Exact and Explicit Traveling Wave Solutions for the Nonlinear Partial Differential Equations

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Abstract: A generalized tanh method has been proposed and discussed. This method has been employed for solving two well-known models of nonlinear equations, namely the Drinfeld-Sokolov (DS) system and PHI-four equation. The exact solutions of these models are obtained. Finally, some wave solutions are exhibited.

Key words: Generalized tanh method • Nonlinear partial differential equations • Nonlinear phenomena • Drinfeld-Sokolov (DS) system • PHI-four equation

INTRODUCTION

Nonlinear partial differential equations (NLPDEs) are mathematical models that are used to describe complex phenomena arising in the world around us. The nonlinear equations appear in many applications of science and engineering such as fluid dynamics, plasma physics, hydrodynamics, solid state physics, optical fibers, acoustics and other disciplines. There are many methods that have been used to generate exact traveling wave solutions for nonlinear equations in the past decades such as the inverse scattering method, Hirota’s bilinear method, similarity transformation method, homogeneous balance method, the sine-cosine method, mapping method, F-expansion method, Riccati equation rational expansion method, Jacobi and Weierstrass elliptic function method, new generalized Jacobi elliptic function expansion method, tanh function method, etc [1-21].

Since 1990 or so, much attention has been devoted to versions of the so-called tanh method that can directly construct the travelling or solitary wave solutions for certain Nonlinear Evolution Equations (NLEEs). The tanh method has been developed for many years. This method is a very effective algorithm that helps to construct exact solutions for a large number of NLEEs [25-35]. Recently, a generalized tanh method is proposed which is aiming the soliton-like solutions [36, 37]. The key idea is to replace the tanh function by the solution of a Riccati equation in the tanh method. We shall describe this method in Section 2.

In this paper, we consider two well-known models of nonlinear equations, namely the Drinfeld-Sokolov (DS) system and PHI-four equation. We shall first deal with the Drinfeld-Sokolov system [6-8]:

\[ u_t + (v^2)_x = 0, \]

\[ v_t - av_{xxx} + 3buv + 3\gamma uv_x = 0, \]

where a, b and \( \gamma \) are constants. This system was introduced by Drinfeld and Sokolov as an example of a system of nonlinear equations possessing Lax pairs of a special form [8]. The solitary wave solutions of the Drinfeld-Sokolov system are obtained and exhibited in Section 3.

Now, we spend some time presenting the PHI-four equation from where it comes. It is known that the nonlinear Klein-Gordon equations (Klein-Fock-Gordon equations) appear in many types of nonlinearities, for example, the Klein-Gordon equation is a relativistic version of the Schrödinger equation. Moreover, any solution to the Dirac equation is automatically a solution to the Klein-Gordon equation [22-24]. These equations play a significant role in many scientific applications such as solid state physics, nonlinear optics and quantum field theory. A particular form of the Klein-Gordon equation is the PHI-four equation:

\[ u_{tt} - \alpha u_{xx} + \beta u + \gamma u^3 = 0, \]
where \( u = u(x,t) \). This equation plays an important role in particle physics where kink and anti-kink solitary waves interact \[24\]. The exact solutions of PHI-four equation are achieved in Section 4.

The Generalized Tanh Method: In this section, we briefly introduce the generalized tanh method and give the needed results for the following sections. We first consider the nonlinear evolution equations

\[
G(u, u_x, u_y, u_{xy}, \ldots) = 0. \tag{1}
\]

Now, we shall show that the travelling waves (or stationary waves) are solutions of Eq. (1). The first step is to unite the independent variables \( x \) and \( t \) into one particular variable through the following new variable

\[
\zeta = x - ct, \quad u(x, y) = U(\zeta),
\]

and simplify Eq. (1) to an ordinary differential equation (ODE)

\[
G(U, U', U'', U^{(3)}, \ldots) = 0. \tag{2}
\]

Secondly, we derive exact solutions (or at least approximate solutions) for the ODE (Eq. (2)). For this purpose, we introduce a new variable \( \Psi = \Psi(\zeta) \), which is a solution of the Riccati equation

\[
\Psi' = k + \Psi^2. \tag{3}
\]

Also, we introduce the following series expansion as a solution of Eq. (1):

\[
u(x, y) = U(\zeta) = \sum_{i=0}^{m} a_i \Psi^i. \tag{4}
\]

The parameter \( m \) is determined by balancing the linear term(s) of highest order with the nonlinear one(s). Moreover, the parameter \( m \) is normally a positive integer, so that an analytic solution in a closed form could be obtained.

Now, recalling Eq. (3), substituting Eq. (4) in Eq. (2) and comparing the coefficients of each power of \( \Psi \) in both sides lead to get an over-determined system of nonlinear algebraic equations with respect to \( k, a_0, a_1, \ldots, a_m \). This system can be solved by using Mathematica programme. The results of such system can be used to derive several types of solutions:

(i) for \( k < 0 \)

\[
\psi = -\sqrt{-k} \coth(\sqrt{-k} \zeta) = -\sqrt{-k} \tanh(\sqrt{-k} \zeta). \tag{5}
\]

(ii) for \( k = 0 \)

\[
\psi = -\frac{1}{\zeta}. \tag{6}
\]

(iii) for \( k > 0 \)

\[
\psi = -\sqrt{k} \cot(\sqrt{k} \zeta) = \sqrt{k} \tan(\sqrt{k} \zeta). \tag{7}
\]

Another advantage of the Riccati equation (Eq. (3)) is that the sign of \( k \) can be used to exactly judge the amount and types of traveling wave solution of Eq. (1).

The Drinfeld-Sokolov System: We first consider the Drinfeld-Sokolov system

\[
u_t + (\nu^2)_x = 0,
\]

\[
u_{_t} - a
u_{_{xxx}} + 3b
u
, \nu + 3 \gamma \nu u_x = 0. \tag{8}
\]

Considering the wave variable \( \zeta = x - ct \), the system (8) can be simplified to a system of ODEs as follows

\[
-c\nu' + (\nu^2)' = 0,
\]

\[
c\nu' + \nu'' - 3b
u' \nu - 3 \gamma \nu \nu_x = 0. \tag{9}
\]

Now, by integrating the first equation in the system (9) and neglecting the constant of integration we deduce \( cu = \nu' \). Thus, the system (9) can be reduced to an ordinary differential equation as follows

\[
c^2 \nu - (2b + \gamma) \nu^3 + ac\nu' = 0. \tag{10}
\]

By balancing the term \( \nu'' \) with the term \( \nu' \), we obtain that the parameter \( m \) is equal to 1. Thus, from (4) with \( m = 1 \), we have

\[
\nu(\zeta) = \sum_{i=0}^{1} a_i \Psi^i = a_0 + a_1 \Psi. \tag{11}
\]

Now, recalling Eq. (3), Substituting Eq. (11) in Eq. (10) and comparing the coefficients of each power of \( \Psi \) in both sides lead to get an over-determined system of nonlinear algebraic equations with respect to \( c,a \). Solving the over-determined system of nonlinear algebraic equations by use of Mathematica yields
Thus, from (5-7) and (12), we obtain

(i) for \( k > 0 \)

\[
v = \frac{c}{\sqrt{2b + \gamma}} \coth(\sqrt{\frac{c}{2a}}(x - ct)),
\]

\[
u = \frac{v^2}{c} = \frac{1}{2b + \gamma} \coth^2(\sqrt{\frac{c}{2a}}(x - ct)),
\]

or

\[
v = \frac{c}{2b + \gamma} \tanh(\sqrt{\frac{c}{2a}}(x - ct)),
\]

\[
u = \frac{v^2}{c} = rac{c}{(2b + \gamma)^2} \tanh^2(\sqrt{\frac{c}{2a}}(x - ct)).
\]

(ii) for \( k = 0 \)

\[
v = \frac{1}{(x - ct)\sqrt{2b + \gamma}}.
\]

\[
u = \frac{1}{(x - ct)^2(2b + \gamma)}.
\]

(iii) for \( k > 1 \)

\[
v = \frac{1}{x - ct} \cot(\sqrt{\frac{-c^2}{2b + \gamma}}(x - ct)),
\]

\[
u = \frac{v^2}{c} = \frac{-1}{(2b + \gamma)^2} \cot^2(\sqrt{\frac{-c}{2a}}(x - ct)),
\]

or

\[
v = \frac{1}{x - ct} \tan(\sqrt{\frac{-c^2}{2b + \gamma}}(x - ct)),
\]

\[
u = \frac{v^2}{c} = \frac{-1}{(2b + \gamma)^2} \tan^2(\sqrt{\frac{-c}{2a}}(x - ct)).
\]

The solitary wave solutions \( v_1 \) and \( u_1 \) of Eq. 13 are displayed in Figures 1 and 2 respectively. The solitary wave solutions \( v_2 \) and \( u_2 \) of Eq. 14 are displayed in figures 3 and 4 respectively.
PHI-Four Equation: We first consider PHI-four the equation

\[ u_{tt} - \alpha u_{xx} + \beta u + \gamma u^3 = 0, \quad (18) \]

where \( u = u(x, t) \). Considering the wave variable \( \zeta = x + ct \) and \( u(x, t) = U(\zeta) \), the Eq. (18) can be simplified to an ODE as follows

\[ (c^2 - \alpha)U'' + \beta U + \gamma U^3 = 0. \quad (19) \]

By balancing the term \( U'' \) with the term \( U^0 \), we obtain that the parameter \( m \) is equal to 1. Thus, from (4) with \( m = 1 \), we have

\[ U(\zeta) = \sum_{i=0}^{1} a_i \psi^i = a_0 + a_1 \psi. \quad (20) \]

Now, recalling Eq. (3), substituting Eq. (20) in Eq. (19), and comparing the coefficients of each power of \( \psi \) in both sides lead to get an over-determined system of nonlinear algebraic equations with respect to \( c, a, \). Solving the over-determined system of nonlinear algebraic equations by use of Mathematica yields

\[ a_0 = 0, \quad a_1 = \pm \frac{2(\alpha - c^2)}{\gamma} \quad \text{and} \quad k = \frac{\beta}{2(\alpha - c^2)}. \quad (21) \]

Thus, from (5-7) and (21), we obtain ten solutions:

1. \[ u = \mp \frac{2(\alpha - c^2)}{\gamma} \sqrt{\frac{\beta}{2(\alpha - c^2)}}, \quad \beta > 0. \quad (22) \]

2. \[ u = \mp \frac{2(\alpha - c^2)}{\gamma} \frac{1}{(x + ct)}, \quad \beta = 0. \quad (23) \]

3. \[ u = \pm \frac{2(\alpha - c^2)}{\gamma} \sqrt{\frac{\beta}{2(\alpha - c^2)}}, \quad \beta > 0. \quad (24) \]
REFERENCES