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On Associativity of Some Banach *-Algebras

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Abstract: The notion of *partial crossed product* by a discrete group was introduced in K. McClanahan [*K*-theory for partial crossed products by discrete groups, J. Funct. Anal. *130* (1995), 77-117]. In Proposition 2.1 of the above named paper, the author has proved that the algebra L is *associative*, where he employed special properties of C^*-algebras, i.e., approximate identity. (L, will be precisely defined later). If A is a C^*-algebra, S is a *unital* inverse semigroup and α is an action of S on A, the concept of $A \times_\alpha S$, is introduced in N. Sieben [C^*-crossed products by partial actions and actions of inverse semigroups, J. Austral. Math. Soc. Ser. A, *63* (1977), 32-46]. In Proposition 5.1 of Sieben's work, also, the associativity of the Banach *-algebra, L, is proved, by using properties of *approximate identity*. In this paper, we will prove that, both algebras are *associative* without using *approximate identity*. It should be noted that we will consider a not *necessarily unital* inverse semigroup.

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INTRODUCTION

The notion of partial crossed product of a C^*-algebra by the group of integers introduced by R. Exel in [1]. McClanahan generalized this idea and defined in [2] *partial actions* and crossed products of C^*-algebras by partial actions of discrete groups. Lastly, R. Exel extended his definitions to the more general case of a twisted partial action of a locally compact group on a C^*-algebra. The general definitions of partial actions and crossed products by partial actions can be seen in [3].

In the notion of partial crossed product of a C^*-algebra by the group of integers, in fact, the automorphism that used in the definition of a crossed product of a C^*-algebra A by the group of integers was replaced by an isomorphism between two ideals of A and was called partial automorphism. Since the set of partial automorphisms of A makes a unital inverse semigroup and we can not talk about a homomorphism from a group into an inverse semigroup; a partial action is an appropriate generalization. Or equivalently, the idea of a partial action is to replace the automorphism group by the inverse semigroup of partial automorphisms.

It should be noted that, in making an inverse semigroup from the set of partial automorphisms, we should make extensive use of the facts like ideals of ideals of a C^*-algebra are, themselves, ideals of that algebra and that the intersection of two ideals equals their product.

The organization of this paper is as follows.

In section 1 we study multiplier algebras and some of their properties. Also, we introduce an auxiliary multiplier which helps us in proving the main theorems without using properties of approximate identities.

Section 2 is devoted to the partial action of a discrete group on a C^*-algebra and introducing the Banach *-algebra, L, where we will prove the main result of this section.

In section 3 the definition of action of an inverse semigroup on a C^*-algebra is given. Also, the Banach *-algebra, L, in semigroup case is introduced and the two main results are proved.

An Auxiliary Double Centralizer: The main goal of this section is to present a double centralizer (multiplier) which helps us in proving the associativity of some algebras.

Definition 1.1: A *double centralizer* for a C^*-algebra A is a pair (L, R) of bounded linear maps on A, such that for all $a,b \in A$

$$L(ab) = L(a)b,\ R(ab) = aR(b) \text{ and } R(a)b = aL(b).$$

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As an example, if $x \in A$ and L_x, R_x are the linear maps on A defined by $L_x(a) = xa$ and $R_x(a) = ax$, then (L_x, R_x) is a double centralizer on A.

Lemma 1.2: If (L, R) is a double centralizer on a C^*-algebra a, then $\|L\|=\|R\|$.

Proof: [4, Lemma 2.1.4]. 0

If A is a C^*-algebra, the set of all double centralizers of A is denoted by $M(A)$. Define the norm of the double centralizer (L, R) to be $\|L\|=\|R\|$. It is not hard to see that $M(A)$ is a closed vector subspace of $B(A) \oplus B(A)$.

For given (L, R) and $(L', R') \in M(A)$ define their product by

$$(L, R)(L', R') = (LL', R'R).$$

It is easy to check that, this product is again a double centralizer of A. That is, $M(A)$ is an algebra under this multiplication.

For $L: A \to A$, define $L^*: A \to A$ by $L^*(a) = (L(a^*))^*$. We will see that L^*. is linear and the map $L \to L^*$ is an isometric conjugate-linear map from $B(A)$ to itself such that $(L^*)^* = L$ and $(LL')^* = (L')^* L^*$. This shows that, if (L, R) is an element of $M(A)$, then so is $(L, R)^* = (R^*, L^*)$. That is, the map $(L, R) \mapsto (L, R)^*$ is an involution on $M(A)$.

Theorem 1.3: If A is a C^*-algebra, then $M(A)$ is a C^*-algebra under the multiplication, involution and norm defined above.

Proof: [5, II. 7.3.4]. 0

The algebra $M(A)$ is called the *multiplier algebra* of A. Also, L is called a *left multiplier* of A and R is called a *right multiplier* of A.

Let I be an ideal of a C^*-algebra A. Define the map $\phi : I \to M(I)$ by putting $\phi(x) = (Lx, Rx)$, $x \in I$. This is an algebra homomorphism and moreover, $L_{xy} = L_x o L_y$, $R_{xy} = R_y o R_x$, which gives

$$\phi(xy) = (L_x o L_y, R_y o R_x) = \phi(x)\phi(y)$$

In what follows we give the double centralizer promised at the beginning of this section.

Lemma 1.4: If I and J are ideals in a C^*-algebra A, $\pi : I \to J$ is a C^*-algebra isomorphism and $(L, R) \in M(I)$ then

$$(\pi o L o \pi^{-1}, \pi o R o \pi^{-1}) \in M(J).$$

Proof: For arbitrary elements a,b of J there are x, y in I such that $a = \pi(x)$, $b = \pi(y)$ and

$(\pi o L o \pi^{-1})(ab) = (\pi o L)(xy) = \pi(L(xy))\pi(L(x)y)$
$= \pi(L(x))\pi(y) = (\pi o L)(x)b = (\pi o R o \pi^{-1})(a)b;$

$(\pi o R o \pi^{-1})(ab) = (\pi o R)(xy) = \pi(R(xy))\pi(R(x)y)$
$= \pi(R(x))\pi(y) = (\pi o R)(x)b = (\pi o R o \pi^{-1})(a)b;$

and

$(\pi o R o \pi^{-1})(a)b = (\pi o R)(x)b = \pi(R(x))\pi(y)$
$= \pi(R(x)y) = \pi(xL(y)) = a(\pi o L)(y)$
$= a(\pi o R o \pi^{-1})(b)$

That is, $(\pi o R o \pi^{-1}, \pi o R o \pi^{-1})$ satisfies in conditions of Definition 1.1. 0□

Since closed ideals in a C^*-algebra are idempotent, we can use the following proposition in the proof of our main result.

Proposition 1.5: If I is an idempotent algebra, that is $I^2 = I$ and
$(L, R), (L', R') \in M(I)$ then
$(R' oL)(a) = (LoR')(a)$ for all $a \in I$.

Proof: Let a, b be arbitrary elements of I. By using the definition of double centralizers we have

$(R' oL)(ab) = R'(L(ab)) = R'(L(a)b)$
$= L(a)R'(b) = L(aR'(b)) = L(R'(ab))$
$= (LoR')(ab)$

Since I is an idempotent algebra, it is easy to see that $(R'oL)(a) = (LoR')(a)$ for all $a \in I$. 0 □

Partial Action of Groups and Associativity of an Algebra: In this section we will review the basic facts about partial actions. Also, following [2] we will introduce a Banach *-algebra, L, whose certain completion is called *partial crossed product of C^*-algebra by discrete group*. The major new result of this section is Theorem 2.3, where we prove the associativity of L without using approximate identity. For that purpose, let A be a C^*-algebra considered fixed throughout this section.

Definition 2.1: A *partial automorphism* of A is a triple $\Theta = (\theta, I, J)$ where I and J are closed two-sided ideals in A and $\theta : I \to J$ is a C^*-algebra isomorphism.

The following is a crucial concept to be studied throughout this work.

Definition 2.2: Let G be a discrete group with identity e. A *partial action* of G on A is a collection $\{(\theta_g, D_{g^{-1}}, D_g): g \in G\}$ of partial automorphisms such that

- $D_e = A$ and θ_e is the identity map on A;
- $\theta_h^{-1}(D_h \cap D_{g^{-1}}) \subseteq D_{(gh)^{-1}}$;
- $\theta g(\theta h(x)) = \theta gh(x)$ for all $x \in \theta_h^{-1}(D_h \cap D_{g^{-1}})$.

It is not hard to see that $\theta_{h^{-1}} = \theta_h^{-1}, \theta_h o \theta_h^{-1}$ is the identity on D_h and conditions (i)-(iii) are equivalent to the following:

- $D_e = A$ and θ_e is the identity map on A:
- $\theta_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$;
- $\theta g(\theta h(x)) = \theta gh(x)$, for all $x \in D_{h^{-1}} \cap D_{(gh)^{-1}}$.

Let $L = \{a \in \ell^1(G, A) : a(g) \in D_g\}$. Also, let L have the norm, scalar multiplication and addition inherited from the algebra $\ell^1(G, A)$. Before we define multiplication and involution on L, let us agree on the following convention: We shall denote by $a_g\delta_g$ the element of L which is identically zero except for its gth component which is equal to a_g. Any element of L, say x, is therefore given by

$$x = \sum_{g \in G} a_g \delta_g$$

Where the sum has finitely many nonzero terms.

Based on [1] and [2], the definition of multiplication and involution on L is given by

$$(a_g\delta_g)(b_h\delta_h) = \theta_g(\theta_{g^{-1}}(a_g)b_h)\delta_{gh}$$
$$(a_g\delta_g)^* = \theta_{g^{-1}}(a_g^*)\delta_{g^{-1}}$$

Here we will prove the main theorem of this section. That is, we will prove that, L with the above defined multiplication is associative, without using properties of approximate identity.

Theorem 2.3: L is associative.

Proof: Obviously, L is associative if and only if

$(a_g\delta_g\, b_h\, \delta_h)c_k\delta_k = a_g\, \delta_g(b_h\, \delta_h\, c_k\, \delta_k)$

for given g, h, k of G and $a_g \in D_g$, $b_h \in D_h$, $c_k \in D_k$. The left hand side of the above equality is nothing but

$$(a_g\delta_g b_h\delta_h)c_k\delta_k = \theta_g((\theta_{g^{-1}})(a_g)b_h)\delta_{gh}c_k\delta_k$$
$$= \theta_{gh}\{\theta_{(gh)^{-1}}[\theta_g(\theta_{g^{-1}}(a_g)b_h)]c_k\}\delta_{ghk}$$

Since $ag \in Dg$ and $\theta_{g^{-1}} : D_g \to D_{g^{-1}}$ we have $\theta_{g^{-1}}, (ag) \in D_{g^{-1}}$ as a consequence $\theta_{g^{-1}}(a_g)\, b_h \in D_{g^{-1}}\, D_h = D_{g^{-1}} \cap Dh$, therefore $\theta_g(\theta_{g^{-1}}(a_g)b_h) \in D_g \cap D_{gh}$ and we can split $\theta_{(gh)^{-1}}$, which gives

$$\theta_{(gh)^{-1}}[\theta_g(\theta_{g^{-1}}(a_g)b_h)] = \theta_{h^{-1}}(\theta_{g^{-1}}(a_g)b_h).$$

Also, since $\theta_{g^{-1}}(\theta_{g^{-1}}(a_g)b_h)$ is an element of $D_{h^{-1}} \cap D_{h^{-1}g^{-1}}$ we can split θ_{gh} and consequently

$$(a_g\delta_g b_h\delta_h)c_k\delta_k = \theta_{gh}[\theta_{h^{-1}}(\theta_{g^{-1}}(a_g)b_h)c_k]\delta_{ghk}$$
$$= \theta_g\{\theta_h[\theta_{h^{-1}}(\theta_{g^{-1}}(a_g)b_h)c_k]\}\delta_{ghk}.$$

A nd, the right hand side is

$$a_g\delta_g(b_h\delta_h c_k\delta_k) = a_g\delta_g[\theta_h(\theta_{h^{-1}}(b_h)c_k)\delta_{hk}]$$
$$= \theta_g\{\theta_{g^{-1}}(a_g)[\theta_h(\theta_{h^{-1}}(b_h)c_k)]\}\delta_{ghk}.$$

If we apply $\theta_{g^{-1}}$ on the last equalities of the both sides we see that the equality holds if and only if

$$\theta_h[\theta_{h^{-1}}(\theta_{g^{-1}}(a_g)b_h)c_k] = \theta_{g^{-1}}(a_g)[\theta_h(\theta_{h^{-1}}(b_h)c_k)]$$

is verified for all

$$a_g \in D_g,\ b_h \in D_h,\ c_k \in D_k$$

Since

$$\theta_{g^{-1}} : D_g \to D_{g^{-1}}$$

is an isomorphism, we see that $\theta_{g^{-1}}(a_g)$ runs over $D_{g^{-1}}$ and as a consequence the equality holds if and only if

$$\theta_h[\theta_{h^{-1}}(a_g b_h)c_k] = a_g[\theta_h(\theta_{h^{-1}}(b_h)c_k)]$$

for every $ag \in D_{g^{-1}}$, $b_h \in D_h$, $c_k \in D_k$. If $g = k = e$ then $D_g = D_k = A$ and L is associative if and only if the above equality holds for arbitrary $h \in G$, $a_g, c_k \in A$ and $b_h \in D_h$. Obviously, this equality is equivalent to say that

$$(\theta_h o R_{c_k} o \theta_{h^{-1}}) o L_{a_g} = L_{a_g} o(\theta_h o R_{c_k} o \theta_{h^{-1}})$$

is valid on D_h for every $h \in G$ and all $a_g, c_k \in A$.

Let R_{c_k} be a right multiplier on $D_{h^{-1}}$ and L_{a_g} be a left multiplier on D_h. By Lemma 1.4 we see that $\theta_h o R_{c_k} o \theta_{h^{-1}}$ is a right multiplier on D_h. Since all ideals $D_h (h \in G)$ are idempotent Proposition 1.5 implies that,

$$(\theta_h o R_{c_k} o \theta_{h^{-1}}) o L_{a_g} = L_{a_g} o(\theta_h o R_{c_k} o \theta_{h^{-1}}).$$

Action of an Inverse Semigroup and Associativity of an Algebra: In this section we wish to apply our method to action of an inverse semigroup on a *C**-algebra. Our task here will be to construct an auxiliary double centralizer from an action of an inverse semigroup. Therefore, in the first part of this section we will present the definition of an action of an inverse semigroup on a *C**-algebra, then we will study a Banach *-algebra, *L*, whose completion with respect to a certain norm is called *inverse semigroup crossed product.*

The major new results of this section are Lemma 3.3 and Theorem 3.4.

Definition 3.1: If *X* is any set we denote be *I*(*X*) the inverse semigroup formed by all bijections between subsets of *X*, under the operation given by composition of functions with the largest possible domain.

Let us now give the definition of an action of an inverse semigroup on a *C**-algebra *A*.

Definition 3.2: An action of an inverse semigroup *S* on a *C**-algebra *A* is a semigroup homomorphism

$$\theta : S \to I(A)$$

Such that

- For every $s \in S$, the domain (and hence also the range) of θ_s is a closed two sided ideal of *A* and θ_s is a *-isomorphism.
- The linear span of the union of the domains of all θ_s is dense in *A*.

If *e* is an idempotent element of *S*, we denote by E_e the domain of θ_e. Also, for each $s \in S$, we have that θ_s is a *-isomorphism from E_{s^*} to E_s. See also [6, 3.3]. In fact, E_{s^*} and E_s are closed two-sided ideals in *A* and (θ_s, E_{s^*}, E_s) is a partial automorphism of *A*. Note that if $s \in S$, then from the definition of inverse semigroup we have $ss^* s=s$ and $s^* ss^* = s^*$. By applying θ on both sides of the above relations, we have

$$\theta_{s^*} = \theta_s^{-1}.$$

Throughout this section, by a *semipartial dynamical system* we mean a triple (*A, S,* θ) in which *A* is a *C**-algebra, *S* is a (not necessarily unital) inverse semigroup and θ is an action of *S* on *A*.

Let (*A, S,* θ) be a semipartial dynamical system. Take,

$$L = \{x \in \ell^1(S, A) : x(s) \in E_s\}.$$

It is not hard to see that

$$L = \bigoplus_{s \in S} E_s.$$

If $x \in L$, then $x=(a_s)_{s\in S}$ in which $a_s \in E_s$ and $a_s = 0$ for all but finitely many *s*. For given $s \in S$ and $a_s \in E_s$, let $a_s \delta_s$ be the element of *L* which is identically zero except for its $s^{th.}$ component which is equal to a_s. Therefore any element of *L*, say $x = (a_s)_{s\in S}$ is given by $x = \sum_{s \in S} a_s \delta_s$, where the sum has finitely many nonzero terms. Let *L* have the norm, scalar multiplication and addition inherited from $\ell^1(S, A)$. Based on [1] and [2] multiplication and involution is given by

$$(a_S \delta_S).(b_t \delta_t) = \theta_S(\theta_{S^*}(a_S) b_t)\delta_{St}$$

$$\text{and} (a_S \delta_S)^* = \theta_{S^*}(a_S^*)\delta_{S^*}$$

for every *s,t* in *S*, $a_s \in E_s$ and $b_t \in E_t$. By using [7, Lemma 4.2] it is easy to see that *L* is a normed *-algebra, (see [7, sec. 5]).

Here, let us present an auxiliary double centralizer which will be used in the proof of the main theorem of this section.

Lemma 3.3: Let *A* be a *C**-algebra, *S* an inverse semigroup and (θ_S, E_{S^*}, E_S) be a partial automorphism of *A*. If *L* and *R* are left and right multiplier of E_{S^*} respectively, then

$$(\theta_S o L o \theta_{S^*}, \theta_S o R o \theta_{S^*}) \in M(E_S).$$

Proof: It is obvious that $\theta_S o L o \theta_{S^*}$ and $\theta_S o R o \theta_{S^*}$ are linear transformations on E_s. To prove that $(\theta_S o L o \theta_{S^*}, \theta_S o R o \theta_{S^*})$ is a multiplier of E_s, let $\theta_S o L o \theta_{S^*} = L'$ and $\theta_S o R o \theta_{S^*} = R'$. For $x \in A$ and $a,b \in E_s$ we have

$$L'_x(ab) = \theta_S o L_x(\theta_{S^*}(ab)) =$$
$$\theta_S(x\theta_{S^*}(ab))$$
$$= \theta_S(x)\theta_S(\theta_{S^*}(ab)) = \theta_S(x)(ab),$$

and

$$L'(a)b = [\theta_S o L_x(\theta_{S^*}(a))]b =$$
$$[\theta_S(x\theta_{S^*}(a))]b$$
$$= [\theta_s(x)a]b = \theta_s(x)(ab) = \text{That is, } L'(ab) = L'(a)b.$$

Also,

$$R'(ab) = (\theta_S o R o \theta_{S^*})(ab) =$$
$$\theta_S o R_x(\theta_{S^*}(ab))$$
$$= \theta_S(\theta_{S^*}(ab)x) = (ab)\theta_S(x),$$

and

$$aR'(b) = a(\theta_S o R_x o \theta_{S^*})(b) = a(\theta_S o R_x(\theta_{S^*}(b)))$$
$$= a\theta_S(\theta_{S^*}(b)x) = a(b\theta_S(x)) = (ab)\theta_S(x),$$

therefore $R'(ab) = aR'(b)$.
Finally,

$$R'(a)b = [(\theta_S oR_X o\theta_{S^*})(a)]b = [\theta_S oR_X(\theta_{S^*}(a))]b$$
$$= [\theta_S(\theta_{S^*}(a)x)]b = [a\theta_S(x)]b.$$

Also,

$$aL'(b) = a[(\theta_S oL_X o\theta_{S^*})b] = a[\theta_S oL_X(\theta_{S^*}(b))] =$$
$$a\theta_S(x\theta_{S^*}(b)) = a\theta_S(x)b.$$

That is, $R'(a)b = aL'(b)$ and these facts show that $(\theta_S oLo\theta_{S^*}, \theta_S oRo\theta_{S^*})$ is a multiplier of E_S.□

Now we would like to show that the normed *-algebra L is associative with the multiplication defined following the Definition 3.2. Note that the following result is proved in Proposition 5.1 of [7] and not only we are not assuming that our inverse semigroup is unital, but we will prove the result without the approximate identity.

Theorem 3.4: The normed *-algebra L is associative.

Proof: Let $x = \sum_{r\in S} a_r\delta_r$, $y = \sum_{s\in S} a_s\delta_S$ and $z = \sum_{t\in S} a_t\delta_t$ are arbitrary elements of L. We want to show that

$$(x.\ y).z = x.(y.z).$$

It suffices to show this for $x = a_r\delta_r$, $y = a_s\ \delta_s$ and $z = a_t\delta_t$.

$$(a_r\delta_r.a_S\delta_S).a_t\delta_t = \theta_r(\theta_{r^*}(a_r)a_S)\delta_{rS}.a_t\delta_t$$
$$= \theta_{rS}\{\theta_{(rS)^*}[\theta_r(\theta_{r^*}(a_r)a_S)]a_t\}\delta_{rst}$$
$$= \theta_{rS}\{\theta_{S^*r^*}[\theta_r(\theta_{r^*}(a_r)a_S)]a_t\}\delta_{rst}$$
$$= \theta_{rS}\{\theta_{S^*}[\theta_{r^*}(a_r)a_S]a_t\}\delta_{rst}$$
$$= \theta_r\{\theta_S(\theta_{S^*}[\theta_{r^*}(a_r)a_S]a_t)\}\delta_{rst}.$$

Note that, since $a_r \in E_r$ and $\theta_{r^*} : E_r \to E_{r^*}$ is a C*-algebra isomorphism, we have $\theta_{r^*}(a_r) \in E_{r^*}$ and $\theta_{r^*}(a_r)a_S \in E_{r^*}E_S = E_{r^*} \cap E_S \subset E_{r^*}$. We used the fact that $\theta_{r^*r} = \theta_{r^*}\theta_r$ is the identity map onE_{r^*}.
On the other hand

$$a_r\delta_r.(a_S\delta_S.a_t\delta_t) = a_r\delta_r.[\theta_S(\theta_{S^*}(a_S)a_t)]\delta_{St}$$
$$= \theta_r\{\theta_{r^*}(a_r)[\theta_S(\theta_{S^*}(a_S)a_t)]\}\delta_{rst}.$$

Therefore, L is associative if and only if

$$\theta_r\{\theta_S(\theta_{S^*}[\theta_{r^*}(a_r)a_S]a_t)\}\delta_{rst}$$
$$= \theta_r\{\theta_{r^*}(a_r)[\theta_S(\theta_{S^*}(a_S)a_t)]\}\delta_{rst}.$$

If we apply L to the both sides of the above equality we see that equality holds if and only if

$$\theta_S(\theta_{S^*}[\theta_{r^*}(a_r)a_S]a_t) = \theta_{r^*}(a_r)[\theta_S(\theta_{S^*}(a_S)a_t)].$$

Since $\theta_{r^*} : E_r \to E_{r^*}$ is an isomorphism, $\theta_{r^*}(a)$ runs over E_{r^*} and consequently, the last equality is equivalent to $\theta_S(\theta_{S^*}[aa_S]a_t) = a[\theta_S(\theta_{S^*}(a_S)a_t)]$ for arbitrary elements a in E_{r^*}, $as \in E_s$ and $at \in E_t$. If $r = t$ we have $E_r = E_t$. Therefore it suffices we prove $\theta_S(\theta_{S^*}[aa_S]a_t) = a[\theta_S(\theta_{S^*}(a_S)a_t)]$ for a_r, a_t in E_r, $s \in S$ and $a_s \in E_s$.

Let R_{a_t} be a right multiplier of E_{s^*} and L_a be a left multiplier of E_s. By Lemma 3.3, $\theta_S oR_{a_t} o\theta_{S^*}$ is a right multiplier of E_s. Since A is a C*-algebra and all closed ideals of A are idempotent, by Proposition 1.5 we have

$$[(\theta_s oR_{a_t} o\theta_{S^*})oL_a](a_S) = [L_a o(\theta_S oR_{a_t} o\theta_{S^*})](a_S),$$

Which is nothing but the desired equality. □

The author believe that, by using this method, we can prove many theorems in the C*-algebra context.

REFERENCES

1. Exel, R., 1994. Circle actions on C*-algebras, partial automorphisms and generalized Pimsner-Voiculescu exact sequences, J. Funct. Anal., 122: 361-401.
2. McClanahan, K., 1995. K-theory for partial crossed products by discrete groups, J. Funct. Anal., 130: 77-117.
3. Exel, R., 1997. Twisted partial actions, a classification of regular C*-algebraic bundles, Proc. London Math. Soc., 74: 417-443.
4. Murphy, G.J., 1990. C*-algebras and operator theory, Academic Press, Boston.
5. Blackadar, B., 2006. Operator algebras, theory of C*-algebras and Von Neumann algebras, Springer.
6. Paterson, A.L.T., 1999. Groupoids, inverse semigroups and their operator algebras, Birkhäuser.
7. Sieben, N., 1997. C*-crossed products by partial actions and actions of inverse semigroups, J. Austral. Math. Soc. Ser. A, 63: 32-46.