An Efficient Algorithm for Bratu Type Models

1Syed Tauseef Mohyud-Din, 2Ahmet Yıldırım, 3Muhammad Usman and 4M.M. Hosseini

1Department of Mathematics, HITEC University Taxila Cantt, Pakistan
2Department of Mathematics, Ege University, 35100 Bornova-İzmir, Turkey
3Department of Mathematics, University of Dayton, Dayton, Oh, USA
4Faculty of Mathematics, Yazd University, P. O. Box 89195-74, Yazd, Iran

Abstract: In this paper, we apply modified decomposition method (MDM) for solving Bratu type models which are of utmost importance for physical sciences. Numerical results clearly support the reliability of the proposed algorithm.

Key words: Modified decomposition method · Bratu-type model · Nonlinear problems

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INTRODUCTION

The Bratu-type model [1, 2, 4, 5, 8] in one-dimensional planar coordinates is of the form

$$u'' + \lambda e^u = 0, \quad 0 < x < 1,$$

with boundary conditions

$$u(0) = 0, \quad u(1) = 0,$$

and is used to model a combustion problem in a numerical slab. The Bratu-type models appear in a number of applications such as the fuel ignition of the thermal combustion theory and in the Chandrasekhar model of the expansion of the universe. It stimulates a thermal reaction process in a rigid material where the process depends on the balance between chemically generated heat and heat transfer by conduction, see [1, 2, 4, 5, 8] and the references therein. The basic motivation of this paper is the implementation of modified decomposition method (MDM) [3, 9-12] which is mainly due to Gejji and Jafari for solving Bratu-type models. It is shown that the proposed method provides the solution in a rapid convergent series with easily computable components. The proposed modified decomposition method (MDM) [3] is applied without any discretization, perturbation, transformation or restrictive assumptions and is free from round off errors, calculation of the so-called Adomian's polynomials and the identification of Lagrange multipliers. Numerical results show the complete reliability and efficiency of the proposed technique. Moreover, the proposed MDM is easier to implement and reduces the computational work while still maintaining a higher level of accuracy.

Modified Decomposition Method (MDM): Consider the following general functional equations:

$$f(x) = 0,$$  (1)

To convey the idea of the modified decomposition method [3, 9-12], we rewrite the above equation as:

$$y = N(y) + c,$$  (2)

Where N is a nonlinear operator from a banach space $B \rightarrow B$ and $f$ is a known function. We are looking for a solution of equation (1) having the series form:

$$y = \sum_{i=0}^{\infty} y_i,$$  (3)

The nonlinear operator N can be decomposed as

$$N \left( \sum_{i=0}^{\infty} y_i \right) = N \left( \sum_{i=0}^{\infty} y_i \right) = \sum_{i=0}^{\infty} \left[ N \left( \sum_{j=0}^{i} y_j \right) - N \left( \sum_{j=0}^{i-1} y_j \right) \right].$$  (4)

Corresponding Author: Dr. Syed Tauseef Mohyud-Din, Department of Mathematics, HITEC University Taxila Cantt, Pakistan.  
E-mail: syedtauzeesf@hotmail.com
From equations (3) and (4), equation (2) is equivalent to
\[ \sum_{i=0}^{\infty} y_i = c + N(y_0) + \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} y_j \right) - N(\sum_{j=0}^{i-1} y_j). \]  
\tag{5}

We define the following recurrence relation:
\[
\begin{align*}
  y_0 &= c, \\
  y_1 &= N(y_0), \\
  y_{m+1} &= N(y_0 + \ldots + y_m) - N(y_0 + \ldots + y_{m-1}), & m = 1, 2, 3, \ldots,
\end{align*}
\tag{6}
\]
then
\[ (y_1 + \ldots + y_{m+1}) = N(y_0 + \ldots + y_m), & m = 1, 2, 3, \ldots, \]
and
\[ y = f + \sum_{i=1}^{\infty} y_i, \]
if \( N \) is a contraction, i.e:
\[
\begin{align*}
  \| N(x) - N(y) \| &\leq \| x - y \|, & 0 < K < 1 \\
  \| N(x) - N(y) \| &\leq \| x - y \|, & 0 < K < 1 \\
  \| y_{m+1} \| - \| y_m \| &\leq K \| y_0 \|, & m = 1, 2, 3, \ldots \\
  \| y \| &\leq K^m \| y_0 \|, & m = 1, 2, 3, \ldots,
\end{align*}
\]
and the series \( \sum_{i=1}^{\infty} y_i \) absolutely and uniformly converges to a solution of equation (1) \([3, 9-12]\), which is unique, in view of the Banach fixed-point theorem.

**Numerical Applications:** In this section, we apply MDM for solving Bratu-type models. The results are very encouraging indicating the reliability and efficiency of the proposed method.

**Example 3.1 Consider the Following Bratu-type Model:**
\[ u^{n} - \pi^2 e^u = 0, \quad 0 < x < 1, \]
with initial conditions
\[ u(0) = 0, \quad u(1) = 0. \]

Applying the modified decomposition method (MDM)
\[ u_n(x) = \alpha x + \sum_{n=0}^{\infty} \left( \pi^2 e^{u_n} \right) ds, \]
where \( \alpha = \pi \) and consequently, the following approximants are obtained
\[ u_0(x) = c, \quad u_0(x) = \alpha x, \quad u_1(x) = N(u_0(x)), \quad u_1(x) = \alpha x - \frac{\pi^2}{2} \left( e^{\alpha x} + e^{\alpha x} + 1 \right), \]
\[ u_2(x) = N(u_0(x) + u_1(x)) - N(u_0(x)), \quad u_2(x) = \alpha x - \frac{\pi^2}{2} \left( e^{\alpha x} + e^{\alpha x} + 1 \right) - \frac{\pi^4}{4\alpha^4} \left( e^{2\alpha x} + 4\alpha x e^{\alpha x} - 4e^{\alpha x} + 2\alpha x + 5 \right), \]
\[ u_3(x) = N(u_0(x) + u_1(x) + u_2(x)) - N(u_0(x) + u_1(x)), \quad u_3(x) = \alpha x - \frac{\pi^2}{2} \left( e^{\alpha x} + e^{\alpha x} + 1 \right) - \frac{\pi^4}{4\alpha^4} \left( e^{2\alpha x} + 4\alpha x e^{\alpha x} - 4e^{\alpha x} + 2\alpha x + 5 \right) + \frac{\pi^6}{12\alpha^6} \left( e^{3\alpha x} + 6e^{2\alpha x} (1 - \alpha x) + \left( 3e^{\alpha x} (2\alpha^2 x^2 - 6\alpha x + 5) - 6\alpha x - 22 \right) \right) + \ldots, \]

The series solution is given as
\[ u(x) = e^{-\frac{x^2}{2}} \left( e^{2\alpha x} + 4\alpha x e^{\alpha x} - 4e^{\alpha x} + 2\alpha x + 5 \right) + \frac{\pi^4}{4\alpha^4} \left( e^{2\alpha x} + 4\alpha x e^{\alpha x} - 4e^{\alpha x} + 2\alpha x + 5 \right) + \frac{\pi^6}{12\alpha^6} \left( e^{3\alpha x} + 6 e^{2\alpha x} (1 - \alpha x) + \left( 3 e^{\alpha x} (2\alpha^2 x^2 - 6\alpha x + 5) - 6\alpha x - 22 \right) \right) + \ldots, \]
or equivalently
\[ u(x) = e^{-\frac{x^2}{2}} \left( e^{2\alpha x} + 4\alpha x e^{\alpha x} - 4e^{\alpha x} + 2\alpha x + 5 \right) + \frac{\pi^4}{4\alpha^4} \left( e^{2\alpha x} + 4\alpha x e^{\alpha x} - 4e^{\alpha x} + 2\alpha x + 5 \right) + \frac{\pi^6}{12\alpha^6} \left( e^{3\alpha x} + 6 e^{2\alpha x} (1 - \alpha x) + \left( 3 e^{\alpha x} (2\alpha^2 x^2 - 6\alpha x + 5) - 6\alpha x - 22 \right) \right) + \ldots. \]

Imposing the boundary condition \( u(1) = 0 \), yields \( \alpha = \pi \) and consequently, closed form solution is given as
\[ u(x) = \ln (1 + \sin(1 + \pi x)). \]

**Example 3.2 Consider the Following Bratu-type Model:**
\[ u^{n} - 2 e^u = 0, \quad 0 < x < 1, \]
with initial conditions
\[ u(0) = 0, \quad u(1) = 0. \]

Applying the modified decomposition method (MDM)
$u_{n+1}(x) = \alpha x + \sum_{0}^{x} (-\pi^2 e^{-\pi x}) d\ s\ ds$,

Where $\alpha = u' (0) \neq 0$. Consequently, following approximants are obtained

$u_0(x) = c$,

$u_0(x) = \alpha x$,

$u_1(x) = N u_0(x)$,

$\begin{align*}
  u_2(x) &= N(u_0(x) + u_1(x) - N u_0(x), \\
  u_2(x) &= \alpha x - \pi^2 \alpha^2 \left( e^{-\alpha x} + \alpha x - 1 \right) - \frac{\pi^4}{4\alpha^4} \\
  &\left( e^{-2\alpha x} + 4\alpha x e^{-\alpha x} + 2\alpha x - 5 \right)
\end{align*}$

$\begin{align*}
  u_3(x) &= N(u_0(x) + u_1(x) + u_2(x) - N (u_0(x) + u_1(x)), \\
  u_3(x) &= \alpha x - \pi^2 \alpha^2 \left( e^{-\alpha x} + \alpha x - 1 \right) - \frac{\pi^4}{4\alpha^4} \\
  &\left( e^{-2\alpha x} + 4\alpha x e^{-\alpha x} + 2\alpha x - 5 \right)
\end{align*}$

The series solution is given as

$u(x) = \frac{\pi^6}{12\alpha^6} \left( e^{-3\alpha x} + 6\alpha x e^{-\alpha x} + 2\alpha x - 5 \right)$

or equivalently

$u(x) = \frac{\pi^6}{12\alpha^6} \left( 2\alpha x^2 + 6\alpha x + 5 \right) + 6\alpha x - 22 + ...$.

Imposing the boundary condition $u(1) = 0$ yields $\alpha = \pi$ and consequently, closed form solution is given as

$u(x) = \ln (1 + \sin (1 + \pi x))$.

**Example 3.3 Consider the Following Initial Value Problem of the Bratu-type:**

$u'' - 2e^x = 0, \quad 0 < x < 1$,

with initial conditions

$u(0) = 0, \quad u(1) = 0$.

Applying the modified decomposition method (MDM)

$u_{n+1}(x) = 2\sum_{n=0}^{x} e^{\alpha x} d\ s\ ds$.

Consequently, following approximants are obtained

$u_0(x) = c$,

$u_0(x) = 0$,

$u_1(x) = N u_0(x)$,

$u_1(x) = x^2$,

$\begin{align*}
  u_2(x) &= N(u_0(x) + u_1(x) - N u_0(x), \\
  u_2(x) &= x^2 + \frac{1}{6} x^4
\end{align*}$

$\begin{align*}
  u_3(x) &= N(u_0(x) + u_1(x) + u_2(x) - N (u_0(x) + u_1(x)), \\
  u_3(x) &= x^2 + \frac{1}{6} x^4 + \frac{2}{45} x^6
\end{align*}$

The series solution is given as

$u(x) = \frac{1}{6} x^4 + \frac{2}{45} x^6 + \frac{17}{1260} x^8$.

or equivalently

$u(x) = -2 \left( \frac{1}{12} x^4 - \frac{1}{12} x^4 - \frac{17}{2520} x^8 \right)$.

The exact solution is given by

$u(x) = -2 \ln (\cos(x))$.

**CONCLUSION**

In this paper, we applied modified decomposition method (MDM) for solving Bratu-type models. The method is applied in a direct way without using linearization, transformation, perturbation, discretization or restrictive assumptions. It may be concluded that MDM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. Numerical results explicitly confirm the reliability of the proposed MDM.
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