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Least-Squares Approximations for Solving Differential-Algebraic Equations (DAEs)

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Abstract: In this paper, numerical solution of differential-algebraic equations(DAEs) is considered by Least-Squares Approximations. Two different problems have been solved by using Least-Squares Approximations and solutions have been compared with those obtained by exact solutions. First we calculate power series of the given equations system then transform it into Least-Squares Approximations, which give an arbitrary order for solving differential-algebraic equation numerically.

Key words: Differential-Algebraic Equations(DAEs) · Power Series · Least-Squares Approximations

INTRODUCTION

Much attention has recently been devoted to the development of numerical methods for Differential-Algebraic Equations(DAEs). Some numerical methods have been developed, using BDF, Runge-Kutta, multistep, one-leg methods, implicit Runge-Kutta, Rosenbrock, extrapolation, Padé approximation, Chebyshev approximation, Adomian decomposition, Pseudospectral Methods [1-16]. The well known codes for DAEs are LSODI [17], DASSL [18], LİMEX [2], RADAU5 [19] and DAEIS [20]. The purpose of this paper is to consider the numerical solution of Differential-Algebraic Equations(DAEs) by using Least-Squares Approximations.

Differential-Algebraic Equation(DAEs) can be used to describe the evolution of many interesting and important systems. Differential-Algebraic Equations(DAEs) are a set of differential equations with additional algebraic constraints in the form:

$$F(x,y(x), y'(x)) = 0 \tag{1.1}$$

With singular Fy', where F and y are of the same dimension. Here an in the following we denote partial derivatives by subscripts, so that $Fy' = \partial F/\partial y'$. Equation (1.1) is also called a fully implicit DAE system. We are here especially interested in semi-explicit systems, differential-equations with algebraic constraints of the form

$$\begin{aligned} y'(x) &= f(x,y(x), z(x)) \\ 0 &= g(x, y(x), z(x)) \end{aligned} \tag{1.2}$$

Where y represents the differential variables and z represents the algebraic variables [21, 22]. The numerical methods devised for DAEs take into account the structure of the underlying DAE. We will calculate power series of the given differential-algebraic equations (DAEs) system then transform it into Least-Squares Approximations form, which give an arbitrary order for solving differential-algebraic equation numerically.

The Method

A differential-algebraic equation has the form

$$F(x, y, y') = 0 \tag{2.1}$$

With initial values

$$y(x_0) = y_0, y'(x_0) = y_1$$

Where $F \in R^n$ and $y \in R^n$ are both vector functions for which we assumed sufficient differentiability and the initial values to be consistent, i.e.

$$F(x,y,y') = 0 \tag{2.2}$$

The solutions of (2.1) can be assumed that

$$y = y_0 + y_1 x + ex^2 \tag{2.3}$$

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Where e is a vector function which is the same size as y_0 and y_1. Substitute (2.3) into (2.1) and convert the elementary functions in (2.1) into series in x = 0 and neglect higher order term, we have the linear equation of e in the form

$$Ae = B \tag{2.4}$$

Where A and B are constant matrices. Solving equation (2.4); the coefficients of x^2 in (2.3) can be determined. Repeating above procedure for higher order terms, we can get the arbitrary order power series of the solutions for (2.1) [23, 12, 24,13].

The Power series given by above method can be transformed into Least-Squares Approximations and we have numerical solution of differential-algebraic equation in (2.1).

Remark: When the initial value problem is

$$F(x, y, y') = 0, y(x_0) = y_0 \tag{2.6}$$

The solution of (2.6) can be assumed that

$$y = y_0 + ex \tag{2.7}$$

And repeating above procedure, we can get the solutions of (2.6).

Least-squares Approximations: We must find $p \in \prod_n$ such that $\|f-p\|_2^2 = \int_a^b \left[f(x)-p(x)\right]^2 dx$ is minimized. Writing P in its standard form as

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n \tag{3.1}$$

It follows that;

$$F(c_0, c_1, \ldots, c_n) = \int_a^b \left(f(x) - c_0 - c_1 x - \cdots - c_n x^n\right)^2 dx \tag{3.2}$$

Find $c_0, c_1, \ldots, c_n$ such that F is minimized. Now,

$$\frac{\partial F}{\partial c_i} = -2\int_a^b x^i \left(f(x) - c_0 - c_1 x - \cdots - c_n x^n\right) dx, \quad \forall i. \tag{3.3}$$

So F will be minimized when $\frac{\partial F}{\partial c_i} = 0, \quad \forall i.$
i.e.

$$c_0 \int_a^b x^i dx + c_1 \int_a^b x^{i+1} dx + \cdots + c_n \int_a^b x^{i+n} dx = \int_a^b x^i f(x) dx \tag{3.4}$$

This is called the normal equations and we have a linear system [24].

$$Ac = b \tag{3.5}$$

Where

$$A = \begin{bmatrix} \int_a^b 1dx & \int_a^b x dx & \cdots & \int_a^b x^n dx \\ \int_a^b x dx & \int_a^b x^2 dx & \cdots & \int_a^b x^{n+1} dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b x^n dx & \int_a^b x^i dx & \cdots & \int_a^b x^{2n} dx \end{bmatrix} \quad c = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} \quad b = \begin{bmatrix} \int_a^b f(x) dx \\ \int_a^b x f(x) dx \\ \vdots \\ \int_a^b x^n f(x) dx \end{bmatrix} \tag{3.6}$$

Example: We consider the following differential-algebraic equation[23].

$$\begin{aligned} y'_1 &= e^x + y_2 + x y'_2 \\ y_2 &= \cos x \end{aligned} \tag{4.1}$$

and initial values

$$y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad y'(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{4.2}$$

The exact solution is

$$\begin{aligned} y'_1 &= e^x + x \cos x \\ y_2 &= \cos x \end{aligned} \tag{4.3}$$

using above method [23,12,24,13]. we have

$y_1(x) = 1 + 2x\ 0.50x^2 - 0.3333333333x^3 + 0.04166666667\ x^4 + 0.05x^5 + 0.001388888889x^6 - 0.001190476190x^7 + 0.00002480158730x^8 + 0.00002755731922x^9$

$y_2(x) = -1.05x^2 + 0.04166666667x^4 - 0.0013888888889x^6 + 0.000024801587302x^8$

The Power series y_1, $y_2(x)$ can be transformed into following Least-Squares Approximations

$y_1(x) = 0.99997080400 + 2.0026784250x + 0.44035445520x^2 + 0.22790367600x^3 - 2.7098711640x^4 + 7.7855057280x^5 - 12.929267551x^6 + 12.693443162x^7 - 6.7551850080x^8 + 1.5030824380x^9$

$y_2(x) = 1.0000348415 - 0.0029590624800x - 0.43780150656x^2 - 0.55951631736x^3 + 2.6875234386x^4 - 7.221214404x^5 + 11.774146544x^6 - 11.319939053x^7 + 5.9157376621x^8 - 1.2957307792x^9$

Table 1: Numerical solution of $y_1(x)$ in (4.1).

X	$y_1(x)$	$y^*_1(x)$	$y_1(x) - y^*_1(x)$ |
0.1	1.2046713346	1.2046652367	0.0000060979
0.2	1.4174160738	1.4174179438	0.0000018700
0.3	1.6364597543	1.6364654518	0.0000056975
0.4	1.8602490952	1.8602412024	0.0000078928
0.5	2.0875125516	2.0875126817	0.0000001301
0.6	2.3173201694	2.3173279827	0.0000078133
0.7	2.5491422386	2.5491362799	0.0000059587
0.8	2.7829062960	2.7829047539	0.0000015421
0.9	3.0190520826	3.0190580094	0.0000059268
1.0	3.2585841344	3.2586149652	0.0000308308

Table 2: Numerical solution of $y_1(x)$ in (4.1).

X	$y_2(x)$	$y^*_2(x)$	$y_2(x) - y^*_2(x)$ |
0.1	0.9950041653	0.9950086446	0.0000044793
0.2	0.9800665778	0.9800672187	0.0000006409
0.3	0.9553364891	0.9553297457	0.0000067434
0.4	0.9210609940	0.9210668820	0.0000058880
0.5	0.8775825619	0.8775843013	0.0000017394
0.6	0.8253356149	0.8253291736	0.0000064413
0.7	0.7648421873	0.7648455398	0.0000033525
0.8	0.6967067093	0.6967089722	0.0000022629
0.9	0.6216099683	0.6216051960	0.0000047723
1.0	0.5403023059	0.5402813670	0.0000209389

We show Table 1 and Table 2 for the solution of (4.1) by above numerical method. The numerical values on Table 1 and Table 2 are coinciding with the exact solutions of (4.1).

The graph of $y_1(x)$, $y_2(x)$ and their Least-Squares Approximations are simultaneously shown in Fig. 1 and Fig. 2.

Example: We consider the following differential-algebraic equation [26].

$$\begin{aligned} &y'_1 - xy'_2 + x^2y'_3 + y_1 - (x+1)\,y_2 + (x^2 + 2x)\,y_3 = 0 \\ &y'_2 - xy'_3 - y_2 + (x-1)\,y_3 = 0 \\ &y_3 = \sin x \end{aligned} \tag{5.1}$$

and initial values

$$y(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{2.2}$$

The exact solution is

$$\begin{aligned} y_1(x) &= e^{-x} + xe^x \\ y_2(x) &= e^x + x\sin x \\ y_3(x) &= \sin x \end{aligned} \tag{5.3}$$

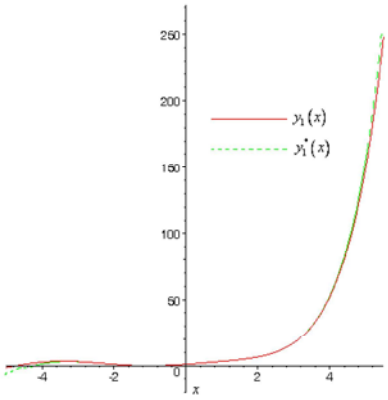

Fig. 1: Graph of $y_1(x)$ and its Least-Squares Approximation

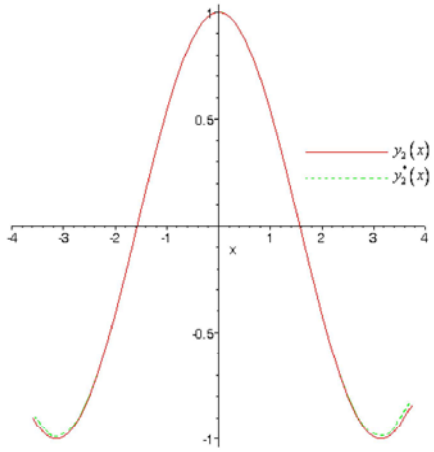

Fig. 2: Graph of $y_2(x)$ and its Least-Squares Approximation

using above method [23,12,24,13]. we have

$$\begin{aligned} y_1(x) &= 1 + 1.5x^2 + 0.33333333333x^3 + 0.20833333333x^4 \\ &+ 0.033333333333x^5 + 0.0097222222222x^6 + 0.0011904761905x^7 \\ &+ 0.00022321428571x^8 + 0.000022045855379x^9 \end{aligned}$$

$$\begin{aligned} y_2(x) &= 1 + x + 1.5x^2 + 0.16666666667x^3 - 0.125x^4 \\ &+ 0.0083333333333x^5 + 0.0097222222222x^6 + \\ &0.00019841269841x^7 - 0.00017361111111x^8 \\ &+ 0.0000027557319224x^9 \end{aligned}$$

$$\begin{aligned} y_3(x) &= x - 0.16666666667x^3 + 0.0083333333333x^5 \\ &- 0.00019841269841x^7 + 0.0000027557319224x^9 \end{aligned}$$

The Power series $y_1(x)$, $y_2(x)$ and $y_3(x)$ can be transformed into following Least-Squares Approximations

$$y_1^*(x) = 0.999987109 + 0.0012157596x + 1.4724683424x^2 + 0.5952618672x^3 - 1.0856817972x^4 + 3.691581894x^5 - 6.1311338088x^6 + 6.0496135128x^7 - 3.2266536588x^8 + 0.7195137664x^9$$

$$y_2^*(x) = 0.999972492 + 1.0024043634x + 1.4483064024x^2 + 0.6402227832x^3 - 2.3981453496x^4 + 6.2906207364x^5 - 10.344537403x^6 + 10.045646582x^7 - 5.2919524944x^8 + 1.1672329812x^9$$

$$y_3^*(x) = 0.00001492134 + 0.99861540768x + 0.03096251928x^2 - 0.4582950372x^3 + 1.4289913638x^4 - 4.0040349149x^5 + 6.6964956458x^6 - 6.563267671x^7 + 3.486396771x^8 - 0.7744232496x^9$$

Table 4: Numerical solution of $y_1(x)$ in (5.1).

X	$y_1(x)$	$y^*_1(x)$	$y_1(x) - y^*_1(x)$
0.1	1.0153545098	1.0153514202	0.0000030896
0.2	1.0630113047	1.0630124555	0.0000011508
0.3	1.1457758630	1.1457784950	0.0000026320
0.4	1.2670499251	1.2670460109	0.0000039142
0.5	1.4308912951	1.4308914539	0.0000001588
0.6	1.6420829163	1.6420867103	0.0000037940
0.7	1.9062121990	1.9062091564	0.0000030426
0.8	2.2297617069	2.2297606384	0.0000010685
0.9	2.6202124598	2.6202142006	0.0000017408
1.0	3.0861612697	3.0861729866	0.0000117169

Table 5: Numerical solution of $y_2(x)$ in (5.1).

X	$y_2(x)$	$y^*_2(x)$	$y_2(x) - y^*_2(x)$
0.1	1.1151542598	1.1151499152	0.0000043446
0.2	1.2611366244	1.2611369532	0.0000003288
0.3	1.4385148696	1.4385202081	0.0000053385
0.4	1.6475920345	1.6475863666	0.0000056679
0.5	1.8884340400	1.8884332153	0.0000008247
0.6	2.1609042844	2.1609101807	0.0000058963
0.7	2.4647050886	2.4647012691	0.0000038195
0.8	2.7994258012	2.7994238402	0.0000019610
0.9	3.1645973299	3.1646007520	0.0000034221
1.0	3.5597528133	3.5597710936	0.0000182803

Table 6: Numerical solution of $y_3(x)$ in (5.1).

X	$y_2(x)$	$y^*_2(x)$	$y_2(x) - y^*_2(x)$
0.1	0.0998334166	0.0998367251	0.0000033085
0.2	0.1986693308	0.1986683329	0.0000009979
0.3	0.2955202067	0.2955170878	0.0000031189
0.4	0.3894183423	0.3894224896	0.0000041473
0.5	0.4794255386	0.4794256521	0.0000001135
0.6	0.5646424734	0.5646383575	0.0000041159
0.7	0.6442176872	0.6442205876	0.0000029004
0.8	0.7173560909	0.7173570538	0.0000009629
0.9	0.7833269096	0.7833238229	0.0000030867
1.0	0.8414709848	0.8414557564	0.0000152284

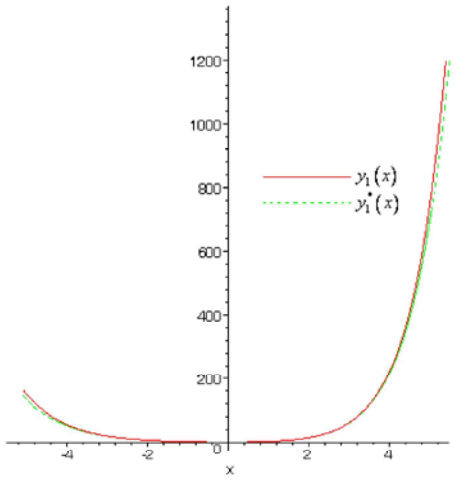

Fig. 4: Graph of $y_2(x)$ and its Least-Squares Approximation

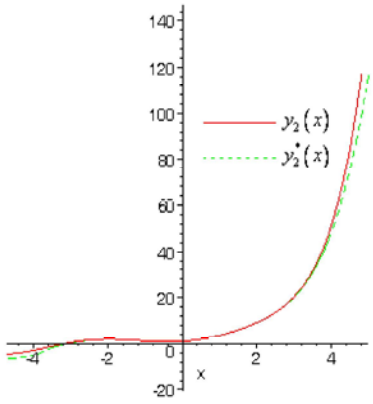

Fig. 5: Graph of $y_2(x)$ and its Least-Squares Approximation

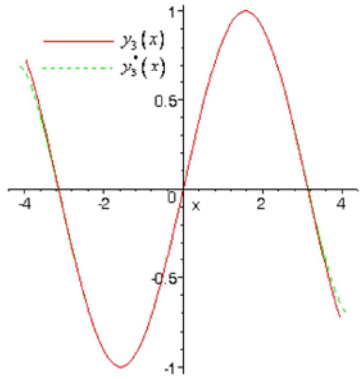

Fig. 6: Graph of $y_3(x)$ and its Least-Squares Approximation

We show Table 3, Table 4 and Table 5 for the solution of (5.1) by above numerical method. The numerical values on Table 3, Table 4 and Table 5 are coinciding with the exact solutions of (5.1).

The graph of $y_1(x)$, $y_2(x)$, $y_3(x)$ and their Least-Squares Approximations are simultaneously shown in Fig. 3, Fig. 4 and Fig. 5.

CONCLUSION

Least-Squares Approximation has proposed for solving differential-algebraic equations in this study. The computations associated with the example discussed above were performed by using Maple 10.

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