World Applied Sciences Journal 10 (6): 637-644, 2010
ISSN 1818-4952

An Iterative Algorithm for Nonlinear Bvps Using Pade´ Approximants

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Abstract: This paper explicitly reveal the efficiency and reliability of a powerful iterative algorithm (ITM) which is mainly due to Geijji and Jafari coupled with the diagonal Pade´ approximants to various singular and nonsingular boundary value problems which arise in engineering and applied sciences. The diagonal *Pade´* approximants prove to be very useful for the understanding of physical behavior of the solution. Numerical results reflect accuracy of the proposed combination.

Key words: Iterative method · Thomas Fermi equation · Flierl-Petviashivili equation · Unsteady flow · Porous medium · Boundary layer problem · Blasius problem · Pade´ approximants

INTRODUCTION

Recently, Geijji and Jafari [1] introduced a very reliable and efficient iterative algorithm (ITM) which has been applied [1, 2-6] to a wide class of diversified linear and nonlinear problems of physical nature. Moreover, through study of literature witnesses the applications of singular and nonsingular initial and boundary value problems in mathematical modeling of diversified physical problems related to engineering and applied sciences. The applications of these problems involve physics, astrophysics, experimental and mathematical physics, nuclear charge in heavy atoms, thermal behavior of a spherical cloud of gas, thermodynamics, population models, chemical kinetics and fluid mechanics, see [1-32] and the references therein. Several techniques [1-32] including decomposition, variational iteration, finite difference, polynomial spline, differential transform, exp-function and homotopy perturbation have been developed for solving such problems. Most of these methods have their inbuilt deficiencies coupled with the major drawback of huge computational work. The basic motivation of this paper is to apply the iterative algorithm developed by Geijji and Jafari [1] coupled with the diagonal Pade´ approximants for singular and nonsingular boundary value problems (BVPS). The Pade´ approximants are applied in order to make the work more concise and for the better understanding of the solution behavior. The use of Pade´ approximants shows real promise in solving BVPS in an infinite domain; see [6]. It is well known in the literature that polynomials are used to approximate the truncated power series. It was observed [6] that polynomials tend to exhibit oscillations that may give an approximation error bounds. Moreover, polynomials can never blow up in a finite plane and this makes the singularities not apparent. To overcome these difficulties, the obtained series is best manipulated by Pade´ approximants for numerical approximations. Using the power series, isolated from other concepts, is not always useful because the radius of convergence of the series may not contain the two boundaries. It is now well known that Pade´ approximants [6] have the advantage of manipulating the polynomial approximation into rational functions of polynomials. By this manipulation, we gain more information about the mathematical behavior of the solution. In addition, the power series are not useful for large values of x. It is an established fact that power series in isolation are not useful to handle boundary value problems (BVPS). This can be attributed to the possibility that the radius of convergence may not be sufficiently large to contain the boundaries of the domain. It is therefore essential to combine the series solution with the

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Pade´ approximants to provide an effective tool to handle boundary value problems on an infinite or semi-infinite domain. We apply this powerful combination of series solution and Pade´ approximants for solving a variety of boundary value problems. Precisely the proposed combination is applied on boundary layer problem, unsteady flow of gas through a porous medium, Thomas Fermi equation, Flierl-Petviashivili (FP) equation and Blasius problem. It is worth mentioning that Flierl-Petviashivili equation has singularity behavior at x = 0 which is a difficult element in this type of equations. We transform the FP equation to a first order initial value problem. The ITM is applied to the reformulated first order initial value problem which leads the solution in terms of transformed variable. The desired series solution is obtained by implementing the inverse transformation. The fact that the ITM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the Adomian's decomposition method.

Iterative Algorithm (ITM): Consider the following general functional equations:

$$f(x) = 0, \tag{1}$$

To convey the idea of the ITM [1], we rewrite the above equation as:

$$y = N(y) + c,$$

Where N is a nonlinear operator from a banach space $B\rightarrow$ and f is a known function. We are looking for a solution of equation (1) having the series form:

$$y = \sum_{i=0}^{\infty} y_i. \tag{2}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=0}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \tag{2a}$$

From equations (2) and (2a),

$$\sum_{i=0}^{\infty} y_i = c + N(y_0) + \sum_{i=0}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \tag{2b}$$

We define the following recurrence relation:

$$\begin{cases} y_0 = c, \\ y_1 = N(y_0), \\ y_{m+1} = N(y_0 + \ldots + y_m) - N(y_0 + \ldots + y_{m-1}), \quad m = 1,2,3,\ldots, \end{cases} \tag{2c}$$

then

$$(y_1 + \ldots + y_{m+1}) = N(y_0 + \ldots + y_m), \ m = 1,2,3,\ldots,$$

and

$$y = f + \sum_{i=1}^{\infty} y_i,$$

if N is a contraction, i.e. $\|N(x) - N(y)\| \le \|x - y\|$ $0<K<1$, then

$$\|y_{m+1}\| = \|N(y_0 + \ldots + y_m) - N(y_0 + \ldots + y_{m-1})\| \le K\|y_m\| \le K^m\|y_0\|,$$
$m = 1,2,3,\ldots,$

and the series $\sum_{i=1}^{\infty} = y_i$ absolutely and uniformly converges to a solution of equation (1) [1, 2-6], which is unique, in view of the Banach fixed-point theorem.

Pade´ Approximants: A Pade´ approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function u(x). The [L/M] Pade approximants to a function y(x) are given by [6].

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)}, \tag{3}$$

Where $P_L(x)$ is polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. the formal power series

$$y(x) = \sum_{i=1}^{\infty} a_i x^i, \tag{4}$$

$$y(x) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1}), \tag{5}$$

determine the coefficients of $P_L(x)$ and $Q_M(x)$ by the equation. Since we can clearly multiply the numerator and denominator by a constant and leave [L/M] unchanged, we imposed the normalization condition.

$$Q_M(x) = 1.0. \tag{6}$$

Finally, we require that $P_L(x)$ and $Q_M(x)$ have non common factors. If we write the coefficient of $P_L(x)$ and $Q_M(x)$ as

$$\left.\begin{array}{l} P_L(x) = p_0 + p_1x + p_2x^2 + \cdots + p_Lx^L, \\ Q_M(x) = q_0 + q_1x + q_2x^2 + \cdots + q_Mx^M \end{array}\right\}. \quad (7)$$

Than by (8) and (9), we may multiply (5) by $Q_M(x)$, which linearizes the coefficient equations. We can write out (7) in more details as.

$$\left.\begin{array}{l} a_{L+1} + a_Lq_1 + \cdots + a_{L-M}q_M = 0, \\ q_{L+2} + q_{L+1}q_1 + \cdots + a_{L-M+2}q_M = 0, \\ \vdots \\ a_{L+M} + a_{L+M-1}q_1 + \cdots + a_Lq_M = 0, \end{array}\right\} \quad (8)$$

$$\left.\begin{array}{l} a_0 = p_0, \\ a_0 + a_0q_1 + \cdots + = p_1, \\ \vdots \\ a_L + a_{L-1}q_1 + \cdots + a_0q_L = p_L \end{array}\right\}. \quad (9)$$

To solve these equations, we start with equation (8), which is a set of linear equations for all the unknown q's. Once the q's are known, then equation (9) gives and explicit formula fro the unknown p's, which complete the solution. If equations (8) and (9) are nonsingular, then we can solve them directly and obtain equation (10) [32], where equation (10) holds and if the lower index on a sum exceeds the upper, the sum is replaced by zero:

$$\left[\frac{L}{M}\right] = \frac{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M}x^j & \sum_{j=M-1}^{L} a_{j-M+1}x^j & \cdots & \sum_{j=0}^{L} a_jx^j \end{bmatrix}}{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{bmatrix}}. \quad (12)$$

To obtain diagonal Pade′ approximants of different order such as [2/2], [4/4] or [6/6], we can use the symbolic calculus software Maple.

Numerical Applications: In this section, we apply the Iterative method (ITM) to solve boundary layer problem, unsteady flow of gas through a porous medium, Thomas Fermi equation, Flierl-Petviashivili equation and Blasius problem. The powerful Pade′ approximants are applied for making the work more concise and to get the better understanding of solution behavior.

Example 4.1: Consider the following nonlinear third order boundary layer problem which appears mostly in the mathematical modeling of physical phenomena in fluid mechanics.

$$f'''(x) + (n-1)f(x)f''(\eta) - 2n(f'(x))^2 = 0,\ n>0,$$

with boundary conditions

$$f(0) = 0, \quad f(0) = 1, \quad f(\infty) = 0,\ n>0,$$

Applying the iterative algorithm (ITM), we get

$$f_{n+1}(x) = f_n(x) - \int_0^x\int_0^x\int_0^x \left((n-1)f_n(x)f_n^{//}(\eta) - 2n\left(f_n^{/}(x)\right)^2\right)dxdx,$$

Consequently following approximants are made

$f_0(x) = c,$
$f_0(x) = x,$
$f_1(x) = Nf_0(x),$
$$f_1(x) = x + \frac{1}{2}\alpha x^2 + \frac{1}{3}x^3,$$
$$f_2(x) = N(f_0(x) + f_1(x)) - N f_0(x),$$

$$f_2(x) = x + \frac{1}{2}\alpha x^2 + \frac{1}{3}x^3 + \frac{1}{24}\alpha(3n+1)x^4 + \frac{1}{30}n(n+1)x^5,$$
$$f_3(x) = N(f_0(x) + f_1(x) + f_2(x)) - N(f_0(x) + f_1(x)),$$

$$f_3(x) = x + \frac{1}{2}\alpha x^2 + \frac{1}{3}x^3 + \frac{1}{24}\alpha(3n+1)x^4 + \frac{1}{30}n(n+1)x^5 + \frac{1}{120}\alpha^2(3n+1)x^5 + \frac{1}{720}\alpha\left(19n^2 + 18n + 3\right)x^6 + \frac{1}{315}n\left(2n^2 + 2n + 1\right)x^7,$$
$\vdots$

The series solution is given as

$$f(x)=x+\frac{\alpha x^2}{2}+\frac{nx^3}{3}+\left(\frac{1}{8}n\alpha+\frac{1}{24}\alpha\right)x^4+$$
$$\left(\frac{1}{30}n^2+\frac{1}{40}n\alpha^2+\frac{1}{120}\alpha^2+\frac{1}{30}n\right)x^5+\left(\frac{19}{720}n^2\alpha+\frac{1}{240}\alpha+\frac{1}{40}n\alpha\right)x^6+$$
$$\left(\frac{1}{120}n\alpha^2+\frac{1}{315}n+\frac{2}{315}n^3+\frac{11}{5040}\alpha^2+\frac{3}{560}n^2\alpha^2+\frac{2}{315}n^2\right)x^7+$$
$$\left(\frac{11}{40320}\alpha^3+\frac{33}{4480}n^2\alpha+\frac{3}{4480}\alpha^3n^2+\frac{23}{5760}n\alpha+\frac{1}{2688}\alpha+\right.$$
$$\left.\frac{167}{40320}n^3\alpha+\frac{1}{960}\alpha^3n\right)x^8+\left(\frac{1}{3780}n+\frac{527}{362880}n^3\alpha^2+\frac{19}{11340}n^3+\right.$$
$$\left.\frac{709}{362880}n\alpha^2+\frac{23}{8064}n^2\alpha^2+\frac{23}{22680}n^2+\frac{13}{22680}n^4+\frac{43}{120960}\alpha^2\right)x^9+\cdots.$$

Example 4.2 Consider the following nonlinear differential equation which governs the unsteady flow of gas through a porous medium.

$$y^{//}(x)+\frac{2x}{\sqrt{1-\alpha y}}y'(x)=0,\quad 0<\alpha<1.$$

with the following boundary conditions

$$y(0)=1,\ \lim_{x\to\infty}y(x)=0.$$

Applying the iterative algorithm (ITM), we get

$$y_{n+1}(x)=y_n(x)-\int_0^x\int_0^x\left(\frac{2x}{\sqrt{1-\alpha y_n}}y_n^{/}(x)\right)dx\,dx,\ 0<\alpha<1.$$

Where $A = y'(0)$. Consequently, following approximants are obtained

$y_0(x) = c$,
$y_0(x) = 1$,
$y_1(x) = Ny_0(x)$,
$y_1(x) = 1 + Ax$,
$y_2(x) = N(y_0(x)+y_1(x)) - Ny_0(x)$,

$$y_2(x)=1+Ax-\frac{A}{3\sqrt{1-\alpha}}x^3,$$

$y_3(x) = N(y_0(x)+y_1(x)+y_2(x)) - N(y_0(x)+y_1(x))$,

$$y_3(x)=1+Ax-\frac{A}{3\sqrt{1-\alpha}}x^3-\frac{\alpha A^2}{12(1-\alpha)^{3/2}}x^4+\frac{A}{10(1-\alpha)}x^5,$$
$\vdots$.

The series solution is given as

$$y(x)=1+Ax-\frac{A}{3\sqrt{1-\alpha}}x^3-\frac{\alpha A^2}{12(1-\alpha)^{3/2}}x^4+$$
$$\left(\frac{A}{10(1-\alpha)}-\frac{3\alpha^2A^3}{80(1-\alpha)^{5/2}}\right)x^5+$$
$$\left(\frac{\alpha A^2}{15(1-\alpha)^2}-\frac{\alpha^3A^4}{48(1-\alpha)^{7/2}}\right)x^6+\cdots.$$

The diagonal Pade´ approximants can be applied to analyze the physical behavior. Based on this, the [2/2] Pade´ approximants produced the slope A to be

$$A=-\frac{2(1-\alpha)^{1/4}}{\sqrt{3\alpha}},\tag{13}$$

and using [3/3] Pade´ approximants we find

$$A=-\frac{\sqrt{(-4674\alpha+8664)\sqrt{1-\alpha}-144\gamma}}{57\alpha},\tag{14}$$

where

$$\gamma=\sqrt{5(1-\alpha)(1309\alpha^2-2280\alpha+1216)}.\tag{15}$$

Using (13)-(15) gives the values of the initial slope $A = y'(0)$ listed in the Table 4.3. The formulas (13) and (14) suggest that the initial slope $A = y'(0)$ depends mainly on the parameter α, where $0<\alpha<1$.

Example 4.3: Consider the following Thomas-Fermi (T-F) equation which arises in the mathematical modeling of various models in physics, astrophysics, solid state physics, nuclear charge in heavy atoms and applied sciences.

$$y''(x)=\frac{y^{3/2}}{x^{1/2}},$$

with boundary conditions

$$y(0)=1,\ \lim_{x\to\infty}y(x)=0.$$

Applying the iterative algorithm (ITM), we get

$$y_{n+1}(x)=y_0(x)+\int_0^x\int_0^x\left(x^{-1/2}y_n^{3/2}\right)dx\,dx.$$

Consequently, following approximants are obtained

$y_0(x) = c$,

$y_0(x) = 1,$

$y_1(x) = Ny_0(x),$

$$y_1(x) = 1 + Bx + \frac{4}{3}x^{3/2},$$

$$y_2(x) = 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3,$$

$$y_2(x) = 1 + Bx + \frac{4}{3}x^{3/2} + \frac{2}{5}Bx^{5/2} + \frac{1}{3}x^3,$$

$y_3(x) = N(y_0(x)+y_1(x) + y_2(x)) - N(y_0(x) + y_1(x)),$

$$y_3(x) = 1 + Bx + \frac{4}{3}x^{3/2} + \frac{2}{5}Bx^{5/2} + \frac{1}{3}x^3 + \frac{3}{70}B^2x^{7/2} + \frac{2}{15}Bx^4 + \frac{2}{27}x^{9/2} + \frac{1}{3}x^3,$$

$\vdots$.

The series solution is given as

$$y(x) = 1 + Bx + Bx + Bx + \frac{4}{3}x^{3/2} + \frac{2}{5}Bx^{5/2} + \frac{1}{3}x^3 + \frac{3}{70}B^2x^{7/2} + \frac{2}{15}Bx^4 + \frac{2}{27}x^{9/2} + \frac{3}{70}B^2x^{7/2} - \frac{1}{252}B^3x^{9/2} + \frac{1}{175}B^2x^5 + \frac{2}{27}x^{9/2} + \frac{3}{70}B^2x^{7/2} + \frac{1}{1056}B^4x^{11/2} + \frac{4}{1575}B^3x^6 + \frac{557}{100100}B^2x^{13/2} + \frac{4}{693}Bx^7 + \frac{101}{52650}x^{15/2} - \frac{3}{9152}B^5x^{13/2} - \frac{29}{24255}B^4x^7 - \frac{512}{351000}B^3x^{15/2} - \frac{46}{45045}B^2x^8 - \frac{113}{1178100}Bx^{17/2} + \frac{23}{473850}x^9 \cdots,$$

$\vdots$.

Setting $x^{1/2} = t$, the series solution is obtained as

$$y(t) = 1 + Bt^2 + \frac{4}{3}t^3 + \frac{2}{5}Bt^5 + \frac{1}{3}t^6 + \frac{3}{70}B^2t^7 + \frac{2}{15}Bt^8 + \left(-\frac{1}{252}B^3 + \frac{2}{27}\right)t^9 + \frac{1}{175}B^2t^{10} + \left(\frac{1}{1056}B^4 + \frac{31}{1485}B\right)t^{11} + \left(\frac{4}{1575}B^3 + \frac{4}{405}\right)t^{12} + \left(-\frac{3}{9152}B^5 + \frac{557}{100100}B^2\right)t^{13} + \left(-\frac{29}{24255}B^4 + \frac{4}{693}B\right)t^{14} + \left(\frac{7}{499}B^6 - \frac{623}{351000}B^3 + \frac{101}{52650}\right)t^{15} + \left(\frac{68}{105105}B^4 - \frac{46}{45045}B^2\right)t^{16} + \left(-\frac{3}{43520}B^7 + \frac{153173}{116424000}B^4 - \frac{113}{1178100}B\right)t^{17} + \cdots.$$

The diagonal Pade´ approximants can be applied in order to study the mathematical behavior of the potential $y(x)$ and to determine the initial slope of the potential $y'(0)$.

Example 4.4: Consider the generalized variant of the Flierl-Petviashivili equation

$$y'' + \frac{1}{x}y' - y^n - y^{n+1} = 0,$$

with boundary conditions

$$y''(0) = \alpha, \quad y'(0) = 0, \quad y(\infty) = 0.$$

Using the transformation $u(x) = xy'(x)$, the generalized FP equation can be converted to the following first order initial value problem

$$u'(x) = x\left(\int_0^x \left(\frac{u(x)}{x}\right)^n + \left(\frac{u(x)}{x}\right)^{n+1} dx\right),$$

with initial conditions

$$u(0) = 0, \quad u'(0) = 0.$$

Proceeding as before, the series solution after four iterations is given by

$$u(x) = \frac{(\alpha^n + \alpha^{n+1})}{2}x^2 + \frac{(\alpha^n + \alpha^{n+1})(n\alpha^n + (n+1)\alpha^{n+1})}{16\alpha}x^4 + \frac{(\alpha^n + \alpha^{n+1})(2n(3n-1)\alpha^{2n} + 2n(3n+1)\alpha^{2n+1} + (3n+1)(n+1)\alpha^{2n+2})}{384\alpha^2}x^6 + \frac{(\alpha^n + \alpha^{n+1})(n(18n^2 - 29n + 12)\alpha^{3n} + n(54n^2 - 33n + 7)\alpha^{3n+1} + (18n^2 + 7n + 1)(3n\alpha^{3n+2} + (n+1)\alpha^{3n+3}))}{18432\alpha^3}x^8$$

$+ \cdots,$

and the inverse transformation will yield

$$y(x) = \alpha + \frac{(\alpha^n + \alpha^{n+1})}{4}x^2 + \frac{(\alpha^n + \alpha^{n+1})(n\alpha^n + (n+1)\alpha^{n+1})}{64\alpha}x^4 + \frac{(\alpha^n + \alpha^{n+1})(2n(3n-1)\alpha^{2n} + 2n(3n+1)\alpha^{2n+1} + (3n+1)(n+1)\alpha^{2n+2})}{2304\alpha^2}x^6 + \frac{(\alpha^n + \alpha^{n+1})(n(18n^2 - 29n + 12)\alpha^{3n} + n(54n^2 - 33n + 7)\alpha^{3n+1} + (18n^2 + 7n + 1)(3n\alpha^{3n+2} + (n+1)\alpha^{3n+3}))}{147456\alpha^3}x^8$$

$+ \cdots.$

Table 4.1: Numerical values for $\alpha = f(0)$ for 0<n<1 by using diagonal Pade´ approximants

n	[2/2]	[3/3]	[4/4]	[5/5]	[6/6]
0.2	-0.3872983347	-0.3821533832	-0.3819153845	-0.3819148088	-0.3819121854
1/3	-0.5773502692	-0.5615999244	-0.5614066588	-0.5614481405	-0.561441934
0.4	-0.6451506398	-0.6397000575	-0.6389732578	-0.6389892681	-0.6389734794
0.6	-0.8407967591	-0.8393603021	-0.8396060478	-0.8395875381	-0.8396056769
0.8	-1.007983207	-1.007796981	-1.007646828	-1.007646828	-1.007792100

Table 4.2: Numerical values for $\alpha = f(0)$ for n>1 by using diagonal Pade´ approximants

n	α
4	-2.483954032
10	-4.026385103
100	-12.84334315
1000	-40.65538218
5000	-104.8420672

Table 4.3: Exhibits the initial slopes A=y(0) for various values of α.0

α	$B_{[2/2]}$=)ý(0)	$y_{[3/3]}$
0.1	-3.556558821	-1.957208953
0.2	-2.441894334	-1.786475516
0.3	-1.928338405	-1.478270843
0.4	-1.606856838	-1.231801809
0.5	-1.373178096	-1.025529704
0.6	-1.185519607	-0.8400346085
0.7	-1.021411309	-0.6612047893
0.8	-0.8633400217	-0.4776697286
0.9	-0.6844600642	-0.2772628386

Table 4.4: Exhibits the values of y(x) for y(x) for x = 0.1 *to* 1.0..

x	*y* kidder	$y_{[2/2]}$	$y_{[3/3]}$
0.1	0.8816588283	0.8633060641	0.8979167028
0.2	0.7663076781	0.7301262261	0.7985228199
0.3	0.6565379995	0.6033054140	0.7041129703
0.4	0.5544024032	0.4848898717	0.6165037901
0.5	0.4613650295	0.3761603869	0.5370533796
0.6	0.3783109315	0.2777311628	0.4665625669
0.7	0.3055976546	0.1896843371	0.4062426033
0.8	0.2431325473	0.1117105165	0.3560801699
0.9	0.1904623681	0.04323673236	0.3179966614
1.0	0.1587689826	0.01646750847	0.2900255005

Table 4.5: Pade´ approximants and initial slopes $y^{l}(0)$

Pade approximants	Initial slope $y^{l}(0)$	Error (%)
[2/2]	-1.211413729	23.71
[4/4]	-1.550525919	2.36
[7/7]	-1.586021037	12.9×10^{-2}
[8/8]	-1.588076820	3.66×10^{-4}
[10/10]	-1.588076779	3.64×10^{-4}

Table 4.6: Roots of the Pade´ approximants monopole α, n = 1

Degree	Roots
[2/2]	-1.5
[4/4]	-2.50746
[6/6]	-2.390278
[8/8]	-2.392214

Table 4.7: Roots of the Pade´ approximants monopole α, n = 2

Degree	Roots
[2/2]	-2.0
[4/4]	-2.0
[6/6]	-2.0
[8/8]	-2.0

Table 4.8 Roots of the Pade´ approximants monopole α, n = 3

Degree	Roots
[2/2]	0.0
[4/4]	-2.197575908
[6/6]	-1.1918424398
[8/8]	-1.848997181

Table 4.9: Roots of the Pade´ approximants [8/8] monopole α for several values of n

n	[8/8] Roots	n	[8/8] Roots
1	-2.392213866	7	-1.000708285
2	-2.0	8	-1.000601615
3	-1.848997181	9	-1.000523005
4	-1.286025892	10	-1.000462636
5	-1.001101141	11	-1.000262137
6	-1.000861533	$n \to \infty$	-1.0

Table 4.10: Pade´ approximants and numerical value of α

Pade´ approximant	α
[2/2]	0.5778502691
[3/3]	0.5163977793
[4/4]	0.5227030798

Diagonal Pade´ approximants can be applied to find the roots of the FP monopole α for $n \geq 1$.

Table 4.9 shows that the roots of the monopole α converge to -1 as n increases.

Example 4.5: Consider the two dimensional nonlinear inhomogeneous initial boundary value problem for the integro differential equation related to the Blasius problem.

$$y''(x) = \alpha - \frac{1}{2}\int_0^x y(t)y''(t)dt, \qquad -\infty < x < 0$$

with boundary conditions

$$y(0) = 0, \quad y'(0) = 1.$$

and

$$\lim_{x\to\infty} y'(x) = 0,$$

Where the constant α is positive and defined by

$$y''(0) = \alpha \qquad \alpha>0.$$

Applying the iterative algorithm (ITM), we get

$$y_{n+1}(x) = y_n(x) + \alpha - \int_0^x\int_0^x \left(\frac{1}{2}\int_0^s y_n(x)y_n''(x)\right)dxdx. \qquad -\infty<x<0$$

Proceeding as before, the series solution is given as

$$y(x) = x + \frac{1}{2}\alpha x^2 - \frac{1}{48}\alpha x^4 - \frac{1}{240}\alpha^2 x^5 + \frac{1}{960}\alpha x^6 + \frac{11}{20160}\alpha^2 x^7 + \left(\frac{11}{161280}\alpha^3 + \frac{1}{960}\alpha\right)x^8 - \frac{43}{967680}\alpha^2 x^9 + \left(\frac{1}{52960}\alpha - \frac{5}{387072}\alpha^3\right)x^{10} + \left(\frac{587}{212889600}\alpha^2 - \frac{5}{4257792}\alpha^4\right)x^{11} + \left(-\frac{1}{16220160}\alpha + \frac{1}{7257792}\alpha^3\right)x^{12} + \cdots,$$

and consequently

$$y'(x) = 1 + \alpha x - \frac{1}{12}\alpha x^3 - \frac{1}{48}\alpha^2 x^4 + \frac{1}{160}\alpha x^5 + \frac{11}{2880}\alpha^2 x^6 \left(\frac{11}{20160}\alpha^3 - \frac{1}{2688}\alpha\right)x^7 - \frac{43}{107520}\alpha^2 x^8 + 10\left(\frac{1}{552960}\alpha - \frac{5}{387072}\alpha^3\right)x^9 + 11\left(\frac{587}{212889600}\alpha^2 - \frac{5}{4257792}\alpha^4\right)x^{10} + 12\left(-\frac{1}{16220160}\alpha + \frac{1}{725760}\alpha^3\right)x^{11} + \cdots.$$

Now, we apply the diagonal Pade′ approximants to determine a numerical value for the constant α by using the given condition. Pade′ approximant of $y'(x)$ usually converges on the entire real axis. Moreover, $y'(x)$ is free of singularities on the real axis. Substituting the boundary conditions $y'(-\infty)$ in each Pade′ approximant which vanishes if the coefficient of x with the highest power in the numerator vanishes. By solving the resulting polynomials of these coefficients, we obtain the values of α listed in Table 4.10.

CONCLUSION

In this paper, we applied a reliable combination of iterative algorithm (ITM) and the diagonal Pade′ approximants for obtaining approximate solutions of various singular and nonsingular boundary value problems of diversified physical nature. The proposed ITM is employed without using linearization, discretization, perturbation or restrictive assumptions. The fact that the ITM solves nonlinear problems without using the Adomian's polynomials is a clear advantage of this technique over the decomposition method.

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