Middle-East Journal of Scientific Research 8 (2): 541-543, 2011
ISSN 1990-9233


# Differential Transform Method for Solving One Dimensional Non-homogeneous Parabolic Problems

*H. Zareamoghaddam*

Kashmar branch, Islamic Azad University, Kashmar, Iran

**Abstract:** The aim of this paper is to apply Differential Transform Method (DTM) to solve one dimensional non-homogeneous parabolic equation with a variable coefficient. Examples are presented to show the ability of the method for solving these problems. The results reveal that the method is very effective and simple.

**Key words:** Differential transform method . non-homogeneous parabolic equation

## INTRODUCTION

The non-homogeneous parabolic model is the integral part of applied sciences and arises in various physical phenomena. This paper investigates for the first time the applicability and effectiveness of differential transform method on non-homogeneous parabolic equations. The basic idea of Differential Transform Method (DTM) was initially introduced by Zhou [1] in 1986. Its main application therein was to solve both linear and nonlinear initial value problems arising in electrical circuit analysis. DTM is a technique that uses Taylor series for the solution of differential equations in the form of a polynomial. It is different from the high-order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally expensive especially for high order equation. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. In recent years researchers have applied the method to various linear and nonlinear problems for example it was applied to partial differential equations [2], to integro-differential equations [3], to two point boundary value problems [4], to differential-algebraic equations [5], to the KdV and mKdV equations [6], to the Schrödinger equations [7] and to fractional differential equations [8]. In recent years, special equations of composite type have received attention in many papers. In this paper, we consider to one-dimensional non-homogeneous parabolic partial differential equations with a variable coefficient of the form [10]

$$\frac{\partial u}{\partial t} = \mu(x)\frac{\partial^2 u}{\partial x^2} + \phi(x,t),\ 0 \prec x \prec 1,\ t \succ 0$$

with initial condition

$$u(x,0) = f(x)$$

The layout of the article is as follows: In section 2, the basic idea of differential transform method is introduced. The results of numerical experiments are presented in section 3. And section 4 is dedicated to a brief conclusion.

## BASIC IDEA OF DIFFERENTIAL TRANSFORM METHOD

The basic definitions and fundamental operations of the two-dimensional differential transform are defined in [2-9] as follows. The differential transform function of a function say u(x,y) is in the following form

$$U(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,y)}{\partial x^k \partial y^h}\right]_{(x_0,\, y_0)} \tag{1}$$

where u(x,y) is the original function and U(k,h) is the transformed function.

The differential inverse transform of U(k,h) is defined as:

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)(x-x_0)^k(y-y_0)^h \tag{2}$$

in a real application and when $(x_0, y_0)$ are taken as (0,0), then the function u(x,y) is expressed by a finite series and Eq. (2) can be written as:

**Corresponding Author:** H. Zareamoghaddam, Kashmar Branch, Islamic Azad University, Kashmar, Iran

Table 1: The fundamental mathematical operations performed by two-dimensional differential transform method

| Original function | Transformed function |
|---|---|
| $u(x,y)=g(x,y)\pm h(x,y)$ | $U(k,h)=G(k,h)\pm H(k,h)$ |
| $u(x,y)=\lambda g(x,y)$ | $U(k,h)=\lambda G(k,h)$ |
| $u(x,y)=\partial g(x,y)/\partial x$ | $U(k,h)=(k+1)G(k+1,h)$ |
| $u(x,y)=\partial g(x,y)/\partial y$ | $U(k,h)=(h+1)G(k,h+1)$ |
| $u(x,y)=x^m y^n$ | $U(k,h)=\delta(k-m,h-n)=\begin{cases}1, k=m,h=n\\ \text{ootherwise}\end{cases}$ |
| $u(x,y)=\partial^{r+s}g(x,y)/\partial x^r\partial y^s$ | $U(k,h)=(k+1)\ldots(k+r)(h+1)\ldots(h+s)G(k+r,h+s)$ |
| $u(x,y)=g(x,y)h(x,y)$ | $U(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}G(r,h-s)H(k-r,s)$ |

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,y)}{\partial x^k\partial y^h}\right]x^k y^h \tag{3}$$

Eq. (3) implies that the concept of two-dimensional differential transform is derived from two-dimensional Taylor series expansion. In this study we use the lower case letters to represent the original functions and upper case letters to stand for the transformed functions (T-functions).

The fundamental mathematical operations performed by two-dimensional differential transform method can readily be obtained and are listed in Table 1.

## EXAMPLES

To illustrate capability, reliability and simplicity of the method, three examples for different cases of the equation will be discussed here.

**Example 1:** Consider the following equation

$$u_t=u_{xx}+e^{-x}(\cos t-\sin t) \tag{7}$$

subject to the following initial condition

$$u(x,0)=x \tag{8}$$

With the exact solution

$$u(x,t)=x+e^{-x}\sin t$$

Taking the differential transform of (7), then

$$(h+1)U(k,h+1)=(k+1)(k+2)U(h+2,k)+\frac{(-1)^k}{k!}\frac{1}{h!}\cos(\frac{\pi h}{2})-\frac{(-1)^k}{k!}\frac{1}{h!}\sin(\frac{\pi h}{2}) \tag{9}$$

From the initial condition given by Eq. (8), we have

$$U(k,0)=\begin{cases}1 & k=1\\ 0 & \text{otherwise}\end{cases} \tag{10}$$

Substituting Eqs. (10) into Eq. (9) and by recursive method, the results are listed as follows

$$U(k,h)=\begin{cases}1 & \text{for } k=1 \text{ and } h=0\\ \frac{(-1)^k}{k!}\frac{1}{h!}\sin(\frac{\pi h}{2}) & \text{otherwise}\end{cases}$$

We obtained the series solution as

$$u(x,t)=\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}U(k,h)x^k t^h=x+e^{-x}\sin t$$

which is the exact solution of the problem given in Eqs. (7)-(8).

**Example 2:** Consider the following equation

$$u_t=u_{xx}+e^{-x}(\cosh t-\sinh t) \tag{11}$$

subject to the initial condition

$$u(x,0)=\frac{x^3}{6} \tag{12}$$

With the exact solution

$$u(x,t)=xt+\frac{x^3}{6}+e^x\sinh t$$

Taking the differential transform of (11), then

$$(h+1)U(k,h+1)=(k+1)(k+2)U(h+2,k)+\frac{(-1)^h}{h!}\frac{1}{k!} \tag{13}$$

From the initial condition given by Eq. (12), we obtain

$$U(k,0)=\begin{cases}\frac{1}{6} & k=3\\ 0 & \text{otherwise}\end{cases} \tag{14}$$

Substituting (14) in (13), all spectra can be found as

$$U(k,h)=\begin{cases}\frac{1}{6} & \text{for } k=3 \text{ and } h=0\\ 2 & \text{for } k=1 \text{ and } h=1\\ \frac{1}{2}\frac{1}{k!}(\frac{1}{h!}-\frac{(-1)^h}{h!}) & \text{otherwise}\end{cases}$$

By Substituting U(k,h) into Eq. (3), we obtain

$$u(x,t)=\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}U(k,h)x^k t^h = xt+\frac{x^3}{6}+e^x\sinh t$$

which is the expansion of the function

$$xt+\frac{x^3}{6}+e^x\sinh t$$

and is the exact solution of the problem given in Eqs. (11)-(12).

**Example 3:** Consider the following equation

$$u_t=-xu_{xx}+e^{-x}(1+x(x+t)-2x) \qquad (15)$$

with initial condition,

$$u(x,0)=xe^{-x}+\frac{x^2}{2} \qquad (16)$$

Taking the differential transform of (15), then

$$\begin{aligned}&(h+1)U(k,h+1)\\&=-\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)(k-r+2)\delta(r-1)\delta(h-s)U(k-r+2,s)\\&-\frac{(-1)^k}{k!}\delta(h)+\sum_{r=0}^{k}\sum_{s=0}^{h}\frac{(-1)^r}{r!}\delta(h-s)\delta(k-r-2)\delta(s)\\&+\sum_{r=0}^{k}\sum_{s=0}^{h}\frac{(-1)^r}{r!}\delta(h-s)\delta(k-r-1)\delta(s-1)\\&-2\sum_{r=0}^{k}\sum_{s=0}^{h}\frac{(-1)^r}{r!}\delta(h-s)\delta(k-r-1)\delta(s)\end{aligned} \qquad (17)$$

Substituting U(k,h) into Eq. (3), we have series solution for u and follow closed form solution

$$u(x,t)=\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}U(k,h)x^k t^h=(x+t)e^{-x}-xt+\frac{1}{2}x^2$$

which is the exact solution.

## CONCLUSION

In this work, we have achieved an exact solution of one dimensional non-homogeneous parabolic equation by applying differential transform method. Numerical results reveal that the DTM is a powerful tool for solving linear and nonlinear initial and boundary value problems. The fact that suggested technique solves problems without using Adomian's polynomials is a clear advantage of this algorithm over the decomposition method. The computations associated with examples are performed using Maple 13.

## REFERENCES


1. Zhou, J.K., 1986. Differential transformation and its application for electrical circuits. Huarjung University Press, Wuuhahn, China.
2. Jang, M.J., C.L. Chen and Y.C. Liy, 2001. Two-dimensional differential transform for Partial differential equations. Appl. Math. Comput., 121: 261-270.
3. Arikoglu, A. and I. Ozkol, 2005. Solution of boundary value problems for integro-differential equations by using differential transform method. Appl. Math. Comput., 168: 1145-1158.
4. Chen, C.L. and Y.C. Liu, 1998. Solution of two point boundary value problems using the differential transformation method. J. Opt. Theory Appl., 99: 23-35.
5. Fatma Ayaz, 2004. Applications of differential transform method to differential-algebraic equations. Applied Mathematics and Computation, 152: 649-657.
6. Figen Kangalgil and Fatma Ayaz, 2009. Solitary wave solutions for the KdV and mKdV equations by differential transform method Chaos. Solitons and Fractals, 41 (1): 464-472.
7. Ravi Kanth, S.V. and K. Aruna, 2009. Two-dimensional differential transform method for solving linear and non-linear Schrödinger equations Chaos. Solitons and Fractals, 41 (5): 2277-2281.
8. Arikoglu, A. and I. Ozkol, 2007. Solution of fractional differential equations by using differential transform method. Chaos Soliton. Fract., 34: 1473-1481.
9. Chen, C.K. and S.H. Ho, 1999. Solving partial differential equations by two-dimensional differential transform method. Appl. Math. Comput., 106: 171-179.
10. Bongsoo Jang, 2007. Exact solutions to one dimensional non-homogeneous parabolic problems by the homogeneous Adomian decomposition method. Applied Mathematics and Computation, 186: 969-979.