A New Method for Time-fractional Coupled-KDV Equations
with Modified Riemann-Liouville Derivative

1Mehmet Merdan, 2Syed Tauseef Mohyud-Din

1Department of Mathematics Engineering, Gümüşhane University, 29100-Gümüşhane, Turkey
2Department of Mathematics, HITEC University, Taxila Cantt, Pakistan

Abstract: In this article, we performed a new application of fractional variational iteration method (FVIM) for solving nonlinear fractional coupled-KDV equations with modified Riemann-Liouville derivative. The effects of fractional derivatives for the time-fractional coupled-KDV equations under take into account conditions are examined. It is indicated that the solutions obtained by the FVIM are reliable and effective method for strongly nonlinear partial equations with modified Riemann-Liouville derivative. In this method, to obtain analytic and approximate solutions of different types of fractional differential equations can be used as an alternative.

Key words: Fractional variational iteration method, the time-fractional coupled-KDV equations, Riemann-Liouville derivative

INTRODUCTION

In recent years, considerable interest in fractional calculus used in many fields such as electrical networks, control theory of dynamical systems, probability and statistics, electrochemistry of corrosion, chemical physics, optics, engineering, acoustics, material science and signal processing can be successfully modelled by linear or nonlinear fractional order differential equations [1-8]. In plasma physics these equations give rise to the ion acoustic solutions [9-11]. However, the physical situations in which the KdV equations arise tend to be highly idealized due to the assumption of constant coefficients. In addition, the He’s variational iteration method is applied to solve the non-linear coupled-KdV equations [12].

The variational iteration method (VIM), which proposed by Ji-Huan He [13-25], was successfully applied to autonomous ordinary and partial differential equations and other fields. Ji-Huan He [19] was the first to apply the variational iteration method to fractional differential equations. In recent years, a new modified Riemann-Liouville left derivative is suggested by G. Jumarie [26-30].

In this paper, we extend the application of the VIM in order to derive analytical approximate solutions to nonlinear time fractional coupled-KDV equations

\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + 6\alpha u_x - 2b v v_x + \alpha u_{xxx} = 0
\]

(1)

with the initial conditions

\[
u(x,0) = g(x), \quad v(x,0) = h(x)
\]

(2)

This coupled system is used to describe iterations of water waves proposed by Hirota and Satsuma [31]

The goal of this paper is to extend the application of the variational iteration method to solve nonlinear time fractional coupled-KDV equations with modified Riemann-Liouville derivative.

This paper is organized as follows:

In section 2, we are given definitions related to the fractional calculus theory briefly. In section 3, we define the solution procedure of the fractional variational iteration method to show in efficiency of this method. We present the application of the FVIM for the nonlinear time fractional coupled-KDV with modified Riemann-Liouville derivative and numerical results in Section 4. The conclusions are then given in the final section 5.

BASIC DEFINITIONS

Here, some basic definitions and properties of the fractional calculus theory which can be found in [26-30].
Definition 1: Assume \( f: \mathbb{R} \rightarrow \mathbb{R}, x \rightarrow f(x) \) denote a continuous (but not necessarily differentiable) function and let the partition \( h>0 \) in the interval \([0, 1] \). Jumarie’s derivative is defined through the fractional difference [29]:

\[
\Delta^{(a)} = (\text{FW} - 1)^{a} f(x) = \sum_{k=0}^{\infty} (-1)^{k} \frac{\Gamma(a-k)}{a-k} f(x + (\alpha - k)h) \tag{3}
\]

where \( \text{FW} f(x) = f(x + h) \)

Fractional derivative is defined as the following limit form [1-6, 35-38]

\[
f^{(\alpha)} = \lim_{h \to 0} \frac{\Delta^{(a)} f(x) - f(0)}{h^\alpha} \tag{4}
\]

This definition is close to the standard definition of derivatives (calculus for beginners) and as a direct result, the \( \alpha \)th derivative of a constant, \( 0 < \alpha < 1 \), is zero.

Definition 2: The left-sided Riemann-Liouville fractional integral operator of order \( \alpha \geq 0 \), of a function \( f \in C_{\mu}, \mu \geq -1 \) is defined as

\[
J_{\alpha}^{\mu} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (x - \tau)^{\alpha-1} f(\tau) \, d\tau \tag{5}
\]

for \( \alpha > 0, x > 0 \) and \( J_{\alpha}^{\mu} f(x) = f(x) \)

The properties of the operator \( J_{\alpha}^{\mu} \) can be found in [1-6, 35-38].

Definition 3: The modified Riemann-Liouville derivative [28-29] is defined as

\[
_{0}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_{0}^{x} (x - \tau)^{n-\alpha-1} f(\tau) \, d\tau \tag{6}
\]

where \( x \in [0,1], n - 1 \leq \alpha < n \) and \( n \geq 1 \).

In addition, we want to give as the following some properties of the fractional modified Riemann-Liouville derivative.

Fractional Leibniz product law:

\[
_{0}D_{x}^{\alpha} (uv) = u^{[\alpha]}v + uv^{[\alpha]} \tag{7}
\]

Fractional Leibniz Formulation:

\[
_{0}I_{x}^{\alpha} _{0}D_{x}^{\alpha} (uv) = f(x) - f(0), 0 < \alpha \leq 1 \tag{8}
\]

Fractional the integration of part:
According to the VIM, we can build a correct functional for Eqs. (16), (17) as follows

\[
u_n(t) = u_n(t) + \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t-\tau)^{\alpha-1} \lambda_n(\tau) \left( \frac{\partial^{\alpha} u_n(t, \tau)}{\partial \tau^\alpha} + \left[ 6\alpha u_n(t, \tau) \frac{\partial u_n(t, \tau)}{\partial x} - 2b v_n(t, \tau) \frac{\partial v_n(t, \tau)}{\partial x} + \alpha \frac{\partial^{\alpha} u_n(t, \tau)}{\partial x} \right] \right) d\tau
\]  

(18)

\[
u_n(t) = u_n(t) + \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t-\tau)^{\alpha-1} \lambda_n(\tau) \left( \frac{\partial^{\alpha} v_n(t, \tau)}{\partial \tau^\alpha} + \left[ 3\beta u_n(t, \tau) \frac{\partial v_n(t, \tau)}{\partial x} + \beta \frac{\partial^{\alpha} v_n(t, \tau)}{\partial x} \right] \right) d\tau
\]  

(19)

Using Eq. (5), we obtain a new correction functional

\[
u_n(t) = u_n(t) + \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t-\tau)^{\alpha-1} \lambda_n(\tau) \left( \frac{\partial^{\alpha} u_n(t, \tau)}{\partial \tau^\alpha} + \left[ 6\alpha u_n(t, \tau) \frac{\partial u_n(t, \tau)}{\partial x} - 2b v_n(t, \tau) \frac{\partial v_n(t, \tau)}{\partial x} + \alpha \frac{\partial^{\alpha} u_n(t, \tau)}{\partial x} \right] \right) d\tau
\]  

(20)

\[
u_n(t) = u_n(t) + \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t-\tau)^{\alpha-1} \lambda_n(\tau) \left( \frac{\partial^{\alpha} v_n(t, \tau)}{\partial \tau^\alpha} + \left[ 3\beta u_n(t, \tau) \frac{\partial v_n(t, \tau)}{\partial x} + \beta \frac{\partial^{\alpha} v_n(t, \tau)}{\partial x} \right] \right) d\tau
\]  

(21)

It is obvious that the sequential approximations \( u_k, k \geq 0 \) can be established by determining \( \lambda \), a general Lagrange’s multiplier, which can be identified optimally with the variational theory. The function \( u_0 \) is a restricted variation which means \( \delta u_0 = 0 \). Therefore, we first designate the Lagrange multiplier \( \lambda \) that will be identified optimally via integration by parts. The successive approximations \( u_{n+1}(t), n \geq 0 \) of the solution \( u(t) \) will be easily obtained upon using the obtained Lagrange multiplier and by using any selective function \( u_0 \). The initial values are usually used for selecting the zeroth approximation \( u_0 \). With \( \lambda \) determined, then several approximations \( u_k, k \geq 0 \) follows immediately [35-37, 38]. As a result, the exact solution may be procured by using

\[
u(t) = \lim_{n \to \infty} u_n(t)
\]  

(22)

Approximate solutions of the fractional coupled-KDV equations

In this section, we present the solution of the fractional coupled-KDV equations as the applicability of FVIM. According to the FVIM, we construct a correction functional for the Eqs. (1)-(2). We have

\[
u_n(t) = u_n(t) + \frac{1}{\Gamma(q_1 + 1)} \int_0^t (t-\tau)^{q_1} \lambda_n(\tau) \left( \frac{\partial^{q_1} u_n(t, \tau)}{\partial \tau^{q_1}} + \left[ 6\alpha u_n(t, \tau) \frac{\partial u_n(t, \tau)}{\partial x} - 2b v_n(t, \tau) \frac{\partial v_n(t, \tau)}{\partial x} + \alpha \frac{\partial^{\alpha} u_n(t, \tau)}{\partial x} \right] \right) d\tau
\]  

(23)

\[
u_n(t) = u_n(t) + \frac{1}{\Gamma(q_1 + 1)} \int_0^t (t-\tau)^{q_1} \lambda_n(\tau) \left( \frac{\partial^{q_1} v_n(t, \tau)}{\partial \tau^{q_1}} + \left[ 3\beta u_n(t, \tau) \frac{\partial v_n(t, \tau)}{\partial x} + \beta \frac{\partial^{\alpha} v_n(t, \tau)}{\partial x} \right] \right) d\tau
\]  

(24)

we have

\[ \delta u_{n+1}(x,t) = \delta u_n(x,t) + \frac{1}{\Gamma(q_1+1)} \delta \frac{\partial^{n+1} u_n(x,t)}{\partial x^{n+1}} \left( \frac{\partial^q u_n(x,t)}{\partial x^q} + 6\alpha u_n(x,t) \frac{\partial u_n(x,t)}{\partial x} - 2b v_n(x,t) \frac{\partial v_n(x,t)}{\partial x} \right) \left( \frac{\partial^q v_n(x,t)}{\partial x^q} + \alpha \frac{\partial^2 v_n(x,t)}{\partial x^2} \right) \right] \] \tag{25}

\[ \delta v_{n+1}(x,t) = \delta v_n(x,t) + \frac{1}{\Gamma(q_2+1)} \delta \frac{\partial^{n+1} v_n(x,t)}{\partial x^{n+1}} \left( \frac{\partial^{n+1} v_n(x,t)}{\partial x^{n+1}} + 3\beta u_n(x,t) \frac{\partial v_n(x,t)}{\partial x} + \beta \frac{\partial^2 v_n(x,t)}{\partial x^2} \right) \right] \] \tag{26}

Similarly, we can get the coefficients of \( \delta u_n \) to zero:

\[ 1 + \lambda_1(x,t) \bigg|_{t=0} = 0.1 + \lambda_2(x,t) \bigg|_{t=0} = 0, \quad \frac{\partial^{n+1}\lambda_1(x,t)}{\partial x^{n+1}} = 0, \quad \frac{\partial^{n+1}\lambda_2(x,t)}{\partial x^{n+1}} = 0 \] \tag{27}

The generalized Lagrange multiplier can be identified by the above equations,

\[ \lambda_1(x,t) = -1, \quad \lambda_2(x,t) = -1 \] \tag{28}

substituting Eq. (28) into Eqs. (23)-(24) produces the iteration formulation as follows

\[ u_{n+1}(x,t) = u_n(x,t) - \frac{1}{\Gamma(q_1+1)} \int_0^t \left\{ \frac{\partial^{n+1} u_n(x,t)}{\partial x^{n+1}} + 6\alpha u_n(x,t) \frac{\partial u_n(x,t)}{\partial x} - 2b v_n(x,t) \frac{\partial v_n(x,t)}{\partial x} \right\} \left( \frac{\partial^q v_n(x,t)}{\partial x^q} + \alpha \frac{\partial^2 v_n(x,t)}{\partial x^2} \right) \right] \] \tag{29}

\[ v_{n+1}(x,t) = v_n(x,t) - \frac{1}{\Gamma(q_2+1)} \int_0^t \left\{ \frac{\partial^{n+1} v_n(x,t)}{\partial x^{n+1}} + 3\beta u_n(x,t) \frac{\partial v_n(x,t)}{\partial x} + \beta \frac{\partial^2 v_n(x,t)}{\partial x^2} \right\} \right] \] \tag{30}

Taking the initial value

\[ u_n(x,0) = u_n(0,0) = \frac{\lambda}{\alpha} \tanh \left[ \frac{\lambda}{2 \sqrt{\alpha}} x \right], \quad v_n(x,0) = v_n(0,0) = \frac{\lambda}{\sqrt{2}\alpha} \tanh \left[ \frac{\lambda}{2 \sqrt{\alpha}} x \right] \]

we can derive

\[ u_n(x,t) = u_n(x,t) - \frac{1}{\Gamma(q_1+1)} \int_0^t \left\{ \frac{\partial^{n+1} u_n(x,t)}{\partial x^{n+1}} + 6\alpha u_n(x,t) \frac{\partial u_n(x,t)}{\partial x} - 2b v_n(x,t) \frac{\partial v_n(x,t)}{\partial x} \right\} \right] \] \tag{31}
\[ v_1(x,t) = v(x,t) - \frac{1}{\Gamma(q_1+1)} \left\{ \frac{\partial^q v_0(x,t)}{\partial t^q} + 3b u_0(x,t) \frac{\partial v_0(x,t)}{\partial x} \right\} (dt)^q \]

\[ = \frac{\lambda}{\sqrt{2} \alpha} \text{sech}^2 \left[ \frac{1}{2} \sqrt{\frac{\lambda}{\alpha}} x \right] + \frac{\lambda^2 \alpha}{2 \alpha} \tanh \left[ \frac{1}{2} \sqrt{\frac{\lambda}{\alpha}} x \right] \text{sech}^2 \left[ \frac{1}{2} \sqrt{\frac{\lambda}{\alpha}} x \right] \frac{1}{\Gamma(q_1+1)} \] (32)

and so on, in the same manner the rest of the components of the iteration formulae (29)-(31) can be calculated by Maple.

Then, the approximate solutions in a series form are

\[ u(x,t) = \lim_{n \to \infty} u_n(x,t) = \frac{\lambda}{\alpha} \text{sech}^2 \left[ \frac{1}{2} \sqrt{\frac{\lambda}{\alpha}} x \right] + \frac{\lambda^2 \alpha}{\sqrt{2} \alpha} \tanh \left[ \frac{1}{2} \sqrt{\frac{\lambda}{\alpha}} x \right] \text{sech}^2 \left[ \frac{1}{2} \sqrt{\frac{\lambda}{\alpha}} x \right] \frac{1}{\Gamma(q_1+1)} + \ldots \] (33)

\[ v(x,t) = \lim_{n \to \infty} v_n(x,t) = \frac{\lambda}{\alpha} \text{sech}^2 \left[ \frac{1}{2} \sqrt{\frac{\lambda}{\alpha}} x \right] + \frac{\lambda^2 \alpha}{\sqrt{2} \alpha} \tanh \left[ \frac{1}{2} \sqrt{\frac{\lambda}{\alpha}} x \right] \text{sech}^2 \left[ \frac{1}{2} \sqrt{\frac{\lambda}{\alpha}} x \right] \frac{1}{\Gamma(q_1+1)} + \ldots \] (34)

For the special case \( q_1 = q_2 = 1, b = 3, \alpha = \beta \) is

\[ u(x,t) = \frac{\lambda}{\alpha} \text{sech}^2 \left[ \frac{1}{2} \sqrt{\frac{\lambda}{\alpha}} (x-\lambda t) \right] \] (35)

\[ v(x,t) = \frac{\lambda}{\sqrt{2} \alpha} \text{sech}^2 \left[ \frac{1}{2} \sqrt{\frac{\lambda}{\alpha}} (x-\lambda t) \right] \] (36)

which is an exact solution to the coupled-KDV (1) and (2) equation.

Fig. 1: Continued
Fig. 1: The surface indicates the solution $u(x,t)$ for Eq. (1) when $\alpha = 1$, $\lambda = 1$, $b = 3$, $\beta = 1$. (a) Exact solution (b) $u_1(x,t)$-approximate solution (c) $u_2(x,t)$-approximate solution and (d) $u_3(x,t)$-approximate solution for $q_1 = q_2 = 1$.

Fig. 2: The surface indicates the solution $v(x,t)$ for Eq. (2) when $\alpha = 1$, $\lambda = 1$, $b = 3$, $\beta = 1$. (a) Exact solution (b) $v_1(x,t)$-approximate solution (c) $v_2(x,t)$-approximate solution and (d) $v_3(x,t)$-approximate solution.
Fig. 3: The surface indicates the solution \( u(x,t) \) for Eq. (1) when \( \alpha = 1, \lambda = 1, b = 3, \beta = 1 \). (a) approximate solution \( u_2(x,t) \) for \( q_1 = 0.9, q_2 = 0.8 \), (b) approximate solution \( u_2(x,t) \) for \( q_1 = 0.8, q_2 = 0.9 \) (c) approximate solution \( u_2(x,t) \) for \( q_1 = 0.7, q_2 = 0.5 \) and (d) approximate solution \( u_2(x,t) \) for \( q_1 = 0.5, q_2 = 0.5 \)

Fig. 4: Continued
Fig. 4: The surface indicates the solution $v(x,t)$ for Eq. (2) when $\alpha = 1$, $\lambda = 1$, $b = 3$, $\beta = 1$. (a) approximate solution $v_2(x,t)$ for $q_1 = 0.9$, $q_2 = 0.8$ (b) approximate solution $v_2(x,t)$ for $q_1 = 0.8$, $q_2 = 0.9$ (c) approximate solution $v_2(x,t)$ for $q_1 = 0.7$, $q_2 = 0.5$ and (d) approximate solution $v_2(x,t)$ for $q_1 = 0.5$, $q_2 = 0.5$

Fig. 5: Approx. solution $u_2(x,t)$ for $x = 2$

Fig. 6: Approx. solution $v_1(x,t)$ for $x = 2$

Fig. 7: Approx. solution $u_2(x,t)$ for $t = 0.3$

Fig. 8: Approx. solution $v_2(x,t)$ for $t = 0.3$
Figures 1 and 2 are plotted for approximate solution of fractional coupled-KDV equations.

Figures 3 and 4 are given approximate solution of Eqs. (1)-(2) for different value \( q_1, q_2 \).

Figures 3 and 4 indicate that a decrease in the fractional order \( \alpha \) by selecting the fixed \( x = 5 \) corresponds to an increase in the function. It is seen from Fig. 58 that five sequential values of \( q_1 = q_2 = 0.6, 0.7, 0.8, 0.9, 1 \). Figure 5 and 6 indicate that a decrease in the fractional order \( q_1, q_2 \) by selecting the fixed \( x = 2 \) corresponds to a decrease in the function.

Figure 7 shows that a decrease in the fractional order \( q_1, q_2 \) by selecting the fixed \( t = 0.3 \) corresponds to an increase in the function. Figure 8 indicates that a decrease in the fractional order \( q_1, q_2 \) by selecting the fixed \( t = 0.3 \) corresponds to a decrease in the function.

CONCLUSIONS

In this paper, we obtain the approximate analytical solutions of the time-fractional coupled-KDV equations using He's variational iteration method. Variational iteration method known as very powerful and an effective method for solving nonlinear problems and ordinary, partial, fractional, integral equations. In this paper, we have discussed modified variational iteration method having integral w.r.t. \((d\tau)^\beta\) used for the first time by Jumarie. The obtained results show that this method is powerful and meaningful for solving the nonlinear fractional differential equations.

REFERENCES


