Homotopy Perturbation Method for a Class of Eighth-Order BVPS

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Abstract: In this paper, we apply the homotopy perturbation method (HPM) to solve a class of eighth-order boundary value problems. The suggested algorithm is quite efficient and is practically well suited for use in these problems. The proposed iterative scheme finds the solution without any discretization, linearization or restrictive assumptions. Several examples are given to verify the reliability and efficiency of the method. The fact that the proposed technique solves nonlinear problems without using Adomian’s polynomials can be considered as a clear advantage of this algorithm over the decomposition method.

Keywords: Homotopy perturbation method · Higher order boundary value problems · Initial value problems

INTRODUCTION

In this paper, we consider general class of eighth-order boundary value problems [1-13] of the type:

\[ u^{(8)}(x) = \phi(x) u(x) = \Psi(x), \quad a \leq x \leq b, \]  

with boundary conditions

\[ u(a) = A_a, \quad u^{(i)}(a) = A_{ai}, \quad u^{(i)}(b) = A_{bi}, \quad i = 0, 1, \ldots, 8. \]

Where:

\[ u = u(x), \quad \phi(x) \text{and} \quad \Psi(x) \]  

are continuous functions defined on \([a, b]\) and \(A_i\) and \(B_i\) are real finite constants. A class of characteristic-value problems of higher order (as higher as twenty four) is known to arise in hydrodynamic and hydro magnetic stability. In addition, it is well known that when a layer of fluid is heated from below and is subject to the action of rotation, instability may set in as over stability which may be modeled by an eighth-order boundary value problem, see [1-11] and the references therein. The boundary value problems of higher order have been investigated due to their mathematical importance and the potential for applications in hydrodynamic and hydro magnetic stability. Several techniques [1-13] including finite-difference, polynomial and non polynomial spline and decomposition have been employed to tackle such problems and the references therein. Most of these techniques have their inbuilt deficiencies, like divergence of the results at the points adjacent to the boundary and calculation of the so-called Adomian’s polynomials. Moreover, the performance of most of the methods used so far is well known that they provide the solution at grid points only. Inspired and motivated by the ongoing research in this area, we apply homotopy perturbation method (HPM) [14-28] for finding the solution of a class of eighth-order boundary value problems. It is worth mentioning that the proposed algorithm (HPM) is applied without any discretization, restrictive assumption or transformation and is free from round off errors. Unlike the method of separation of variables that require initial and boundary conditions, the HPM provides an analytical solution by using the initial conditions only. The proposed method work efficiently and the results so far are very reliable. The fact that the homotopy perturbation method (HPM) solves nonlinear problems without using Adomian’s polynomials is a clear advantage of this technique over the decomposition method.

Homotopy Perturbation Method (HPM): To explain the homotopy perturbation method [14-28], we consider a general equation of the type,

\[ L(u) = 0, \]

Where:

\( L \) is any integral or differential operator. We define a convex homotopy \( H(u, p) \) by

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\[ H(u, p) = (1 - p) F(u) = p L(u). \]  

(3)

Where:

\( F(u) \) is a functional operator with known solutions \( v_0 \), which can be obtained easily. It is clear that, for

\[ H(u, p) = 0, \]

(4)
we have

\[ H(u, 0) = F(u), \quad H(u, 1) = L(u). \]

This shows that \( H(u, p) \) continuously traces an implicitly defined curve from a starting point \( H(v_0, 0) \) to a solution function \( H(f, 1) \). The embedding parameter monotonically increases from zero to unit as the trivial problem \( F(u) = 0 \) is continuously deformed to the original problem \( L(u) = 0 \). The embedding parameter \( p \in (0, 1] \) can be considered as an expanding parameter [14-28]. The homotopy perturbation method uses the homotopy parameter \( p \) as an expanding parameter [14-28] to obtain

\[ u = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots, \]

(5)

if \( p = 1 \), then (5) corresponds to (3) and becomes the approximate solution of the form,

\[ f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i. \]

(6)

It is well known that series (6) is convergent for most of the cases and also the rate of convergence is dependent on \( L(u) \); see [14-28]. We assume that (6) has a unique solution. The comparisons of like powers of \( p \) give solutions of various orders. In sum, according to [14], He's HPM considers the solution, \( u(x) \), of the homotopy equation in a series of \( p \) as follows:

\[ u(x) = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + \cdots, \]

and the method considers the nonlinear term \( N(u) \) as

\[ N(u) = \sum_{i=0}^{\infty} p^i H_i = H_0 + p H_1 + p^2 H_2 + \cdots, \]

Where:

\( H_i \)'s are the so-called He's polynomials [14], which can be calculated by using the formula

\[ H_n(u_0, \ldots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left( N \left( \sum_{i=0}^{n} p^i u_i \right) \right) \bigg|_{p=0} \quad n = 0, 1, 2, \ldots. \]

Numerical Applications: In this section, we apply homotopy perturbation method (HPM) to solve a class of eighth-order boundary value problems.

Example 3.1: Consider equation (1) with

\[ [a, b] = [-1, 1], \quad \phi(x) = -x, \quad \Psi(x) = -(55 + 17x + x^2)e^x, \]

and the boundary conditions

\[ A_0 = 0, \quad A_1 = 2, \quad A_2 = -4, \quad A_3 = -18, \]

\[ B_0 = 0, \quad B_1 = -6, \quad B_2 = -20, \quad B_3 = -42. \]

Applying the convex homotopy assuming the initial approximation as

\[ u_0(x) = a + bx + cx^2 = dx^2 + ex^4 + fx^6 + gx^8 + hx_{10}, \]

\[ u_0 + pu_1 + p^2 u_2 + \cdots = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 \]

\[ + \sum_{i=0}^{\infty} \left( \frac{\partial}{\partial x} \left( u_0 + pu_1 + p^2 u_2 + \cdots \right) \right), \]

\[ \sum_{i=1}^{\infty} \int \int \int \int \int \int \left( 55 + 17x + x^2 \right) e^x dx dx dx dx dx dx. \]

Where:

\( a, b, c, d, e, f, g, h \) are constants to be determined. Comparing the co-efficient of like powers of \( p \)

\[ p^{(0)} : u_0(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7, \]

\[ p^{(1)} : u_1(x) = 711 + a + (496 - b)x + c \left( 331 \times \frac{1}{2} \right) x^2 \]

\[ + (d + 35)x^3 + \left( e + \frac{23}{24} \right)x^4 + \left( f + \frac{19}{30} \right)x^5 \]

\[ + (g + \frac{17}{240})x^6 + \left. \right. \]

\[ + \frac{(h + x^0)}{2520} + \frac{ax^9}{362880} + \frac{bx^{10}}{1814400} + \frac{cx^{11}}{6652800} + \frac{dx^{12}}{19958400} + \frac{ex^{13}}{51891840} + \frac{fx^{14}}{121089600} + \frac{gx^{15}}{259459200}, \]

\[ \left. \right. \]

\[ + \frac{hx^{16}}{518918400} \left( -25x^2 + 215x - 711 + x^3 \right) e^x. \]

Impose the boundary conditions will yield
\[ a = 1, \quad b = 1, \quad c = -\frac{1}{2}, \quad d = -\frac{5}{6}, \]
\[ e = -\frac{11}{24}, \quad f = -\frac{19}{120}, \quad g = -\frac{29}{720}, \quad h = -\frac{41}{5040}. \]

The first order approximate solution is given as
\[ u_0(x) = 1 + x - \frac{x^2}{2} - \frac{5}{6} x^3 - \frac{11}{3} x^4 - \frac{19}{120} x^5 - \frac{29}{720} x^6 - \frac{41}{5040} x^7. \]

Proceeding as before, the second-order approximate solution is given as
\[ u_1(x) = 1 + x - \frac{x^2}{2} - \frac{5}{6} x^3 - \frac{11}{3} x^4 - \frac{19}{120} x^5 - \frac{29}{720} x^6 - \frac{41}{5040} x^7 + \frac{19958400}{479001600} + \frac{518918400}{1245404160} + \frac{87178291200}{118879488000} + O(x^{16}). \]

This gives the solution in a closed form as \((1-x^2)e^x\).

**Example 3.2:** Consider equation (1) with
\[ [a, b] = [-1, 1], \quad \phi(x) = -1, \quad \Psi(x) = -8[2\cos(x) + 7\sin(x)], \]
and the boundary conditions
\[ A_1 = 0, \quad A_2 = -4\cos(1) - 2\sin(1), \]
\[ A_3 = -4\cos(1) - 2\sin(1), \quad A_4 = -12\cos(1) - 30\sin(1), \]
\[ B_1 = 0, \quad B_2 = 4\cos(1) + 2\sin(1), \]
\[ B_3 = -8\cos(1) - 12\sin(1), \quad B_4 = 12\cos(1) + 30\sin(1). \]

Applying the convex homotopy assuming the initial approximation as
\[ u_0(x) - a + bx + cx^2 - dx^3 - ex^4 + fx^5 + gx^6 + hx^7. \]
\[ u_0 + pu_1 + p^2 u_2 + \cdots = a + bx + cx^2 + dx^3 + \]
\[ + ex^4 + fx^5 + gx^6 + hx^7 + p \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{n!} \frac{d^n}{dx^n} \frac{d^k}{dx^k} \]
\[ \left( u_0 + pu_1 + p^2 u_2 + \cdots \right) \] \[ dx \cdot dx \cdot dx \cdot dx \cdot dx \cdot dx \cdot dx. \]

**Where:**
\[ a, b, c, d, e, f, g, h \] are constants to be determined.

Comparing the co-efficient of like powers of \( p \)
\[ p^{(0)}: u_0(x) = a + bx = cx^2 + dx^3 + ex^4 = fx^5 + gx^6 + hx^7, \]
\[ p^{(1)}: u_1(x) = 711 + a + (496 - b) x + (e + \frac{331}{2}) x^2 + \]
\[ + (d + 35) x^3 + (e + \frac{127}{24}) x^4 + (f + \frac{19}{30}) x^5 + (g + \frac{17}{240}) x^6 + \]
\[ +(h + \frac{23}{2520}) x^7 + \frac{ax}{362800} + \frac{bx}{1814400} + \frac{cx}{6652800} + \frac{dx}{19958400} + \frac{ex}{518918400} + \frac{fx}{121080960} + \frac{gx}{259459200} + \frac{hx}{518918400} \]
\[ + O(x^{16}). \]

Imposing the boundary conditions will yield
\[ a = 1, \quad b = 1, \quad c = -\frac{1}{2}, \quad d = -\frac{5}{6}, \quad e = -\frac{11}{24}, \]
\[ f = -\frac{19}{120}, \quad g = -\frac{29}{720}, \quad h = -\frac{41}{5040}. \]

The first order approximate solution is given as
\[ u_1(x) = 1 + x - \frac{x^2}{2} - \frac{5}{6} x^3 - \frac{11}{3} x^4 - \frac{19}{120} x^5 - \frac{29}{720} x^6 - \frac{41}{5040} x^7. \]

Proceeding as before, the second approximate solution is given as follows:
\[ u_2(x) = 1 + x - \frac{x^2}{2} - \frac{5}{6} x^3 - \frac{11}{3} x^4 - \frac{19}{120} x^5 - \frac{29}{720} x^6 - \frac{41}{5040} x^7 + \frac{19958400}{479001600} + \frac{518918400}{1245404160} + \frac{87178291200}{118879488000} + O(x^{16}). \]

This gives the solution in a closed form by \((x^2 - 1)\cos(x)\).

**Example 3.3:** Consider equation (1) with
\[ [a, b] = [-1, 1], \quad \phi(x) = -1, \quad \Psi(x) = -8[2\cos(x) + 7\sin(x)], \]
and the boundary conditions
Applying the homotopy perturbation method assuming the initial approximation as

\[ u_0(x) = a + bx + cx^2 = dx^4 + f(x) + gx^6 + hx^7. \]

\[ u_0 + pu_1 + p^2u_2 + \cdots = a + bx + cx^2 + dx^4 + f(x) + gx^6 + hx^7. \]

\[ + g(x^6) + h(x^7) + p \sum_{n=0}^{\infty} \frac{g(x^6) + h(x^7)}{n!} \left( u_0 + pu_1 + p^2u_2 + \cdots \right) + 8(2 \sin(x) - 7 \cos(x)) \]

\[ dx \, dx \, dx \, dx \, dx \, dx \, dx \, dx. \]

**Example 3.4:** Consider equation (1) with

\[ [a, b] = [0, 1], \quad \phi(x) = x, \quad \psi(x) = -(48 - 15x + x^3)e^x, \]

and the boundary conditions

\[ A_3 = 0, \quad A_4 = 0, \quad A_4 = \frac{9}{8}, \quad A_5 = -2. \]

\[ B_1 = 0, \quad B_2 = -4e, \quad B_3 = -16e, \quad B_4 = -36e. \]

Applying the homotopy perturbation method assuming the initial approximation as

\[ u_0(x) = a + bx + cx^2 = dx^4 + f(x) + gx^6 + hx^7. \]

\[ u_0 + pu_1 + p^2u_2 + \cdots = a + bx + cx^2 + dx^4 + f(x) + gx^6 + hx^7. \]

\[ + g(x^6) + h(x^7) + p \sum_{n=0}^{\infty} \frac{g(x^6) + h(x^7)}{n!} \left( u_0 + pu_1 + p^2u_2 + \cdots \right) + 48 + 15x + x^3 \]

\[ dx \, dx \, dx \, dx \, dx \, dx \, dx \, dx. \]

**Where:**

\[ a, b, c, d, e, f, g, h \] are constants to be determined.

Comparing the co-efficient of like powers of \( p \)

\[ p_0^0 = u_0(x) = a + bx + cx^2 = dx^4 + f(x) + gx^6 + hx^7. \]

\[ p_1^0 = u_1(x) = -72a + 4 + 20b \sin^2(x) + 20c \sin^2(x) + (e - \frac{1}{3})k \sin^2(x). \]

\[ + f(x) + \frac{5}{3} + \frac{5}{3} \sin^2(x) + g(x) + h(x) + \frac{1}{3} \sin^2(x) + \frac{1}{2} \sin^2(x). \]

\[ + \frac{1}{12} \sin^2(x) + \frac{1}{2} \sin^2(x) + \frac{1}{3} \sin^2(x) + \frac{1}{4} \sin^2(x) + \frac{1}{5} \sin^2(x) + \frac{1}{6} \sin^2(x). \]

Imposing the boundary conditions will yield

\[ a = -1, \quad b = 0, \quad c = \frac{3}{2}, \quad d = 0, \quad e = \frac{31}{24}, \quad f = 0, \quad g = \frac{31}{720}, h = 0. \]

The first approximate solution is given as

\[ u_1(x) = -1 + \frac{3}{2} x^2 - \frac{13}{24} x^4 + \frac{31}{720} x^6. \]

Proceeding as before, the second approximate solution is given as

\[ u_2(x) = -1 + \frac{3}{2} x^2 - \frac{13}{3} x^4 + \frac{3}{2} x^6 + \frac{3}{2} x^8 + \frac{3}{2} x^{10} - \frac{19}{6842800} + O(x^{10}). \]

This gives the solution in a closed form by \((x^2 - 1)\cos(x) - 1).
Proceeding as before, the second approximate solution is given as

\[
\alpha_2(x) = x - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{30} + \frac{x^8}{144} - \frac{x^{10}}{840} + \frac{x^{12}}{5760} - \frac{x^{14}}{43560} + O(x^{15})
\]

This gives the solution in a closed form as \(x(1-x)e^x\).

CONCLUSION

In this paper, we applied the homotopy perturbation method (HPM) for finding the solution of a class of eighth-order boundary value problems. The proposed method is employed without using linearization, discretization or restrictive assumptions. It may be concluded that the homotopy perturbation method (HPM) is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. The fact that the HPM solves nonlinear problems without using the Adomian’s polynomials is a clear advantage of this technique over the decomposition method.

REFERENCES


