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Analytical Study on Nonlinear Fifth Order Korteweg-De Vries Equation

E. Alibeigi and A. Neyrame

Islamic Azad University of Gonbad Kavoos Branch, Iran

Abstract: In this paper, He's homotopy perturbation method is applied to nonlinear fifth order KdV equation. The procedure of the method is systematically illustrated. The results reveal that the proposed method is very effective and simple for obtaining approximate solutions of nonlinear partial differential equations.

Key words: Fifth order KdV equation · Homotopy perturbation method

INTRODUCTION

During the past decades, both mathematicians and physicists have devoted considerable effort to the study of exact and numerical solutions of the nonlinear ordinary or partial differential equations corresponding to the nonlinear problems. Many powerful methods have been presented. The homotopy perturbation method, proposed first by He in 1998 and was further developed and improved by He [1-5]. The application of the homotopy perturbation method (HPM) in nonlinear problems has been devoted by scientists and engineers, because this method is to continuously deform a simple problem which is easy to solve into the under study problem which is difficult to solve. It can be said that He's homotopy perturbation method is a universal one, is able to solve various kinds of nonlinear functional equations. In this method the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solutions.

For the purpose of applications illustration of the methodology of the proposed method, using homotopy perturbation method, we consider the following nonlinear differential equation,

$$A(u) - f(r) = 0, \tag{1}$$

$$B(u, \partial u / \partial n) = 0, \tag{2}$$

Where A is a general differential operator, $f(r)$ is a known analytic function, B is a boundary condition and Γ is the boundary of the domain Ω.

The operator A can be generally divided into two operators, L and N where L is a linear, while N is a nonlinear operator. Equation (1) can be, therefore, written as follows:

$$L(u) + N(u) - f(r) = 0, \tag{3}$$

Using the homotopy technique, we construct a homotopy $U(r,p)\colon \Omega \times [1,0] \rightarrow \mathbb{R}$ which satisfies:

$$\begin{aligned} H(U,P) &= (1-P)[L(U) - L(u_0)] \\ &+ p[A(U) - f(r)] = 0,\ p \in [0,1],\ \ r \in \Omega \end{aligned} \tag{4}$$

$$H(U,p) = L(U) - L(u_0) + pL(u_0) + p[N(U) - f(r)] = 0, \tag{5}$$

Where $p \in [0, 1]$, is called homotopy parameter and u_0 is an initial approximation for the solution of Eq.(1), which satisfies the boundary conditions. Obviously from Esq. (4) and (5) we will have.

$$H(U,0) = L(U) - L(u_0) = 0, \tag{6}$$

$$H(U,1) = A(U) - f(r) = 0, \tag{7}$$

We can assume that the solution of (4) or (5) can be expressed as a series in p, as follows:

$$U = U_0 + pU_1 + p^2U_2 + \dots \tag{8}$$

Setting $p = 1$, results in the approximate solution of Eq. (1)

$$u = \lim_{p \to 1} U = U_0 + U_1 + U_2 + \dots \tag{9}$$

HPM for Fifth Order KdV Equation: The Korteweg-de Vries (KdV) equation, given here in canonical form, [6]

$$\frac{\partial u}{\partial t} + 45u^2\frac{\partial u}{\partial x} - 15\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - 15u\frac{\partial^3 u}{\partial x^3} + \frac{\partial^5 u}{\partial x^5} = 0. \tag{10}$$

With initial condition,

Corresponding Author: E. Alibeigi, Islamic Azad University of Gonbad Kavoos Branch, Iran

$$u(x,0) = a_0 - 2c_2 \sec h^2(\sqrt{c_2}x). \tag{11}$$

and the exact solution

$$u = a_0 - 2c_2 \sec h^2(\sqrt{c_2}(x - (45a_0^2 - 60a_0c_2 + 16c_2^2)t)), \quad c_2 > 0.$$

Where a_0, c_2 are arbitrary constants.

To solve Eq. (10) by homotopy perturbation method, we construct the following homotopy.

$$(1-p)(\frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t}) + p(\frac{\partial U}{\partial t} + 45U^2\frac{\partial U}{\partial x} - 15\frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x^2} - 15U\frac{\partial^3 U}{\partial x^3} + \frac{\partial^5 U}{\partial x^5}) = 0,$$

or

$$\frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t} + p(\frac{\partial u_0}{\partial t} + 45U^2\frac{\partial U}{\partial x} - 15\frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x^2} - 15U\frac{\partial^3 U}{\partial x^3} + \frac{\partial^5 U}{\partial x^5}) = 0, \tag{12}$$

Suppose the solution of Eq. (12) has the following form

$$U = U_0 + pU_1 + p^2U_2 \tag{13}$$

Substituting (13) into (12) and equating the coefficients of the terms with the identical powers of p leads to

$$p^0: \frac{\partial U_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0,$$

$$p^1: \frac{\partial U_1}{\partial t} + 45{U_0}^2\frac{\partial U_0}{\partial x} - 15\frac{\partial U_0}{\partial x}\frac{\partial^2 U_0}{\partial x^2} - 15U_0\frac{\partial^3 U_0}{\partial x^3} + \frac{\partial^5 U_0}{\partial x^5} = 0,$$

$$p^2: \frac{\partial U_2}{\partial t} + 90U_1U_0\frac{\partial U_0}{\partial x} + 45{U_0}^2\frac{\partial U_1}{\partial x} - 15\frac{\partial U_1}{\partial x}\frac{\partial^2 U_0}{\partial x^2} - 15\frac{\partial U_0}{\partial x}\frac{\partial^2 U_1}{\partial x^2} - 15U_1\frac{\partial^3 U_0}{\partial x^3} - 15U_0\frac{\partial^3 U_1}{\partial x^3} + \frac{\partial^5 U_1}{\partial x^5} = 0,$$

$$\vdots$$

$$p^j: \frac{\partial U_j}{\partial t} + 45\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1} U_iU_k\frac{\partial U_{j-k-i-1}}{\partial x} - 15\sum_{k=0}^{j-1}\frac{\partial U_k}{\partial x}\frac{\partial^2 U_{j-1-k}}{\partial x^2} - 15\sum_{k=0}^{j-1}U_k\frac{\partial^3 U_{j-1-k}}{\partial x^3} + \frac{\partial^5 U_j}{\partial x^5} = 0,$$

$$\vdots$$

We take

$$U_0 = u_0 = a_0 - 2c_2 \sec h^2(\sqrt{c_2}x). \tag{14}$$

We have the following recurrent equations for j = 1,2,3,....

$$U_j = -\int_0^t(45\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}U_iU_k\frac{\partial U_{j-k-i-1}}{\partial x} - 15\sum_{k=0}^{j-1}\frac{\partial U_k}{\partial x}\frac{\partial^2 U_{j-1-k}}{\partial x^2} - 15\sum_{k=0}^{j-1}U_k\frac{\partial^3 U_{j-1-k}}{\partial x^3} + \frac{\partial^5 U_j}{\partial x^5})dt = 0. \tag{15}$$

With the aid of the initial approximation given by Eq. (14) and the iteration formula (15) we get the other of component as follows

$$\begin{aligned} U_1 = {} & 720ta_0\sqrt{c_2^5}\sec h^2(\sqrt{c_2}x)\tanh^3(\sqrt{c_2}x) - 480ta_0\sqrt{c_2^5}\sec h^2(\sqrt{c_2}x)\tanh(\sqrt{c_2}x) - 2160t\sqrt{c_2^7}\sec h^4(\sqrt{c_2}x)\tanh^3(\sqrt{c_2}x) \\ & + 1200t\sqrt{c_2^7}\sec h^4(\sqrt{c_2}x)\tanh(\sqrt{c_2}x) - 180\sqrt{c_2^3}\sec h^2(\sqrt{c_2}x)\tanh(\sqrt{c_2}x)ta_0^2 + 720\sqrt{c_2^5}\sec h^4(\sqrt{c_2}x)\tanh(\sqrt{c_2}x)ta_0 \\ & - 720\sqrt{c_2^7}\sec h^6(\sqrt{c_2}x)\tanh(\sqrt{c_2}x)t - 1440\sqrt{c_2^7}\sec h^2(\sqrt{c_2}x)\tanh^5(\sqrt{c_2}x)t + 1920\sqrt{c_2^7}\sec h^2(\sqrt{c_2}x)\tanh^3(\sqrt{c_2}x)t \\ & - 544\sqrt{c_2^7}\sec h^2(\sqrt{c_2}x)\tanh(\sqrt{c_2}x)t, \end{aligned}$$

$$\begin{aligned} U_2 = {} & 353792t^2c_2^6\sec h^2(\sqrt{c_2}x) + 162000t^2a_0^3c_2^3\sec h^2(\sqrt{c_2}x)\tanh^4(\sqrt{c_2}x) + 238080t^2a_0c_2^5\sec h^2(\sqrt{c_2}x) \\ & - 43480800t^2c_2^6\sec h^6(\sqrt{c_2}x)\tanh^6(\sqrt{c_2}x) - 74606400t^2c_2^6\sec h^4(\sqrt{c_2}x)\tanh^8(\sqrt{c_2}x) \\ & - 130394880t^2c_2^6\sec h^2(\sqrt{c_2}x)\tanh^6(\sqrt{c_2}x) + 85680t^2a_0^2c_2^4\sec h^2(\sqrt{c_2}x) + 156470400t^2c_2^6\sec h^4(\sqrt{c_2}x)\tanh^6(\sqrt{c_2}x) \\ & + 5702400t^2c_2^6\sec h^8(\sqrt{c_2}x)\tanh^2(\sqrt{c_2}x) - 9504000t^2c_2^6\sec h^8(\sqrt{c_2}x)\tanh^4(\sqrt{c_2}x) \end{aligned}$$

$$+119750400t^2c_2^6\sec h^2(\sqrt{c_2}x)\tanh^8(\sqrt{c_2}x) - 17540640t^2c_2^6\sec h^6(\sqrt{c_2}x)\tanh^2(\sqrt{c_2}x) - 129600t^2c_2^5\sec h^8(\sqrt{c_2}x)a_0$$
$$-10830336t^2c_2^6\sec h^2(\sqrt{c_2}x)\tanh^2(\sqrt{c_2}x) + 61036800t^2c_2^6\sec h^2(\sqrt{c_2}x)\tanh^4(\sqrt{c_2}x) + 97200t^2c_2^4\sec h^6(\sqrt{c_2}x)a_0^2$$
$$-172800t^2c_2^4\sec h^4(\sqrt{c_2}x)a_0^2 - 544320t^2c_2^5\sec h^4(\sqrt{c_2}x)a_0 + 21600t^2a_0^3c_2^3\sec h^2(\sqrt{c_2}x) + 4050t^2a_0^4c_2^2\sec h^2(\sqrt{c_2}x)$$
$$+432000t^2c_2^5\sec h^6(\sqrt{c_2}x)a_0 + 23326080t^2c_2^6\sec h^4(\sqrt{c_2}x)\tanh^2(\sqrt{c_2}x) - 32400t^2c_2^3\sec h^4(\sqrt{c_2}x)a_0^3$$
$$-39916800t^2c_2^6\sec h^2(\sqrt{c_2}x)\tanh^{10}(\sqrt{c_2}x) - 712800t^2c_2^6\sec h^{10}(\sqrt{c_2}x)\tanh^2(\sqrt{c_2}x)$$
$$+55663200t^2c_2^6\sec h^6(\sqrt{c_2}x)\tanh^4(\sqrt{c_2}x) - 104843520t^2c_2^6\sec h^4(\sqrt{c_2}x)\tanh^4(\sqrt{c_2}x)$$
$$-5068800t^2a_0c_2^5\sec h(\sqrt{c_2}\tanh^2(\sqrt{c_2}x) + 19353600t^2a_0c_2^5\sec h^2(\sqrt{c_2}x)\tanh^4(\sqrt{c_2}x)$$
$$-25401600t^2a_0c_2^5\sec h(\sqrt{c_2}\tanh^6(\sqrt{c_2}x) - 1587600t^2c_2^4\sec h^2(\sqrt{c_2}x)\tanh^6(\sqrt{c_2}x)a_0^2$$
$$+10886400t^2a_0c_2^5\sec h(\sqrt{c_2}\tanh^8(\sqrt{c_2}x) + 18273600t^2c_2^5\sec h^4(\sqrt{c_2}x)\tanh^6(\sqrt{c_2}x)a_0$$
$$-26481600t^2a_0c_2^5\sec h^4(\sqrt{c_2}x)\tanh^4(\sqrt{c_2}x) + 9720000t^2a_0c_2^5\sec h^4(\sqrt{c_2}x)\tanh^2(\sqrt{c_2}x)$$
$$-2268000t^2c_2^4\sec h^4(\sqrt{c_2}x)\tanh^4(\sqrt{c_2}x)a_0^2 + 1836000t^2c_2^4\sec h^4(\sqrt{c_2}x)\tanh^2(\sqrt{c_2}x)a_0^2$$
$$+8553600t^2c_2^5\sec h^6(\sqrt{c_2}x)\tanh^4(\sqrt{c_2}x)a_0 - 5875200t^2c_2^5\sec h^6(\sqrt{c_2}x)\tanh^2(\sqrt{c_2}x)a_0$$
$$-162000t^2a_0^3c_2^3\sec h^2(\sqrt{c_2}x)\tanh^2(\sqrt{c_2}x) - 12150t^2a_0^4c_2^2\sec h^2(\sqrt{c_2}x)\tanh^2(\sqrt{c_2}x)$$
$$-1164240t^2a_0^2c_2^4\sec h^2(\sqrt{c_2}x)\tanh^2(\sqrt{c_2}x) + 2646000t^2a_0^2c_2^4\sec h^2(\sqrt{c_2}x)\tanh^4(\sqrt{c_2}x)$$
$$+162000t^2c_2^3\sec h^4(\sqrt{c_2}x)a_0^3\tanh^2(\sqrt{c_2}x) - 680400t^2c_2^4\sec h^6(\sqrt{c_2}x)a_0^2\tanh^2(\sqrt{c_2}x)$$
$$+1166400t^2c_2^5\sec h^8(\sqrt{c_2}x)a_0\tanh^2(\sqrt{c_2}x) - 868800t^2c_2^6\sec h^4(\sqrt{c_2}x) + 64800t^2c_2^6\sec h^{10}(\sqrt{c_2}x)$$
$$+796320t^2c_2^6\sec h^6(\sqrt{c_2}x) - 345600t^2c_2^6\sec h^8(\sqrt{c_2}x),$$

Approximate solution of (10) can be obtained by setting

$$u = \lim_{p\to 1} U = U_0 + U_1 + U_2 + \ldots \quad (16)$$

In the same manner, the rest of components of the iteration were obtained using Maple Package. Using Taylor series into (16), we find the closed form solution

$$u = a_0 - 2c_2\sec h^2(\sqrt{c_2}(x - (45a_0^2 - 60a_0c_2 + 16c_2^2)t)),$$

Which is exactly same as obtained by Fan[6].

CONCLUSION

In this article, the homotopy perturbation method has been successfully used for finding the solution of the fifth-order KdV equations. The results show that the homotopy perturbation method is a powerful mathematical tool to solving a KdV equation, it is also a promising method to solve other nonlinear equations. In our work, we use the maple package to carry the computations.

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