Fractional Derivative of a Product of Some Special Functions

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Abstract: Our aim in the present paper is to study certain fractional calculus operator pertaining to many families of expansion involving two some product of special function with H-function of several complex variables. The main results of our paper are unified in nature and capable of yielding a very large number of corresponding results (new and known) involving simpler special functions and polynomials (of one or more variables) as special cases of our formulae. The present study unifies and extends a number of composition formulae lying scattered in the literature hitherto.

Key words: H- function • G- function • Gauss hypergeometric function

INTRODUCTION

The fractional derivatives of certain product of special functions, which are Gauss hypergeometric function [1], generalized Lauricella functions [2] a general class of multivariable polynomials [3] and the multivariable H-function is established. The generalized Leibnitz rule has been used to derive the main result pertaining to the product of these functions. The multivariable H-function which was introduced and studied systematically in a series of papers by [4, 5] is an extension of multivariable G-function. The multivariable H-function includes Fox’s H-function and Meijer’s G-function of one and two variables, the generalized Lauricella function of [2] Appell functions, the Whittaker functions etc.. Therefore the results derived in this paper are of general character from which several known and new results can be deduced.

Definition and Notations: Following [6, 7, 8] we define the fractional derivative of a function \( f(x) \) of complex order \( \mu \) (or alternatively, \( a - \mu^n \) order fractional integral of \( f(x) \)) by

\[
\frac{aD_x^\mu \{f(x)\}}{dx^m} = \frac{1}{\Gamma(-\mu)} \int_x^\infty (x - y)^{-\mu-1} f(y) \, dy, \quad \text{Re}(\mu) < 0,
\]

or

\[
\frac{d^m}{dx^m} \frac{aD_x^{\mu-m} \{f(x)\}}{dx^m}, \quad 0 \leq \text{Re}(\mu) < m,
\]

where \( m \) is a positive integer.

The well-known binomial expansion

\[
(1 - \xi)^{\lambda} = (-\xi)^{\lambda} \sum_{m=0}^{\infty} \binom{\lambda}{m} \left( -\frac{t}{\xi} \right)^m, \quad -\frac{t}{\xi} < 0.
\]

The special case of fractional derivative [6] is

\[
D_x^{\xi} \{f(x)\} = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \mu + 1)} t^{\lambda-\mu}, \quad \text{Re}(\lambda) > -1.
\]

The following known result of [7]

\[
\text{If } \mu \geq 0, 0 < t < 1, \quad \text{Re}(1+p) > 0, \quad \text{Re}(q) > -1, \quad \mu > 0 \text{ and } \Delta > 0, \text{ or } \Delta = 0 \text{ and } |z_i| < a_i, i = 1,2,\ldots,r \text{ then}
\]

\[
x^{\mu} \begin{bmatrix} \begin{array}{c} z_1 \cdots z_r \\ x \end{array} \end{bmatrix} = \sum_{\mu=0}^{\infty} \frac{(1+p+q+2\lambda)(-\mu)\lambda(1+p)\mu}{M!} \frac{\Gamma_M[a_i]}{M + \mu + q + a_i + 1} f_M[z_i, \ldots, z_r] \begin{bmatrix} \begin{array}{c} -M \ni \mu + q + M + 1 \\ \mu \end{array} \end{bmatrix},
\]

where

\[
\Gamma_M [z_1, \ldots, z_r]
\]

Where

\[
P_x^{\lambda; \mu; \nu; \alpha; \beta; \gamma; \delta; \epsilon; \zeta} \left( \begin{array}{c} \lambda, \mu, \nu, \alpha, \beta, \gamma, \delta, \epsilon, \zeta \\ \alpha, \beta, \gamma, \delta, \epsilon, \zeta \\ x \end{array} \right)
\]

Where \( M \geq 0 \).
For the sake of brevity, we use the following notation
\[
\begin{bmatrix}
\rho_{1} \\
\vdots \\
\rho_{p}
\end{bmatrix} = F
\begin{bmatrix}
\mu_{1} \\
\vdots \\
\mu_{r}
\end{bmatrix}
\]
(6)
The Gauss hypergeometric function [1] is defined as
\[
\mathbf{2F}_{1}(\rho; \sigma ; \mathbf{x}) = \sum_{\lambda=0}^{\infty} \frac{\prod_{i=0}^{p} (\rho_{i})_{\lambda}}{\prod_{i=0}^{q} (\sigma_{i})_{\lambda}} x^{\lambda},
\]
(7)
for \( \varepsilon \) neither zero nor a negative integer and Re \( (\varepsilon - \rho - \sigma) > 0 \).

The multivariable H-function is defined by [4, 5], see also [9] in the following manner
\[
H[z_{1}, \ldots, z_{r}]
= \mathcal{H}_{AC} \begin{bmatrix}
[\omega_{1}; \omega_{2}, \ldots, \omega_{p}] \\
[\eta_{1}; \eta_{2}, \ldots, \eta_{q}]
\end{bmatrix} \sum_{k_{1}, \ldots, k_{r}=0}^{\infty} \frac{B(N; k_{1}, \ldots, k_{r})}{k_{1}! \ldots k_{r}!} \zeta \left( \cdots \left( \sum_{k_{1}, \ldots, k_{r}=0}^{\infty} \frac{B(N; k_{1}, \ldots, k_{r})}{k_{1}! \ldots k_{r}!} \zeta \left( \cdots \left( \sum_{k_{1}, \ldots, k_{r}=0}^{\infty} \frac{B(N; k_{1}, \ldots, k_{r})}{k_{1}! \ldots k_{r}!} \zeta \left( \cdots \left( \sum_{k_{1}, \ldots, k_{r}=0}^{\infty} \frac{B(N; k_{1}, \ldots, k_{r})}{k_{1}! \ldots k_{r}!} \zeta \left( \cdots \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right).
\]
(8)
Where \( i = \sqrt{-1} \).

The convergence conditions and other details of the H-function of several complex variables \( H[z_{1}, \ldots, z_{r}] \) are given by [9]. For general class of multivariable polynomials [3]
\[
S_{M_{1}, \ldots, M_{r}}^{(M)}[w_{1}, \ldots, w_{r}]
= \sum_{k_{1}, \ldots, k_{r}=0}^{\infty} \frac{(-N)_{M_{1}, \ldots, M_{r}} B(N; k_{1}, \ldots, k_{r})}{k_{1}! \ldots k_{r}!} (a_{1})_{k_{1}} \cdots (a_{r})_{k_{r}}.
\]
(9)
Where \( M_{1}, \ldots, M_{r} \) are arbitrary positive integers and the coefficients \( B(N; k_{1}, \ldots, k_{r}) \). \( N \); \( k_{r} \geq 0, r = 1, \ldots, s \) are arbitrary constants, real or complex.

**Main Result:** The fractional derivative to be established here is
\[
D_{x}^{\alpha} \left( (1-x)^{\alpha} (y-1)^{\alpha} \right)^{\beta} = \mathbf{2F}_{1}(u; v, R (1-x)^{A}, (y-1)^{B}).
\]
(10)
Where
\[
\Delta = \left( -1 \right)^{p+K+B+K_{1}} \frac{(1 + p + q + M)(-\sigma)_{M}(1+p)}{M! (1+ p + q + M)_{\alpha + 1}}
\]
and
\[
\mathbf{F}_{M}[t_{1}, \ldots, t_{r}] = \frac{(-M)^{r} (1 + p + q + M)_{K}}{(1+p+q+M)_{K} \Gamma(\alpha + 1) \Gamma(\beta + 1)} \left( \sum_{k_{1}, \ldots, k_{r}=0}^{\infty} \frac{B(N; k_{1}, \ldots, k_{r})}{k_{1}! \ldots k_{r}!} \zeta \left( \cdots \left( \sum_{k_{1}, \ldots, k_{r}=0}^{\infty} \frac{B(N; k_{1}, \ldots, k_{r})}{k_{1}! \ldots k_{r}!} \zeta \left( \cdots \left( \sum_{k_{1}, \ldots, k_{r}=0}^{\infty} \frac{B(N; k_{1}, \ldots, k_{r})}{k_{1}! \ldots k_{r}!} \zeta \left( \cdots \right) \right) \right) \right) \right) \right) \right).
\]
(10)

**Proof:** To prove (10), replacing the Gauss hypergeometric function, Lauricella function, general class of polynomials and the multivariable H-function by (7), (5), (9) and (8) respectively and collecting the powers of \((1-x)\) and \((y-1)\). Finally, making use of the result (2) and (3), we arrive at (10).
Interesting Special Cases:

For $\lambda = A - C = 0$, the multivariable H-function split into product of Fox’s H-function and consequently there holds the following result

\[
\begin{align*}
D^{\lambda}_{\eta^r} \left( (t-x)^{\rho} \eta^r (y-t)^{\rho+p} \right) & = \sum_{K_1, K_2, \ldots, K_r = 0}^{\infty} \sum_{k_1, k_2, \ldots, k_r = 0}^{N} \left( -N_{M_{k_1} + \ldots + M_{k_r}} / K_1! \ldots K_r! \right) B(N, k_1, \ldots, k_r) \Delta \\
& = \sum_{K_1, K_2, \ldots, K_r = 0}^{\infty} \sum_{k_1, k_2, \ldots, k_r = 0}^{N} \left( -N_{M_{k_1} + \ldots + M_{k_r}} / K_1! \ldots K_r! \right) B(N, k_1, \ldots, k_r) \Delta
\end{align*}
\]

Which holds true under the same conditions as given in (10).

Taking $B(N; k_1, \ldots, k_r) = 1(\theta)$ in (9) where

\[
\begin{align*}
I(\theta) & = \prod_{j=1}^{E} \left( u_j \right)_{k_j, \alpha_j} \prod_{j=1}^{V^0} \left( v_j \right)_{k_j, \alpha_j} \prod_{j=1}^{U^0} \left( u_j \right)_{k_j, \alpha_j} \\
& = \prod_{j=1}^{E} \left( u_j \right)_{k_j, \alpha_j} \prod_{j=1}^{V^0} \left( v_j \right)_{k_j, \alpha_j}
\end{align*}
\]

valid under the conditions surrounding (10). When $\phi$ (i = 1, \ldots, r) in (10), we find the following

\[
\begin{align*}
& = \sum_{K_1, K_2, \ldots, K_r = 0}^{\infty} \sum_{k_1, k_2, \ldots, k_r = 0}^{N} \left( -N_{M_{k_1} + \ldots + M_{k_r}} / K_1! \ldots K_r! \right) B(N, k_1, \ldots, k_r) \Delta \\
& = \sum_{K_1, K_2, \ldots, K_r = 0}^{\infty} \sum_{k_1, k_2, \ldots, k_r = 0}^{N} \left( -N_{M_{k_1} + \ldots + M_{k_r}} / K_1! \ldots K_r! \right) B(N, k_1, \ldots, k_r) \Delta
\end{align*}
\]
The results established here are quite general in nature and a large number of (known or new) results can be obtained by specializing the parameters suitably of the various functions involved in them.

Fractional derivative formula is used to solve many differential equations in mathematics, physics and engineering science.

REFERENCES