Involute Curve of a Biharmonic Curve in the Heisenberg Group Heis³

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Abstract: In this paper, we study involute curve of biharmonic curve in the Heisenberg group Heis³. We characterize involute curve of biharmonic curve in terms of curvature and torsion of biharmonic curve in the Heisenberg group Heis³ Finally, we construct parametric equations of involute curve of biharmonic curve.

Key words: Heisenberg group · Biharmonic curve · Involute curve
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INTRODUCTION

The idea of a string involute is due to C. Huygens (1658), who is also known for his work in optics. He discovered involutes while trying to build a more accurate clock [1]. The involute of a given curve is a well-known concept in Euclidean-3 space E^3.

An evolute and its involute, are defined in mutual pairs. The evolute of any curve is defined as the locus of the centers of curvature of the curve. The original curve is then defined as the involute of the evolute. The simplest case is that of a circle, which has only one center of curvature (its center), which is a degenerate evolute. The circle itself is the involute of this point.

In recent years, the theory of degenerate submanifolds has been treated by researchers and some classical differential geometry topics have been extended to Lorentz manifolds. For instance, in [2], the authors extended and studied spacelike involute-evolute curves in Minkowski space-time.

A smooth map $\phi : N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |T(\phi)|^2 \, dv_h,$$

Where $T(\phi) := \mathrm{tr} \nabla^\phi d\phi$ is the tension field of ϕ.

The Euler-Lagrange equation of the bienergy is given by $T_2(\phi) = 0$. Here the section $T_2(\phi)$ is defined by

$$T_2(\phi) = -\Delta_\phi T(\phi) + \mathrm{tr} R(T(\phi), d\phi) d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps. Therefore many authors [3-16] studied biharmonic curves and its applications.

In this paper, we study involute curve of biharmonic curve in the Heisenberg group Heis³. We characterize involute curve of biharmonic curve in terms of curvature and torsion of biharmonic curve in the Heisenberg group Heis³ Finally, we construct parametric equations of involute curve of biharmonic curve.

Heisenberg Group Heis³: Heisenberg group Heis³ can be seen as the space R^3 endowed with the following multipilcation:

$$(\bar{x}, \bar{y}, \bar{z})(x, y, z) = (\bar{x} + x, \bar{y} + y, \bar{z} + z - \frac{1}{2}\bar{x}y + \frac{1}{2}x\bar{y}) \tag{2.1}$$

Heis³ is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g = dx^2 + dy^2 + (dz + \frac{y}{2}dx - \frac{x}{2}dy)^2.$$

The Lie algebra of Heis³ has an orthonormal basis

$$e_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \, e_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \, e_3 = \frac{\partial}{\partial z}, \tag{2.2}$$

for which we have the Lie products

$$[e_1, e_2] = e_3, [e_2, e_3] = [e_3, e_1] = 0$$

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With

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1.$$

We obtain

$$\nabla_{e_1}e_1 = \nabla_{e_2}e_2 = \nabla_{e_3}e_3 = 0,$$

$$\nabla_{e_1}e_2 = -\nabla_{e_2}e_1 = \frac{1}{2}e_3,$$

$$\nabla_{e_1}e_3 = \nabla_{e_3}e_1 = -\frac{1}{2}e_2,$$

$$\nabla_{e_2}e_3 = \nabla_{e_3}e_2 = \frac{1}{2}e_1.$$

We adopt the following notation and sign convention for Riemannian curvature operator on Heis³ defined by

$$R(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z,$$

While the Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = g(R(X,Y)Z,W),$$

Where X,Y,Z,W are smooth vector fields on Heis³

The components $\{R_{ijkl}\}$ of R relative to $\{e_1,e_2,e_3\}$ are defined by

$$g(R(e_i,e_j)e_k,e_l = R_{ijkl}$$

The non-vanishing components of the above tensor fields are

$$R_{121} = -\frac{3}{4}e_2, \quad R_{131} = \frac{1}{4}e_3, \quad R_{122} = \frac{3}{4}e_1,$$

$$R_{232} = \frac{1}{4}e_3, \quad R_{133} = -\frac{1}{4}e_1, \quad R_{233} = -\frac{1}{4}e_2,$$

and

$$R_{1212} = -\frac{3}{4}, \quad R_{1313} = R_{2323} = \frac{1}{4}. \tag{2.3}$$

Biharmonic Curves in the Heisenberg Group Heis³: Let $I \subset R$ be an open interval and $\gamma : I \to Heis^3$ be a curve parametrized by arc length on Heisenberg group Heis³. Putting $\mathbf{t} = \gamma'$, we can write the tension field of γ as $\tau(\gamma) = \nabla_{\gamma'}\gamma'$ and the biharmonic map equation (1.1) reduces to

$$\nabla_t^3\mathbf{t} + R(\mathbf{t},\nabla_t\mathbf{t})\mathbf{t} = 0. \tag{3.1}$$

A successful key to study the geometry of a curve is to use the Frenet frames along the curve, which is recalled in the following.

Let $\gamma : I \to Heis^3$ be a curve on Heis³ parametrized by arc length. Let $\{\mathbf{t},\mathbf{n}_1,\mathbf{n}_2\}$ be the Frenet frame fields tangent to Heis³ along γ defined as follows: $\mathbf{t}$ is the unit vector field γ' tangent to γ, $\mathbf{n}_1$, is the unit vector field in the direction of $\nabla_t\mathbf{t}$ (normal to γ) and $\mathbf{b}$ is chosen so that $\{\mathbf{t},\mathbf{n}_1,\mathbf{n}_2\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\nabla_t\mathbf{t} = \kappa\mathbf{t},$$

$$\nabla_t\mathbf{n}_1 = -\kappa\mathbf{t} - \tau\mathbf{n}_2 \tag{3.2}$$

$$\nabla_t\mathbf{n}_2 = \tau\mathbf{n}_1,$$

Where $\kappa = |\nabla_T\mathbf{T}|$ is the curvature of γ and τ is its torsion. With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\mathbf{t} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3,$$

$$\mathbf{n}_1 = n_1^1\mathbf{e}_1 + n_1^2\mathbf{e}_2 + n_1^3\mathbf{e}_3,$$

$$\mathbf{n}_2 = \mathbf{t}\times\mathbf{n}_1 = n_2^1\mathbf{e}_1 + n_2^2\mathbf{e}_2 + n_2^3\mathbf{e}_3.$$

Theorem 3.1: (see [9]) Let $\gamma : I \to Heis^3$ be a non-geodesic curve on Heis³ parametrized by arc length. Then γ is a non-geodesic biharmonic curve if and only if

$$\kappa = \text{constant} \neq 0,$$

$$\kappa^2 + \tau^2 = \frac{1}{4} - (n_2^3)^2, \tag{3.3}$$

$$\tau' = n_1^3 n_2^3.$$

Theorem 3.2: (see [9]) Let $\gamma : I \to Heis^3$ be a non-geodesic curve on the Heisenberg group Heis³ parametrized by arc length. If κ is constant and $n_1^3 n_2^3 \neq 0$, then γ is not biharmonic.

Involute Curve of Biharmonic Curve in Heisenberg Group Heis³: Definition 4.1. Let unit speed curve $\gamma : I \to Heis^3$ and the curve $\beta : I \to Heis^3$ be given. For $\forall s \in I$, then the curve β is called the involute of the curve γ, if the tangent at the point $\gamma(s)$ to the curve γ passes through the tangent at the point $\beta(s)$ to the curve β and

$$g(\mathbf{t}^*(s), \mathbf{t}(s)) = 0. \tag{4.1}$$

Let the Frenet-Serret frames of the curves γ and β be $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2\}$ and $\left\{\mathbf{t}^*, \mathbf{n}_1^*, \mathbf{n}_2^*\right\}$, respectively.

Theorem 4.2: Let $\gamma : I \to Heis^3$ be a unit speed biharmonic curve and β its involute curve on $Heis^3$. Then, the parametric equations of β are

$$x_\beta(s) = (\rho - s)\sin\varphi\cos[\Lambda s + \rho] + \frac{1}{\Lambda}\sin\varphi\sin[\Lambda s + \rho] + a_1,$$

$$y_\beta(s) = (\rho - s)\sin\varphi\sin[\Lambda s + \rho] - \frac{1}{\Lambda}\sin\varphi\cos[\Lambda s + \rho] + a_2,$$

$$\tag{4.2}$$

$$z_\beta(s) = (\cos\varphi + \frac{1}{2\Lambda}\sin^2\varphi)\rho + a_3,$$

Where $\rho, \alpha_1, \alpha_2, \alpha_3$ are constants of integration and

$$\Lambda = \frac{\cos\varphi \pm \sqrt{5(\cos\varphi)^2 - 4}}{2}.$$

Proof: The curve $\beta(s)$ may be given as

$$\beta(s) = \gamma(s) + u(s)\mathbf{t}(s). \tag{4.3}$$

If we take the derivative (4.3), then we have

$$\beta'(s) = \left(1 + u'(s)\right)\mathbf{T}(s) + u(s)\kappa(s)\mathbf{n}_1(s). \tag{4.4}$$

Since the curve β is involute of the curve γ, $g(\mathbf{t}^*(s), \mathbf{t}(s)) = 0$. Then, we get

$$1 + u'(s) = 0 \text{ or } u(s) = \rho - s, \tag{4.5}$$

Where ρ is constant of integration.

Substituting (4.5) into (4.3), we get

$$\beta(s) = \gamma(s) + (\rho - s)\mathbf{T}(s). \tag{4.6}$$

The covariant derivative of the vector field $\mathbf{t}$ is:

$$\nabla_\mathbf{t}\mathbf{t} = (t_1' + t_2 t_3)\mathbf{e}_1 + (t_2' - t_1 t_3)\mathbf{e}_2 + t_3'\mathbf{e}_3. \tag{4.7}$$
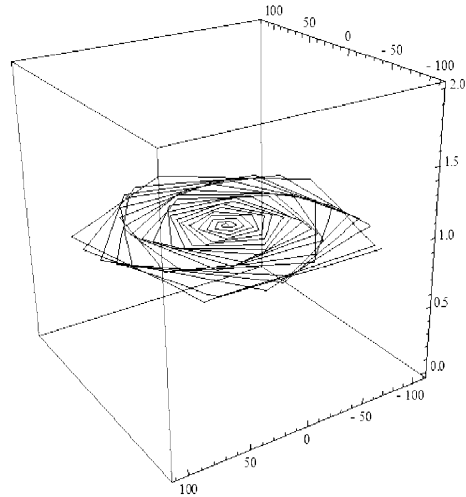

Fig. 1:

Thus using Theorem 3.2, we have

$$\mathbf{t} = \sin\varphi\cos[\Lambda s + \rho]\mathbf{e}_1 + \sin\varphi\sin[\Lambda s + \rho]\mathbf{e}_2 + \cos\varphi\mathbf{e}_3, \tag{4.8}$$

Where $\Lambda = \dfrac{\cos\varphi \pm \sqrt{5(\cos\varphi)^2 - 4}}{2}.$

Using (2.2) in (4.8), we obtain

$$\mathbf{t} = (\sin\varphi\cos[\Lambda s + \rho], \sin\varphi\sin[\Lambda s + \rho],$$

$$\cos\varphi - \frac{1}{2}y(s)\sin\varphi\cos[\Lambda s + \rho] + \frac{1}{2}x(s)\sin\varphi\sin[\Lambda s + \rho]).$$

From (2.2), we get

$$\mathbf{t} = (\sin\varphi\cos[\Lambda s + \rho], \sin\varphi\sin[\Lambda s + \rho], \cos\varphi + \frac{1}{2\Lambda}\sin^2\varphi).$$

$$\tag{4.9}$$

We substitute (4.9) into (4.6), we get (4.2). The proof is completed.

Using Mathematica in Theorem 4.2, yields

Corollary 4.3: Let $\gamma : I \to Heis^3$ be a unit speed biharmonic curve and β its involute curve on $Heis^3$. Then,

$$\kappa = \sqrt{\frac{1}{4} - \left(n_2^3\right)^2}\cos Y,$$

$$\tau = \sqrt{\frac{1}{4} - \left(n_2^3\right)^2}\sin Y.$$

Here Y is arbitrary angle.

Proof: Using second equation of (3.3), we have above system.

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