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Analytical Study on Goursat Problems

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Abstract: The Goursat problems are investigated. The homotopy analysis method (HAM) is used to construct analytical solutions to these problems. The linear and the nonlinear structures are handled in a like manner without any need to restrictive assumptions. The accuracy level of the obtained results reveal the power of this method over existing numerical methods.

Key words: Analytical study · Goursat problems

INTRODUCTION

Most physical phenomena in real world are described through nonlinear equations and these type of equations have attracted lots of attention among scientists. Large class of nonlinear equations do not have a precise analytic solution, so numerical methods have largely been used to handle these equations. There are also some analytic techniques for nonlinear equations. Some of the classic analytic methods are the Lyapunov's artificial small parameter method, perturbation techniques and d-expansion method. In the last two decades, some new analytic methods have been proposed to handle functional equations, among them are Adomian decomposition method, tanh method, sinh–cosh method, homotopy analysis method (HAM), variational iteration method (VIM) and homotopy perturbation method (HPM).

Homotopy analysis method (HAM), first proposed by Liao [1], is an elegant method which has proved its effectiveness and efficiency in solving many types of functional equations, see [2-11] and the references therein. HAM properly overcomes restrictions of perturbation techniques because it does not need any small or large parameters to be contained in the problem. Liao, in his book [2], proves that this method is a generalization of some previously used techniques such as d-expansion method, artificial small parameter method and ADM. Also, it is shown in [13, 14] that HPM [12] is just a special case of HAM. Moreover, unlike previous analytic techniques, the HAM provides with a convenient way to adjust and control the convergence region and rate of approximation series.

The Goursat problem arises in partial differential equations with mixed derivatives.
The standard form of the Goursat problem [15] is given by

$$u_{xt} = f(x, t, u, u_x, u_t),\ 0 \le x \le a,\ 0 \le t \le b,$$
$$u(x,0) = g(x),\ u(0,t) = h(t),\ g(0) = h(0) = u(0,0).$$

This equation has been examined by several numerical methods such as Runge-Kutta method, finite difference method, finite elements method, Adomian decomposition method and geometric mean averaging of the functional values of $f(x, t, u, u_x, u_t)$, In this work the linear and the nonlinear Goursat problems will be examined via the Homotopy analysis method (HAM).

Basic Ideas Homotopy Analysis Method: To describe the basic ideas the homotopy analysis method, we consider the following differential equation,

$$N(u(r,t)] = 0, \qquad (1)$$

Where N is a nonlinear operator, r and t are independent variables, $u(r,t)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [2] constructs the so-called *zero-order* deformation equation.

$$(1-p)L(\varphi(r,t;p) - u_0(r,t)) = p\hbar H(r,t)N(r,t;p), \qquad (2)$$

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Where $P\in[0,1]$ is the embedding parameter, h is a nonzero auxiliary parameter, L is an auxiliary linear operator, $u_0(r, t)$ is an initial guess of $u(r, t)$, $\varphi(r, t; p)$ is a unknown function on independent variables r, t, p. It is important that one has great freedom to choose auxiliary parameter $\hbar$ in homotopy analysis method. If $p = 0$ and $p = 1$, it holds

$$\varphi(r,t;p) = u_0(r,t), \quad \varphi(r,t;1) = u(r,t) \tag{3}$$

Thus, as p increases from 0 to 1, the solution $\varphi(r, t; p)$ varies from the initial guesses $u_0(r, t)$ to the solution $u(r, t)$ Expanding $\varphi(r, t; p)$ in Taylor series with respect to p, we have

$$\varphi(r,t;p) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t)p^m, \tag{4}$$

Where

$$u_m(r,t) = \frac{1}{m!}\frac{\partial^m \varphi(r,t;p)}{\partial p^m}\Big|_{p=0}. \tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function are so properly chosen, the series (4) converges at $p = 1$, then we have

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t). \tag{6}$$

Define the Vector $\vec{u_n} = \{u_0, u_1, \ldots u_n\}$.

Differentiating equation (2) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we obtain the *mth–order* deformation equation

$$L[u_m - \chi_m u_{m-1}] = \hbar H(r,t) R_m(\vec{u}_{m-1}), \tag{7}$$

Where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N(r,t;p)}{\partial p^{m-1}}\Big|_{p=0}, \tag{8}$$

and

$$\chi_m = \begin{cases} 0 & m \le 1, \\ 1 & m > 1. \end{cases} \tag{9}$$

Applying L^{-1}on both side of equation (7), we get

$$u_m(r,t) = \chi_m u_{m-1}(r,t) + \hbar L^{-1}[H(r,t) R_m(\vec{u}_{m-1})]. \tag{10}$$

In this way, it is easily to obtain u_m form $m\ge 1$, at *Mth–order* we have

$$u(r,t) = \sum_{m=0}^{M} u_m(r,t). \tag{11}$$

When $M \to \infty$ we get an accurate approximation of the original equation (1). For the convergence of the above method we refer the reader to Liao [2]. If equation (1) admits unique solution, then this method will produce the unique solution. If equation (1) does not possess unique solution, the homotopy analysis method will give a solution among many other (possible) solutions.

Example 1: We first consider the homogeneous Goursat problem

$$u_{xt} = u, \tag{12}$$

$$u(x,0) = e^x,\ u(0,t) = e^t, u(0,0) = 1. \tag{13}$$

To solve the equation (12) by means of homotopy analysis method, according to the initial conditions denoted in equation (13), it is natural to choose

$$u_0 = e^x \tag{14}$$

We choose the linear operator

$$L(\varphi(x,t;p)] = \frac{\partial\varphi^2(x,t;p)}{\partial x \partial t},$$

With the property $L[c] = 0$. Where c is constant. We now define a nonlinear operator as

$$N[\varphi(x,t;p)] = \frac{\partial^2\varphi(x,t;p)}{\partial x\partial t} - \varphi(x,t;p) = 0.$$

Using above definition, with assumption $H(x,t) = 1$. We construct the zeroth–order deformation equations

$$(1-p)L(\varphi(x,t;p) - u_0(x,t) = p\hbar N(\varphi(x,t;p)),$$

Obviously, when $p = 0$ and $p = 1$

$$\varphi(x,t,0) = u_0(x,t), \quad \varphi(x,t;0) = u(x,t),$$

Thus, we obtain the *mth–order* deformation equation

$$L[u_m - \chi_m u_{m-1}] = \hbar R_m(\vec{u}_{m-1}). \tag{15}$$

Where

$$R_m(\vec{u}_{m-1}) = \frac{\partial^2 U_{m-1}}{\partial x\partial t} - U_{m-1}.$$

Now, the solution of the *mth–order* order deformation equation (15)

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})], \quad (16)$$

Finally, we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t).$$

From equations (14) and (16) and subject to initial condition

$u_m(x,0) = 0,\ m \geq 1.$

We obtain

$u_0 = e^x.$

$u_1 = te^x$

$u_2 = \frac{t^2}{2}e^x,$

$\vdots$

Then, the approximate solution in a series form is

$$u = e^x(1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \ldots) = e^{x+t}.$$

Which is the exact solution of Equation (12).

Example 2: We first consider the homogeneous Goursat problem

$$u_{xt} = -2u, \quad (17)$$

$$u(x,0) = e^x, u(0,t) = e^{-2t},\ u(0,0) = 1. \quad (18)$$

To solve the equation (17) by means of homotopy analysis method, according to the initial conditions denoted in equation (18), it is natural to choose.

$$u_0 = e^x. \quad (19)$$

We choose the linear operator

$$L(\varphi(x,t;p)] = \frac{\partial\varphi^2(x,t;p)}{\partial x\partial t},$$

With the property $L[c] = 0$. Where c is constant. We now define a nonlinear operator as:

$$N[\varphi(x,t;p)] = \frac{\partial^2\varphi(x,t;p)}{\partial x\partial t} + 2\varphi(x,t;p) = 0.$$

Using above definition, with assumption $H(x,t) = 1$. We construct the zeroth– order deformation equations

$$(1-p)L(\varphi(x,t;p) - u_0(x,t) = p\hbar N(\varphi(x,t;p)),$$

Obviously, when $p = 0$ and $p = 1$.

$$\varphi(x,t;0) = u_0(x,t),\ \ \varphi(x,t;0) = u(x,t),$$

Thus, we obtain the *mth–order* deformation equation

$$L[u_m - \chi_m u_{m-1}] = \hbar R_m(\vec{u}_{m-1}). \quad (20)$$

Where

$$R_m(\vec{u}_{m-1}) = \frac{\partial^2 U_{m-1}}{\partial x\partial t} + 2U_{m-1}.$$

Now, the solution of the *mth–order* order deformation equation (20)

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})], \quad (21)$$

Finally, we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t).$$

From equations (19) and (21) and subject to initial condition

$u_m(x,0) = 0,\ \ m \geq 1.$

We obtain

$u_0 = e^x$
$u_1 = 2te^x$
$u_2 = 2t^2e^x$

Then, the approximate solution in a series form is

$$u = e^x(1 - 2t + 2t^2 + \ldots) = e^{x-2t}.$$

Which is the exact solution of Equation (17).

CONCLUSION

In this work we presented an analytic framework to handle the Goursat problems. HAM successfully worked to give exact and approximate solutions to these models. A disadvantage of this new approach is to need an initial value. This technique cannot be employed if the problem does not include initial and boundary conditions.

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