{2(\pi)}\int_0^\pi q(t)dt. \quad (3)$$

The function $S(x,\lambda)$ is the solution of the integral equation

$$S(x,\lambda)=\frac{\sin\rho x}{\rho}+\int_0^x\frac{\sin\rho(x-\tau)}{\rho}\Big(q(t)S(\tau,\lambda)+\int_0^\tau M(\tau-s)S(s,\lambda)ds\Big)d\tau. \quad (4)$$

For $|\rho|\to\infty$ on can establish the asymptotic

$$S(x,\lambda)=\frac{\sin\rho x}{\rho}+O\Big(\frac{1}{|\rho|^2}e^{|\mathrm{Im}\,\rho|x}\Big), \quad (5)$$

uniformly with respect to $x\in[0,\pi]$.

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Lemma 1: The representation

$$S(x,\lambda)=\frac{\sin\rho x}{\rho}+\int_0^x K(x,t)\frac{\sin\rho t}{\rho}dt, \tag{6}$$

holds, where K(x,t) it is a continuous function and K(x,0) = 0.
Proof. See [1].

Lemma 2: The function $\Delta(\lambda)$ is uniquely determined by its zeros. Moreover,

$$\Delta(\lambda)=\pi\prod_{n=1}^{\infty}\frac{\lambda_n-\lambda}{n}. \tag{7}$$

Proof. See [1].

We are now in a position to state and prove the uniqueness theorem for the solution of inverse problem (1). Let $\{\tilde{\lambda}_n\}_{n\geq 1}$ be the eigenvalues of the boundary value problem $\tilde{L}=L(q,\tilde{M})$.

Theorem 1 If $\lambda_n=\tilde{\lambda}_n, n\geq 1$, then $M(x)=\tilde{M}(x), x\in(0,p)$.

Proof. Let the function $S^*(x,\lambda)$ be the solution of the equation

$$^*z(x):=-z''(x)+q(x)z(x)+\int_x^{\pi}M(t-x)z(t)dt=\lambda z(x) \tag{8}$$

under the conditions $S^*(\pi,\lambda)=0$, $S^{*\prime}(\pi,\lambda)=1$. Put $\Delta^*(\lambda)=S^*(0,\lambda)$ and denote $\hat{M}(x):=M(x)-\tilde{M}(x)$. Then

$$\int_0^{\pi}S^*(x,\lambda)\int_0^x\hat{M}(x-t)\tilde{S}(t,\lambda)dtdx$$

$$=\int_0^{\pi}S^*(x,\lambda)l\tilde{S}(x,\lambda)dx-\int_0^{\pi}S^*(x,\lambda)\tilde{l}\tilde{S}(x,\lambda)dx$$

$$=\int_0^{\pi}l^*S^*(x,\lambda)\tilde{S}(x,\lambda)dx-\int_0^{\pi}S^*(x,\lambda)\tilde{l}\tilde{S}(x,\lambda)dx$$

For $\tilde{l}=l$ we have $\Delta^*(\lambda)=\Delta(\lambda)$ and consequently

$$\int_0^{p}S^*(x,\lambda)\int_0^x\hat{M}(x-t)\tilde{S}(t,\lambda)dtdx=\hat{\Delta}(\lambda). \tag{9}$$

Transform (9) into

$$\int_0^x\hat{M}(x)(\int_x^p S^*(t,\lambda)\tilde{S}(t-x,\lambda)dt)dx=\hat{\Delta}(\lambda). \tag{10}$$

Denote $\omega(x,\lambda)=S^*(p-x,\lambda), \hat{N}(x)=\hat{M}(p-x),$

$$\varphi(x,\lambda)=\int_0^x\omega(t,\lambda)\tilde{S}(t-x,\lambda)dt. \tag{11}$$

Then (10) takes the form

$$\int_0^p\hat{N}(x)\varphi(x,\lambda)dx=\hat{\Delta}(\lambda). \tag{12}$$

The representation

$$\varphi(x,\lambda)=\frac{1}{2\rho^2}(-x\cos\rho x+\int_0^x V(x,t)\cos\rho t dt) \tag{13}$$

holds, where V(x,t) is a continuous function. (See [1, Lemma 4.6.3])

Since $\lambda_n=\tilde{\lambda}_n, n\geq 1$, we have by Lemma 1

$$\Delta(\lambda)=\tilde{\Delta}(\lambda).$$

Then, substituting (13) into (12), we obtain

$$\int_0^{\pi}\cos\rho x(-x\hat{N}(x)+\int_x^{\pi}V(t,x)\hat{N}(t)dt)dx=0$$

And consequently,

$$-x\hat{N}(x)+\int_x^{\pi}V(t,x)\hat{N}(t)dt=0. \tag{14}$$

For each fixed $\epsilon>0$, (14) is a homogeneous Volterra integral equation of the second kind in the interval (ϵ,π). Consequently, $\hat{N}(x)=0$ a.e. in (ϵ,π) and, since ϵ is arbitrary, this holds in the whole interval $(0,\pi)$. Thus, $M(x)=\tilde{M}(x)$ a.e. in $(0,\pi)$.

Asymptotic of the Nodal Points. Uniqueness Theorem: The eigenfunctions of the boundary value problem L have the form $y_n(x)=S(x,\lambda_n)$. For the boundary value problem an analog of Sturm's oscillation theorem is true. More precisely, the eigenfunction $y_n(x)$ has exactly n zeros inside the interval $(0,\pi)$. Namely:

$0<x_n^1<x_n^2<...<x_n^j<p, j=1,2,...,n-1.$ The set $\{x_n^j\}_{n\geq 1, j=\overline{1,n}}$ is called the set of nodal points of the boundary value problem. It is shown that the set of all nodal points $\{x_n^j\}$ is dense in $[0,\pi]$.

Theorem 2 The nodal points of the problem (1)-(2) are

$$x_n^j=a_n^j-\frac{A_1}{n^2}a_n^j+O(\frac{1}{n^3}), \tag{15}$$

Where $a_n^j:=\frac{jp}{n}.$

Proof: Substituting (3) into (5) give the following asymptotic formula for $n \to \infty$ uniformly in x:

$$\sin(n+\frac{A_1}{n})x + O(\frac{1}{n^2})=0, \tag{16}$$

$$\sin nx+\cos nx(\frac{A_1}{n})x + O(\frac{1}{n^2}) = 0. \tag{17}$$

We obtain the following asymptotic formulae for the nodal points as $n \to \infty$ uniformly in j:

$$x_n^j = \alpha_n^j - \frac{A_1}{n^2}\alpha_n^j + O(\frac{1}{n^3}), \tag{18}$$

Where

$$a_n^j := \frac{jp}{n}.$$

Now, we will give a uniqueness theorem. It says that the function M(x) is uniquely determined by a dense subset of the nodes and the function q(x).

Theorem 3: The function M is uniquely determined by any dense set of nodal points and given the function q.

Proof: Assume that we have two problems of the (1)-(2) with M, $\tilde{M}$. Let the nodal points $x_n^j, \bar{x}_n^j$ satisfying $x_n^j = \bar{x}_n^j$ form a dense set in $[0,\pi]$. We tke solutions of (1)-(2) as φ_n and $\tilde{\varphi}_n$. It follow from (1) that

$$(\varphi_n'\tilde{\varphi}_n' - \tilde{\varphi}_n'\varphi_n)' = [q - \tilde{q} + \tilde{\rho}_n^2 - \rho_n^2]\varphi_n\tilde{\varphi}_n + \tilde{\varphi}_n\int_0^x M(x-t)j_n(t,\lambda_n)dt$$

$$-\varphi_n\int_0^x \tilde{M}(x-t)\tilde{\varphi}_n(t-\tilde{\lambda}_n)dt. \tag{19}$$

We integrate both sides of (19) from 0 to x_n^j and using the boundary conditions (2) we obtain

$$0 = \int_0^{x_n^j}[q - \tilde{q} + \tilde{\rho}_n^2 - \rho_n^2]\varphi_n(x,\lambda_n)\tilde{\varphi}_n(x,\tilde{\lambda}_n)dx$$

$$+\int_0^{x_n^j}\tilde{\varphi}_n(x,\tilde{\lambda}_n)\int_0^x M(x-t)\varphi_n(t,\lambda_n)dtdx$$

$$-\int_0^{x_n^j}\varphi_n(x,\lambda_n)\int_0^x \tilde{M}(x-t)\tilde{\varphi}_n(t,\tilde{\lambda}_n)dtdx.$$

From the asymptotic forms of $\tilde{\lambda}_n$ and λ_n

$$0 = \int_0^{x_n^j}[q - \tilde{q} - \frac{1}{p}\int_0^p q(t)dt + \frac{1}{p}\int_0^p \tilde{q}(t)dt]\varphi_n\tilde{\varphi}_n dx$$

$$+\int_0^{x_n^j}\int_0^x M(x-t)\tilde{\varphi}_n(x,\tilde{\lambda}_n)\varphi_n(t,\lambda_n)dtdx$$

$$-\int_0^{x_n^j}\int_0^x \tilde{M}(x-t)j_n(x,\lambda_n)\tilde{\varphi}_n(t,\tilde{\lambda}_n)dtdx.$$

Suppose that $A_1 = \tilde{A}_1$. We take a sequence x_n^j accumulating at an arbitrary $a \in [0, \pi\}$ as $n \to \infty$ uniformly in j. Hence,

$$\int_0^a \tilde{\varphi}_n(x,\tilde{\lambda}_n)\int_0^x M(x-t)\varphi_n(t,\lambda_n)dtdx = \int_0^a j_n(x,\lambda_n)\int_0^x \tilde{M}(x-t)\tilde{j}_n(t,\tilde{\lambda}_n)dtdx.$$

Similar to the proof of Theorem 1, we can conclude that $M = \tilde{M}$ almost every where on $[0,\pi]$.

REFERENCES

1. Mclaughlin, J.R., 1988. Inverse spectral theory using nodal points as data-a uniqueness result, J. Differential Equations, 73: 354-362.
2. Yang, X.F., 1997. A solution of the inverse nodal problem, Inverse Problem, 13: 203-213.
3. Law, C.K. and Ching-Fu Yang, 1998. Reconstructing the potential function nad its derivatives using nodal data, Inverse Problem, 14: 299-312.
4. Shen, C.L. and T.M. Tasi, 1995. On a uniform approximation of the density function of a string equation using EV sans nodal points and some related inverse nodal problems, Inverse Problems, 11(5): 1113-1123.
5. Shen, C.L. and C.T. Shieh, 2000. An inverse nodal problem for vectorial sturm-Liouville equation, Inverse Problems, 16: 349-356.
6. Freiling, G. and V.A. Yurko, 2001. Inverse Sturm-Liouville problems and their applications, NOVA science publishers, New York.