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Global Attractivity and Periodic Nature of a Difference Equation

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Abstract: Our goal in this paper is to investigate the global stability character and the periodicity of the solutions of the difference equation

$$x_{n+1} = ax_{n-1} - \frac{bx_{n-l}}{cx_{n-l} - dx_{n-k}}, \qquad n = 0,1,...,$$

Where the initial conditions $x_{-r}, x_{-r+1}, ..., x_0$ are arbitrary positive real numbers, $r = \max\{l, k\}$ is nonnegative integer and a, b, c, d are positive constants.

Key words: Stability・Periodic solutions・Global attractor・Difference equations
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INTRODUCTION

Our goal in this paper is to investigate the global stability character and the periodicity of the solutions of the difference equation.

$$x_{n+1} = ax_{n-1} - \frac{bx_{n-l}}{cx_{n-l} - dx_{n-k}}, \qquad n = 0,1,..., \tag{1}$$

Where the initial conditions $x_{-r}, x_{-r+1}, ..., x_0$ are arbitrary positive real numbers, $r = \max\{l, k\}$ is nonnegative integer and a, b, c, d are positive constants.

Recently there has been a lot of interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations see for example [1-20].

The study of the nonlinear rational difference equations of a higher order is quite challenging and rewarding and the results about these equations offer prototypes towards the development of the basic theory of the global behavior of nonlinear difference equations of a big order, recently, many researchers have investigated the behavior of the solution of difference equations for example: In [7]. Elabbasy *et al.* investigated the global stability character, boundedness and the periodicity of solutions of the difference equation.

$$x_{n+1} = \frac{ax_n + \beta x_{n-1} + \gamma x_{n-2}}{Ax_n + Bx_{n-1} + Cx_{n-2}},$$

Elabbasy *et al.* [8] investigated the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation.

$$x_{n+1} = \frac{ax_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}$$

Elabbasy *et al.* [9] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation.

$$x_{n+1} = \frac{dx_{n-l}x_{n-k}}{cx_{n-s} - b} + a.$$

El-Metwally *et al.* [15] dealed with the following difference equation

$$y_{n+1} = \frac{y_{n-(2k+1)} + p}{y_{n-(2k+1)} + qy_{n-21}}.$$

Saleh *et al.* [31] investigated the difference equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}.$$

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Yalçınkaya [41] has studied the following difference equation

$$x_{n+1} = a + \frac{x_{n-m}}{x_n^k}.$$

For some related work see [21-43].

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$F: I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1},\dots,x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1},\dots,x_{n-k}), \quad n = 0,1,\dots, \qquad (2)$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$

Definition 1: (Equilibrium Point)
A point $\bar{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\bar{x} = F(\bar{x}, \bar{x},\dots,\bar{x}).$$

That is, $x_n = \bar{x}$ *for* $n \geq 0$, is a solution of Eq.(2), or equivalently, $\bar{x}$ is a fixed point of F.

Definition 2: (Periodicity)
A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 3: (Stability)

- The equilibrium point $\bar{x}$ of Eq.(2) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x_{-k}, $x_{-k+1},\dots,x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \delta,$$

We have

$$|x_n - \bar{x}| < \varepsilon \textit{ for all } n \geq -k.$$

- The equilibrium point $\bar{x}$ of Eq.(2) is locally asymptotically stable if $\bar{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all x_{-k}, $x_{-k+1},\dots,x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \gamma,$$

We have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

- The equilibrium point $\bar{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1},\dots,x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

- The equilibrium point $\bar{x}$ of Eq.(2) is globally asymptotically stable if $\bar{x}$ is locally stable and $\bar{x}$ is also a global attractor of Eq.(2).
- The equilibrium point $\bar{x}$ of Eq.(2) is unstable if $\bar{x}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\bar{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\bar{x}, \bar{x},\dots,\bar{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem A [26]: Assume that $p, q \in R$ and $k \in \{0,1,2,\dots\}$. Then

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \qquad n = 0,1,\dots,$$

Remark 1: Theorem A can be easily extended to a general linear equations of the form

$$x_{n+1} + p_1 x_{n+k-1} + \dots + p_k x_n = 0, \qquad n = 0,1,\dots,$$

Where $p1, p2,\dots,pk \in R$ and $k \in \{1,2,\dots\}$. Then Eq.(4) is asymptotically stable provided that

$$\sum_{i=0}^{k} |P_i| < 1.$$

Consider the following equation

$$x_{n+1} = g(x_n, x_{n-1})$$

The following theorem will be useful for the proof of our results in this paper.

Theorem B [27]: Let $[a, \beta]$ be an interval of real numbers and assume that

$$g:[a, \beta]^2 \to [a, \beta],$$

is a continuous function satisfying the following properties:

- $g(x, y)$ is non-decreasing in x in $[a, \beta]$ for each $y \in [a,\beta]$, and is non-increasing in $y \in [a,\beta]$ for each $x \in [a,\beta]$,
- If $(m, M) \in [a, \beta] \times [a, \beta]$ is a solution of the system

$$m = g(m, M) \text{ and } M = g(M, m),$$

then

$$m = M.$$

Then Eq.(5) has a unique equilibrium $\bar{x} \in [a, \beta]$ and every solution of Eq.(5) converges to $\bar{x}$.

The paper proceeds as follows. In Section 2 we show that when $|ac - d| + |ad - d| < |c - d|$, then the equilibrium point of Eq.(1) is locally asymptotically stable. In Section 3 we prove that the there exists a period two solution of Eq.(1). In Section 4 we prove that the equilibrium point of Eq.(1) is global attractor. Finally we give numerical examples of some special cases of Eq. (1) and draw it by using Matlab.

Local Stability of the Equilibrium Point of Eq (1): In this section we study the local stability character of the solutions of Eq.(1).

The equilibrium points of Eq.(1) are given by the relation

$$\bar{x} = a\bar{x} - \frac{b\bar{x}}{c\bar{x} - d\bar{x}}.$$

If $c \neq d$, $a \neq 1$, then the only equilibrium point of Eq.(1) is given by

$$\bar{x} = \frac{b}{(c-d)(a-1)}.$$

Let $f: (0, \infty)^2 \to (0, \infty)$ be a function defined by

$$f(u,v) = au\frac{bu}{cu - dv}.$$

Therefore

$$\frac{\partial f(u,v)}{\partial u} = a + \frac{bdu}{(cu-dv)^2},$$

$$\frac{\partial f(u,v)}{\partial v} = -\frac{bdv}{(cu-dv)^2}.$$

Then we see that

$$\frac{\partial f(\bar{x},\bar{x})}{\partial u} = a + \frac{d(a-1)}{(c-d)} = -c_0,$$

$$\frac{\partial f(\bar{x},\bar{x})}{\partial v} = -\frac{d(a-1)}{(c-d)} = -c_1,$$

Then the linearized equation of Eq.(1) about x is

$$y_{n+1} + C_0 y_{n-l} + C_1 y_{n-k} = 0. \quad (6)$$

Theorem 2.1. Assume that

$$|ac - d| + |ad - d| < |c - d|.$$

Then the positive equilibrium point of Eq.(1) is locally asymptotically stable.

Proof: It is follows by Theorem A that, Eq.(6) is asymptotically stable if $|C_1| + |C_0| < 1$.

$$\left|a + \frac{d(a-1)}{(c-d)}\right| + \left|a - \frac{d(a-1)}{(c-d)}\right| < 1,$$

and so

$$|a(c-d) + d(a-1)| + |-d(a-1)| < |c-d|,$$

or

$$|ac - d| + |ad - d| < |c - d|.$$

This completes the proof.

Periodic Solutions of Eq.(1): In this section we study the existence of periodic solutions of Eq.(1). The following theorem states the necessary and sufficient conditions that this equation has periodic solutions.

Theorem 3.1: Eq.(1) has positive prime period two solutions if and only if

$$(C + d)(a + 1) > 4d,\ ac \neq d \text{ and } k - odd,\ l - even. \quad (7)$$

Proof: First suppose that there exists a prime period two solution

$$\ldots, p, q, p, q, \ldots,$$

of Eq.(1). We will prove that Condition (7) holds.
We see from Eq.(1) (when $k - odd$, $l - even$) that

$$p = aq - \frac{bq}{cq - dp},$$

and

$$q = ap - \frac{bp}{cp - dq}.$$

Then

$$cpq - dp^2 = acq^2 - adpq - bq, \tag{8}$$

and

$$cpq - dp^2 = acp^2 - adpq - bq. \tag{9}$$

Subtracting (8) from (9) gives

$$d(q^2 - p^2) = ac(q^2 - p^2) - b(q - b).$$

Since $p \neq q$, it follows that

$$p + q = \frac{b}{ac - d}. \tag{10}$$

Again, adding (8) and (9) yields

$$2cpq - d(p^2 + q^2) = ac(p^2 + q^2) - 2adpq - b(p + q) \quad (11).$$

It follows by (10), (11) and the relation

$$p^2 + q^2 = (p + q)^2 - 2pq \text{ for all } \quad p,q \in R,$$

that

$$pq = \left(\frac{b^2 d}{(ac - d)^2 (c + d)(a + 1)}\right). \tag{12}$$

Now it is clear from Eq.(10) and Eq.(12) that p and q are the two positive distinct roots of the quadratic equation.

$$(ac - d)t^2 - bt + \frac{b^2 d}{(ac - d)(c + d)(a + 1)} = 0, \tag{13}$$

and so

$$b^2 > \frac{4b^2 d}{(c + d)(a + 1)},$$

thus

$$(c + d)(a + 1) > 4d.$$

Therefore Inequality (7) holds.

Second suppose that Inequality (7) is true. We will show that Eq.(1) has a prime period two solution.

Assume that

$$p = \frac{b + \gamma}{2(ac - d)},$$

and

$$q = \frac{b - \gamma}{2(ac - d)},$$

Where $\gamma = \sqrt{b^2 - \frac{4b^2 d}{(c + d)(a + 1)}}$.

We see from Inequality (7) that

$$(c + d)(a + 1) > 4d \;\Rightarrow\; 1 > \frac{4d}{(c + d)(a + 1)},$$

then after multiplying by b^2 we see that

$$b^2 > \frac{4b^2 d}{(c + d)(a + 1)}.$$

Therefore p and q are distinct real numbers.

Set

$$x_{-1} = q, x_{-l+1} = p, \ldots, x_{-k} = p, x_{-k+1} = q, \ldots, \text{ and } x_0 = q.$$

We wish to show that

$$x1 = x_{-1} = p \text{ and } x_2 = x_0 = q.$$

It follows from Eq.(1) that

$$x_1 = ax_{-1} - \frac{bx_{-l}}{cx_{-l} - dx_{-k}} = aq - \frac{bq}{cq - dp} = aq - \frac{\left[\frac{b - \gamma}{2(ac - d)}\right]}{c\left[\frac{b - \gamma}{2(ac - d)}\right] - d\left[\frac{b + \gamma}{2(ac - d)}\right]}.$$

Multiplying the denominator and numerator of the right side by $2(ac - d)$

$$x_1 = aq - \frac{b(b - \gamma)}{c(b - \gamma) - d(b + \gamma)} = aq - \frac{b(b - \gamma)}{b(c - d) - \gamma(c + d)}.$$

Multiplying the denominator and numerator of the right side by $\{b[c-d]+\gamma[c+d]\}$

$$x_1 = aq - \frac{b(b^2(c-d)+b(c+d)\gamma - b(c-d)\gamma - (c+d)\gamma^2)}{b^2[c-d]^2 - \gamma^2[c+d]^2}$$

$$= aq - \frac{b\left[b^2(c-d)+2bd\gamma - (c+d)\left(b^2 - \frac{4b^2d}{(c+d)(a+1)}\right)\right]}{b^2[c-d] - \gamma^2[c+d]^2\left(b^2 - \frac{4b^2d}{(c+d)(a+1)}\right)}$$

$$= aq - \frac{b\left[b^2[(c-d)-(c+d)] + \frac{4b^2d}{(a+1)} + 2bd\gamma\right]}{b^2([c-d]^2 - [c+d]^2) + \frac{4b^2d(c+d)}{(a+1)}}$$

$$= aq - \frac{b\left[2b^2d + \frac{4b^2d}{(a+1)} + 2bd\gamma\right]}{b^2(-4cd) + \frac{4b^2d(c+d)}{(a+1)}}$$

$$= aq + \frac{2b^2d\left[-b + \frac{2b}{(a+1)} + \gamma\right]}{4b^2d\left(c - \frac{c+d}{a+1}\right)}$$

Multiplying the denominator and numerator of the right side by $(a+1)$ and dividing the denominator and numerator of the right side by $\{2b^2d\}$ gives

$$x_1 = aq + \frac{(-ba - b + 2b + \gamma + \gamma a)}{2(ac + c - c - d)}$$

$$= \frac{ab - a\gamma}{2(ac-d)} + \frac{-ba + b + \gamma + \gamma a}{2(ac-d)}.$$

Thus

$$x_1 = \frac{ab - a\gamma - ba + b + \gamma + \gamma a}{2(ac-d)} = \frac{b+\gamma}{2(ac-d)} = p.$$

Similarly as before one can easily show that

$$x_2 = q.$$

Then it follows by induction that

$$x_{2n} = q \ and \ x_{2n+1} = p \ for \ all \ n \geq -1.$$

Thus Eq.(1) has the positive prime period two solution

$$\ldots,p,q,p,q,\ldots,$$

Where p and q are the distinct roots of the quadratic equation (13) and the proof is complete.

Global Attractor of the Equilibrium Point of Eq.(1): In this section we investigate the global attractivity character of solutions of Eq.(1).

Theorem 4.1: The equilibrium point $\bar{x}$ of Eq.(1) is global attractor.

Proof: Let a, β are a real numbers and assume that $g:[a, \beta]^2 \to [a, \beta]$ be a function defined by

$$g(u,v) = au - \frac{bu}{cu - dv}.$$

We can easily see that the function $g(u, v)$ increasing in u and decreasing in v.
Suppose that (m, M) is a solution of the system

$$m = g(m, M) \quad and \quad M = g(M, m).$$

Then from Eq.(1), we see that

$$m = am - \frac{b}{cm - dM}, \quad 1 - a = -\frac{bM}{cM - dm},$$

That is

$$1 - a = -\frac{b}{cm - dM}, \quad 1 - a = -\frac{b}{cM - dm},$$

or,

$$-\frac{b}{cm - dM} = -\frac{b}{cM - dm}.$$

then

$$(c + d)(m - M) = 0.$$

Thus

$$M = m.$$

It follows by the Theorem B that $\bar{x}$ is a global attractor of Eq.(1) and then the proof is complete.

Numerical Examples: For confirming the results of this paper, we consider numerical examples which represent different types of solutions to Eq. (1).

Example 1: We assume $l = 3$, $k = 2$, $x_{-3} = 12$, $x_{-2} = 7$, $x_{-1} = 9$, $x_0 = 10$, $a = 1.1$, $b = 0.5$, $c = 0.6$, $d = 0.8$. See Fig. 1.

Example 2: See Fig. 2, since

$l = 4$, $k = 3$, $x_{-4} = 12$, $x_{-3} = 7$, $x_{-2} = 9$, $x_{-1} = 10$, $x_0 = 5$, $a = 0.9$, $b = 2$, $c = 7$, $d = 3$.

Example 3: We consider
$l = 3, k = 2, x_{-3} = 12, x_{-2} = 7, x_{-1} = 9, x_0 = 10, a = 0.3, b = 1.5,$
$c = 11, d = 8.$
See Fig. 3.

Example 4: See Fig. 4, since
$l = 3, k = 4, x_{-4} = 12, x_{-3} = 7, x_{-2} = 9, x_{-1} = 10, x_0 = 5, a = 0.6,$
$b = 2, c = 7, d = 4.$

Example 5: Fig. 5. shows the solutions when
$a = 6, b = 2, c = 7, d = 3, l = 4, k = 3, x_{-4} = p, x_{-3} = q, x_{-2} = p, x_{-1} = q, x_0 = p.$

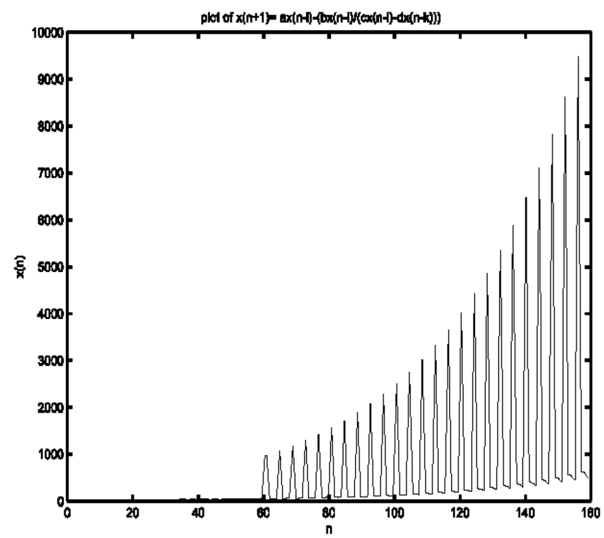

Fig. 1:

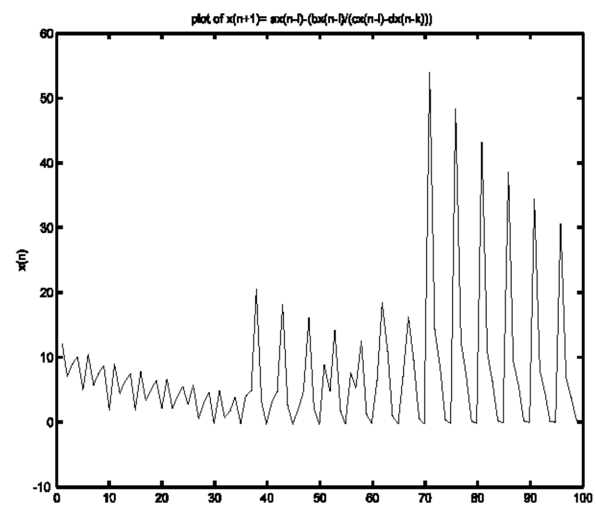

Fig. 2:

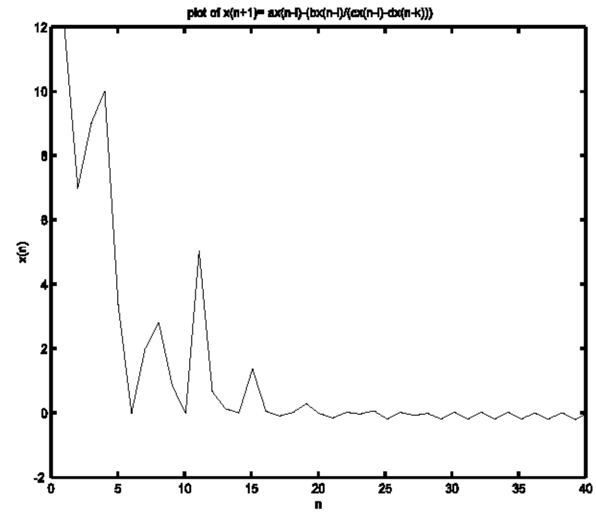

Fig. 3:

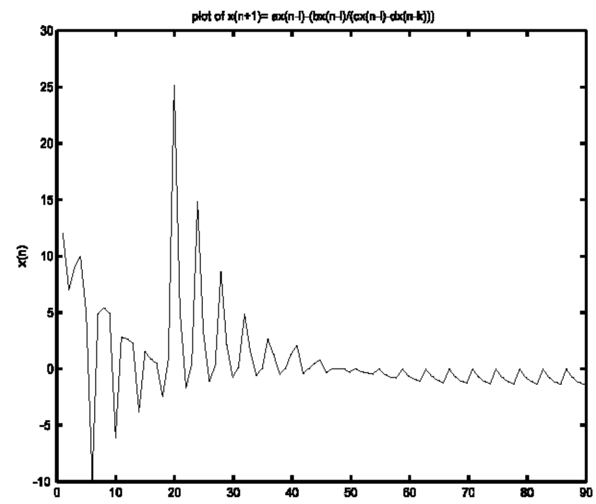

Fig. 4:

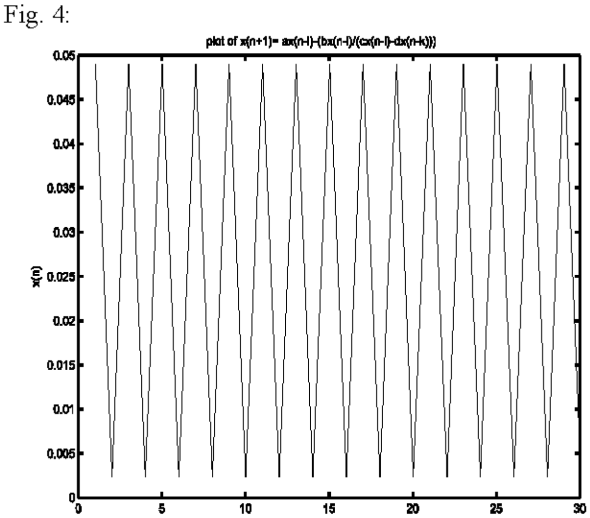

Fig. 5:

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