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Numerical Solution of Mew Equation by Using Finite Difference Method

Pınar Keskin and Dursun Irk

Department of Mathematics and Computer Science,
Eskişehir Osmangazi University, 26480, Eskişehir, Türkiye

Abstract: We present a numerical study of the modified equal width (MEW) equation by using the finite difference method. The method is examined for the motion of single solitary wave and interaction of two solitary waves. The obtained numerical results show that the present method is a remarkably successful numerical technique for solving the MEW equation.

Key words: Numerical solution of PDE, Finite difference method, Solitary wave, Modified equal width equation

INTRODUCTION

The modified equal width wave (MEW) equation based on the equal width wave (EW) equation, which was introduced by Morrison *et al.* [1] as a model for non-linear dispersive waves, has the form with the boundary conditions →0 as $x\to\pm\infty$,

$$u_t + \varepsilon u^2 u_x - \mu u_{xxt} = 0, \quad (1)$$

Where u, t and x denote the amplitude, time and spatial coordinate, respectively, ε and μ are positive constants. Boundary conditions on the region $a \le x \le b$ are chosen from

$$\begin{aligned} &u(a,t) = u(b,t) = 0, \\ &u_x(a,t) = u_x(b,t) = 0, \quad t \in (0,T] \\ &u_{xx}(a,t) = u_{xx}(b,t) = 0, \end{aligned} \quad (2)$$

and the initial condition $u(x, 0) = f(x)$ will be defined in the numerical experiments section.

The MEW equation has an analytical solution with a limited set of boundary and initial conditions. Therefore, many authors have used various kinds of numerical methods to solve Eq. (1). Zaki [2] solved the MEW equation numerically by a Petrov-Galerkin method using quintic B-spline finite elements to investigate migration of a single solitary wave, interaction of two solitary waves and birth of solitons. The numerical solution of the MEW equation was obtained by using a lumped Galerkin method based on quadratic B-spline finite elements in the paper [3]. Quintic B-spline collocation algorithms for numerical solution of the MEW equation have been proposed by Saka [4]. Evans and Raslan presented a collocation method for the MEW equation using quadratic B-splines at midpoints as element shape functions [5]. Esen and Kutluay used a linearized numerical scheme based on finite difference method to obtain solitary wave solutions of MEW equation [6].

Solitary waves are stable and can travel over very large distances without change their shapes and when a taller solitary wave overtakes a shorter solitary wave, they don't combine and add together. In this study, MEW equation is solved numerically using the finite difference method. After the new time discretization of the Eq. (1) is performed, five-point stencils approximating first and second derivatives for the space discretization are used to obtain a system of algebraic equation. The nonlinear part of the resulting system of the method is handled by using an inner iteration. In the numerical experiments section, the propagation of a solitary wave and interaction of two solitary waves test problems are investigated and it is found that obtained numerical results are in good agreement with the analytical solutions.

Time Discretization and Finite Difference Method: For computational work, the space-time plane is discretized by grid with space step length h and the time step Δt. The exact solution of unknown function at the grid point is denoted by

$$u(x_m, t_n) = u_m^n,\ m = 0,1,\ldots,N;\ \ n = 0,1,2,\ldots$$

Corresponding Author: Dursun Irk, Eskişehir, Department of Mathematics and Computer Science, Eski ehir Osmangazi University, 26480, Eski ehir, Türkiye, Tel: +90-222-2393750, E-mail: dirk@ogu.edu.tr.

and the notation U_m^n is used to represent the numerical value of u_m^n.

To solve Eq. (1), numerically, we replace the setting $v(x, t) = u - \mu u_{xx}$. Then Eq. (1) can be written as

$$v_t = -\varepsilon u^2 u_x \qquad (3)$$

Consider applying a Taylor series expansion to find $v(x_m, t_n + \Delta t)$ instead, assuming that all necessary derivatives exist:

$$v_m^{n+1} = v_m^n + \Delta t (v_t)_m^n + \frac{\Delta t^2}{2!}(v_{tt})_m^n + \frac{\Delta t^3}{3!}(v_{ttt})_m^n + O(\Delta t^4).$$

Using the following finite difference approximation for the second and third time derivatives of v:

$$(v_{tt})_m^n \approx \frac{(v_t)_m^{n+1} - (v_t)_m^n}{\Delta t}, \qquad (4)$$

$$(v_{ttt})_m^n \approx \frac{(v_t)_m^{n+1} - 2(v_t)_m^n + (v_t)_m^{n-1}}{\Delta t^2}, \qquad (5)$$

We have

$$v_m^{n+1} - (v_t)_m^{n+1}\left(\frac{\theta_1 \Delta t}{2} + \frac{\theta_2 \Delta t}{6}\right) = v_m^n + (v_t)_m^n\left(\Delta t - \frac{\theta_1 \Delta t}{2} - \frac{\theta_2 \Delta t}{3}\right) + \frac{\theta_2 \Delta t}{6}(v_t)_m^{n-1} \qquad (6)$$

Where θ_1 and θ_2 are the real constants determined in later to have higher accuracy of the proposed method with respect to time discretization. Using the (3) and then after some mathematical manipulations, Eq. (6) can be rewritten as.

$$\begin{aligned} &U_m^{n+1} + \varepsilon\left(\frac{\theta_1 \Delta t}{2} + \frac{\theta_2 \Delta t}{6}\right)\left(U^2\right)_m^{n+1}\left(U_x\right)_m^{n+1} - \mu\left(U_{xx}\right)_m^{n+1} = \\ &U_m^n - \varepsilon\left(\Delta t - \frac{\theta_1 \Delta t}{2} - \frac{\theta_2 \Delta t}{3}\right)\left(U^2\right)_m^n\left(U_x\right)_m^n - \mu\left(U_{xx}\right)_m^n - \frac{\theta_2 \Delta t}{6}\varepsilon\left(U^2\right)_m^{n-1}\left(U_x\right)_m^{n-1}. \end{aligned} \qquad (7)$$

It can be easily seen that the local truncation errors are $O(\Delta t^3)$ with $\theta_1 = 1, \theta_2 = 0$ and $O(\Delta t^4)$ with $\theta_1 = 1, \theta_2 = -1/2$. For the space discretization, we take five-point stencils approximating first and second derivatives

$$(U_x)_m = \frac{U_{m-2} - 8U_{m-1} + 8U_{m+1} - U_{m+2}}{12h} + O(h^4), \qquad (8)$$

$$(U_{xx})_m = \frac{-U_{m-2} + 16U_{m-1} - 30U_m + 16U_{m+1} - U_{m+2}}{12h^2} + O(h^4). \qquad (9)$$

Then, substituting (8-9) into Eq. (7), the resulting algebraic system of equations takes the form

$$\begin{aligned} &U_{m-2}^{n+1}\left(\frac{\alpha_{m1}}{12h} + \frac{\mu}{12h^2}\right) + U_{m-1}^{n+1}\left(-\frac{2\alpha_{m1}}{3h} - \frac{4\mu}{3h^2}\right) + U_m^{n+1}\left(1 + \frac{5\mu}{2h^2}\right) + \\ &U_{m+1}^{n+1}\left(\frac{2\alpha_{m1}}{3h} - \frac{4\mu}{3h^2}\right) + U_{m+2}^{n+1}\left(-\frac{\alpha_{m1}}{12h} + \frac{\mu}{12h^2}\right) = \\ &U_{m-2}^n\left(-\frac{\alpha_{m2}}{12h} + \frac{\mu}{12h^2}\right) + U_{m-1}^n\left(\frac{2\alpha_{m2}}{3h} - \frac{4\mu}{3h^2}\right) + U_m^n\left(1 + \frac{5\mu}{2h^2}\right) + \\ &U_{m+1}^n\left(-\frac{2\alpha_{m2}}{3h} - \frac{4\mu}{3h^2}\right) + U_{m+2}^n\left(\frac{\alpha_{m2}}{12h} + \frac{\mu}{12h^2}\right) + \\ &U_{m-2}^{n-1}\left(-\frac{\alpha_{m3}}{12h}\right) + U_{m-1}^{n-1}\left(\frac{2\alpha_{m3}}{3h}\right) + U_{m+1}^{n-1}\left(-\frac{2\alpha_{m3}}{3h}\right) + U_{m+2}^{n-1}\left(\frac{\alpha_{m3}}{12h}\right) \end{aligned} \qquad (10)$$

Where

$$\alpha_{m1} = \varepsilon\left(\frac{\theta_1 \Delta t}{2} + \frac{\theta_2 \Delta t}{6}\right)\left(U_m^{n+1}\right)^2,$$

$$\alpha_{m2} = \varepsilon\left(\Delta t - \frac{\theta_1 \Delta t}{2} - \frac{\theta_2 \Delta t}{3}\right)\left(U_m^{n}\right)^2,$$

$$\alpha_{m3} = \varepsilon\frac{\theta_2 \Delta t}{6}\left(U_m^{n-1}\right)^2.$$

Whereas the finite difference scheme (10) with $\theta_1 = 1$ and $\theta_1 = 1/2$ has a truncation error of $O(\Delta t^4 + \Delta t h^4)$, the scheme with $\theta_1 = 1$ and $\theta_2 = 0$ has a truncation error of $O(\Delta t^3 + \Delta t h^4)$.

This set of equations is recurrence relationship for unknown parameters vector $\boldsymbol{d}^{n+1} = \left(U_{-2}^{n+1}, U_{-1}^{n+1}, \ldots, U_{N+1}^{n+1}, U_{N+2}^{n+1}\right)$. This pentadiagonal matrix system is made up $N + 1$ equations including $N + 5$ unknown for $m = 0,1,\ldots,N-1,N$. Application of the boundary conditions

$$\left(U_x\right)_0^{n+1} = \left(U_x\right)_N^{n+1} = 0,$$

$$\left(U_{xx}\right)_0^{n+1} = \left(U_{xx}\right)_N^{n+1} = 0$$

enables the elimination of the variables $U_{-2}^{n+1}, U_{-1}^{n+1}, U_{N+1}^{n+1}$ and U_{N+2}^{n+1} from the system (10). Then, the system is reduced to $(N+1)\times(N+1)$ matrix system, which can be solved by using the Thomas algorithm. After $\boldsymbol{d}^{0} = \left(U_{-2}^{0}, U_{-1}^{0}, \ldots, U_{N+1}^{0}, U_{N+2}^{0}\right)$ unknown vector is found from the initial condition, taking $\theta_1 = 1$, $\theta_2 = 0$ in the system (10), we can easily find the unknown vector $\boldsymbol{d}^{1} = \left(U_{-2}^{1}, U_{-1}^{1}, \ldots, U_{N+1}^{1}, U_{N+2}^{1}\right)$. Once the initial vectors d^0 and d^1 are computed, d^{n+1}, $n = 1,2,3,\ldots$ unknown vectors can be found repeatedly by solving the recurrence relation (10) using two previous d^n, d^{n-1} unknown vectors. Note that since the system (10) is an implicit system, we have taken $\left(U_m^{n+1}\right)^2$ as $\left(U_m^{n}\right)^2$ in the coefficient α_{m1} and done an inner iteration for 5 times to increase the accuracy of the system.

Numerical Experiments: In this section, to illustrate the effectiveness of the presented numerical scheme, two test problems are studied for the MEW equation. Accuracy of the method is assessed by computing the difference between the analytical and numerical results at the node points by way of calculation of the L_∞ error norm defined by

$$L_\infty = \max_m \left|u_m - U_m\right|.$$

Since an accurate numerical scheme must keep the conservation properties of evolution equations, we will monitor the three invariants of numerical solution for the MEW equation corresponding to conservation of mass, momentum and energy given by the following integrals [2]:

$$I_1 = \int_{-\infty}^{\infty} u dx,\ I_2 = \int_{-\infty}^{\infty}\left(u^2 + \mu\left(u_x\right)^2\right)dx,\ I_3 = \int_{-\infty}^{\infty} u^4 dx. \quad (11)$$

Integrals are approximated by employing the trapezium rule and the first derivative of u can be computed from (8).

Motion of Single Solitary Wave for MEW Equation: The Eq. (1) has a solitary wave solution of the form.

$$u(x, t) = A\,\mathrm{sech}(k[x - x_0 - vt]) \quad (12)$$

Where $v = \varepsilon\frac{A^2}{6}, k = \frac{1}{\sqrt{\mu}}$. This solution corresponds to a solitary wave of magnitude A, initially centered on the position x_0 propagating towards the right without change of shape at a steady velocity v.

In this test problem, single solitary wave simulation is carried out over the solution domain $0 \le x \le 80$ in the time period $0 \le t \le 20$ with the parameters $\varepsilon = 3, \mu = 1, x_0 = 30$ and the amplitudes $A = 0.25, 1$. Initial condition (12) with $t = 0$ enables the invariants (11) for the MEW equation to be determined analytically as.

$$I_1 = \frac{A\pi}{k}, I_2 = \frac{2A^2}{k} + \frac{2\mu k A^2}{3}, I_3 = \frac{4A^4}{3k}.$$

Firstly, the program is run up to time $t = 20$ with space step $h = 0.1$ and time step $\Delta t = 0.05$. L_∞ error norm, conversation invariants for the proposed method and equivalent results for the previous methods are presented in Table 1. It is clearly seen that the result obtained by the proposed finite difference method is more accurate than the result obtained by some earlier papers. We have also observed that the quantities I_1, I_2 and I_3 remain almost constant at the end of the running time for the algorithm, so propagation of the single solitary wave is represented faithfully.

We have also studied the same problem with $A = 1$ and $h = \Delta t = 0.01$ and recorded the invariants and error norm in Table 2. It can be seen that the maximum error measured by the L_∞ error norm remain satisfactorily small and numerical values of invariants are very close to analytical values.

Table 1: Invariants and error norms for a single solitary wave at time t = 20

$\varepsilon = 30$, $\mu = 1$, $A = 0.25$, $x_0 = 30$, $h = 0.1$, $\Delta t = 0.05$, $[a, b] = [0, 80]$.				
	$L_\infty \times 10^3$	I_1	I_2	I_3
present	0.00240	0.7853982	0.1666654	0.0052083
[3]	0.04655	0.7853893	0.1667614	0.0052082
[5]	0.24989	0.7849545	0.1664765	0.0051995
[6]	0.25700	0.7853997	0.1664735	0.0052083
exact		0.7853982	0.1666667	0.0052083

Table 2: Invariants and error norms for a single solitary wave at time t = 20

$\varepsilon = 3$, $\mu = 1$, $A = 1$, $x_0 = 30$, $h = \Delta t = 0.01$, $[a, b] = [0, 80]$				
	$L_\infty \times 10^3$	I_1	I_2	I_3
present	0.09047	3.1415927	2.6666671	1.3333338
[6]	0.09871	3.1415790	2.6666350	1.3333310
exact		3.1415927	2.6666667	1.3333333

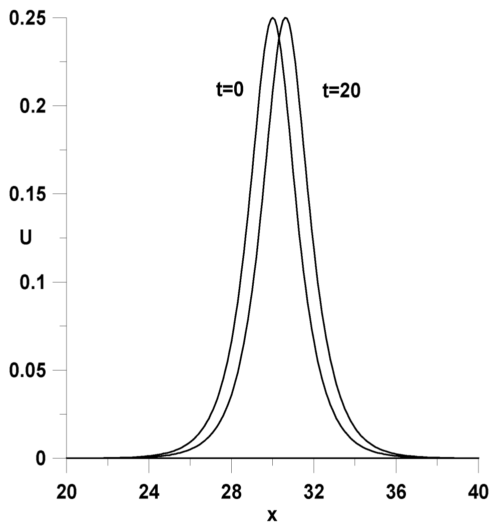

Fig. 1: Single solitary wave at t = 0 and t = 20.

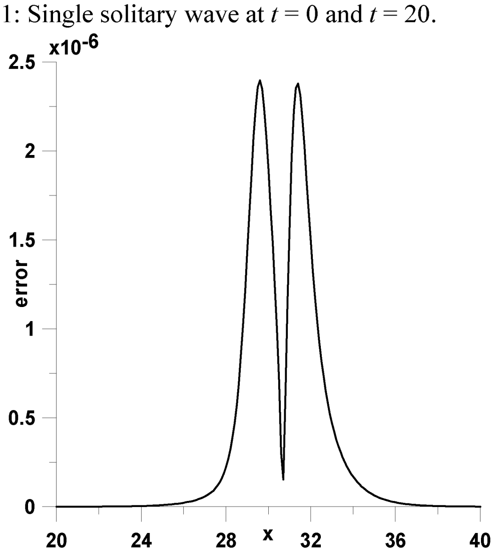

Fig. 2: Absolute error distribution at t = 20.

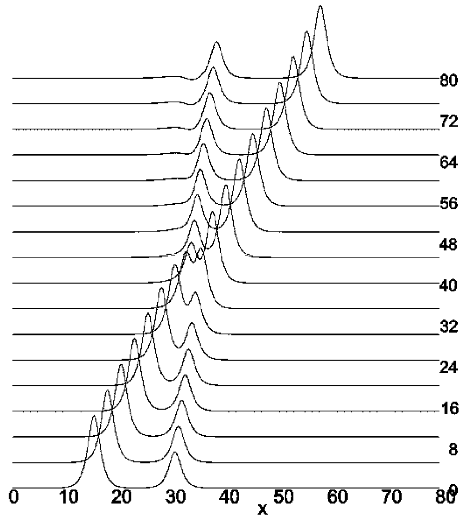

Fig. 3: Interaction of two solitary waves

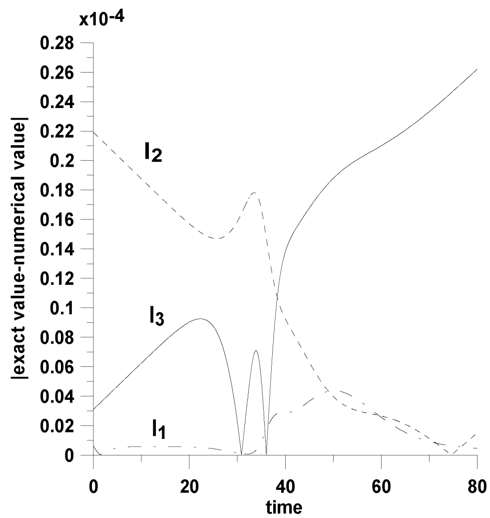

Fig. 4: Error for the invariants

Initial solution and solitary wave profile at time t = 20 are depicted in Fig. 1 with $A = 0.25$, $x_0 = 30$, $h = 0.1$, Δt 0.05 and space interval $[a, b] = [0,80]$. Amplitude of the solitary waves at t = 20 is measured as 0.2499222. When the amplitudes of initial and solitary wave profiles are compared, the difference is 0.0000778 so that there is no degradation between amplitudes. The error distribution of the numerical and analytical solution at time t = 20 is also shown in Fig. 2. As seen from the figure, absolute value of the maximum error occurred at about peak of solitary wave.

Interaction of Two Solitary Waves for MEW Equation: We consider the interaction of two solitary waves using the following initial condition

$$u(x, 0) = A_1\text{sech}(k[x - x_1]) + A_2\text{sech}(k[x - x_2]) \quad (13)$$

In this test problem, all computations are done for the parameter $\varepsilon = 3, \mu = 1, k = \frac{1}{\sqrt{\mu}}$, $A_1 = 1$, $A_2 = 0.5$, $x_1 = 15$ and $x_2 = 30$ over the region $0 \le x \le 80$. These parameters provide two solitary waves of magnitudes 1 and 0.5 and peak positions of them are located at $x = 15$ and 30, respectively. The analytical values of the invariants can be found as

$$I_1 = \frac{\pi}{k}(A_1 + A_2) \approx 4.7123889,$$
$$I_2 = \frac{2}{k}\left(A_1^2 + A_2^2\right) + \frac{2\mu k}{3}\left(A_1^2 + A_2^2\right) \approx 3.3333333,$$
$$I_3 = \frac{4}{3k}\left(A_1^4 + A_2^4\right) \approx 1.4166667.$$

The program is run up to time $t = 80$ with $h = 0.1$ and $\Delta t = 0.025$. The interaction process can be observed clearly from the graph of the time-space-amplitude in Figure 3. After the nonlinear interaction takes place between about time 30 and 50, two solitary waves regain the original shape. The amplitude of the larger wave is 1.000014 at the point $x = 56.9$ at the time $t = 80$, whereas the amplitude of the smaller wave is 0.498759 at the point $x = 37.7$. The absolute difference in amplitude is only 0.001241 for the smaller wave and 0.000014 for the larger wave. Figure 4 shows that the absolute value of the differences between the exact and numerical values of invariants.

CONCLUSION

The MEW equation is solved numerically using finite difference method by using new time discretization. The efficiency of the method is tested on the problems of propagation of single solitary wave and interaction of two solitary waves. Propagation of the single solitary wave and interaction of two solitary waves are simulated well with the proposed algorithms and conservation invariants do not change much during the computer run. Since the proposed method is an accurate and efficient numerical technique and also application of the method is easier than many other numerical techniques, this technique can be considered reliably in obtaining the numerical solution of the similar type of nonlinear equations.

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