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On Bivariate Concomitants of Order Statistics for Pseudo Exponential Distribution

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Abstract: The distribution of vector of two concomitants and joint distribution of two vectors of concomitants of order statistics for Pseudo Exponential Distribution has been obtained. Expression for *p*-th moment of vector of concomitants and product moment for pair of vectors of concomitants has been obtained.

Key words: Order Statistics · Bivariate Concomitants · Pseudo Exponential Distribution

INTRODUCTION

Suppose a random sample of size n is available from an absolutely continuous distribution function $F(x)$ and suppose that these observations are arranged in increasing order of magnitude as $x_{1:n} \le x_{2:n} \le .. \le x_{n:n}$. The *r-th* observation in this arranged sequence is called the *r-th* order statistics and is denoted by $x_{r:n}$. The density function of *r-th* order statistics is given by David and Nagaraja [1] as:

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r}; \quad (1.1)$$

The joint density function of two order statistics $x_{r:n} \le x_{s:n}$ is obtained as:

$$f_{r,s:n}(x_1,x_2) = C_{r,s,n} f(x_1) f(x_2)\left[F(x_1)\right]^{r-1} \left[F(x_2)-F(x_1)\right]^{s-r-1}\left[1-F(x_2)\right]^{n-s}; \quad (1.2)$$

Where:

$$C_{r,s,n} = \frac{n!}{(r-1)!(s-r)!(n-s)!}. \quad (1.3)$$

When sample from a bivariate distribution function $F(x,y)$ is available and the sample observations are arranged with respect to random variable X then automatically shuffled observations of random variable Y are called the concomitants of order statistics. The density function of concomitant associated with *r-th* order statistics is given by David and Nagaraja [2] as:

$$g_{[r:n]}(y) = \int_{-\infty}^{\infty} f(y|x) f_{r:n}(x)\,dx; \quad (1.4)$$

The joint distribution of two concomitants is given as:

$$g_{[r,s:n]}(y_1,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} f(y_1|x_1) f(y_2|x_2) f_{r,s:n}(x_1,x_2)\,dx_1 dx_2; \quad (1.5)$$

Where:

$f_{r:n}$ is defined in (1.1) and $f_{r,s:n}(x_1,x_2)$ is given in (1.2).

Concomitants have been extensively studied in literature by various people. The distribution of rank of concomitant of order statistics has also been discussed by David *et el.* [3] have obtained the distribution of rank of concomitant. The asymptotic behavior of the distribution of concomitant of order statistics has been studied by Chu *et el.* [1]. Distribution of concomitant in case of multivariate distributions has been studied by Wang *et el.* [8]. Shahbaz *et al.* [6] have studied the distribution of concomitants for bivariate pseudo Exponential distribution with some of its properties. In the following we extend the definition of concomitants given in (1.4) and (1.5).

Bivariate Concomitants of Order Statistics: Suppose a random sample of n observations is available from a trivariate distribution function $F(x,y_1,y_2)$ and the sample is arranged in increasing order with respect to variable X. We define the joint distribution of two k-th concomitants of order statistics as:

$$f_{[r:n]}(\mathbf{y}_r) = \int_{-\infty}^{\infty} f(y_1,y_2|x) f_{r:n}(x)\,dx \quad (2.1)$$

Where:

$f_{r:n}$ is defined in (1.1), $\mathbf{y}=[y_1\ y_2]$ and $y_1 = y_{1/k};y_2 = y_{2k}$. The joint distribution of two bivariate concomitants is defined as:

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$$f_{[r:n]}(\mathbf{y}_r, \mathbf{y}_s) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} f(\mathbf{y}_1, \mathbf{y}_2|x) f_{r,s:n}(x_1, x_2)\, dx_1 dx_2 \qquad (2.2)$$

Where:

$\mathbf{y}_1 = [y_{1r}\, y_{2r}]'$, $\mathbf{y}_2 = [y_{1s}\, y_{2s}]'$ and $f_{r,s:n}(x_1,x_2)$ is defined in (1.2). We will find the distributions (2.1) and (2.2) in the following section.

Bivariate Distribution of *r-th* pair of Concomitants: The density function of trivariate Pseudo-Exponential distribution is defined by Shahbaz *et al.* [7] as:

$$f(x, y_1, y_2) = f(x) f(y_1|x) f(y_2|x, y_1)$$
$$= \alpha e^{-\alpha x}\phi_1(x) exp[-\phi_1(x) y_1]\phi_2(x, y_1) exp[-\phi_2(x, y_1) y_2]$$

Where:

$\phi_1(x)$ is some function of variable X and $\phi_2(x,y_2)$ is some function of random variables X and Y. Using $\phi_1(x) = x$ and $\phi_2(x, y_1) = xy_1$, the above density can be written as:

$$f(x, y_1, y_2) = \alpha x^2 y_1 exp[-x(\alpha + y_1 + y_2)] \qquad (3.1)$$

The distribution of bivariate concomitants for (3.1) can be obtained by using (2.1). From (3.1) we can readily see that:

$$f(x) = \alpha e^{-\alpha x}; \alpha, x > 0 \qquad (3.2)$$

and

$$f(y_1, y_2|x) = x^2 y_1 exp[-xy_1(1 + y_2)]; y_1, y_2 > 0 \qquad (3.3)$$

The density of $f_{r:n}(x)$ for (3.1) is given as:

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}\alpha e^{-\alpha x}\left[1 - e^{-\alpha x}\right]^{r-1}\left[e^{-\alpha x}\right]^{n-r} \qquad (3.4)$$
$$= \frac{\alpha n!}{(r-1)!(n-r)!}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h} e^{-\alpha x(n-r+h+1)}.$$

Using (3.3) and (3.4) in (2.1), the distribution of bivariate concomitants of order statistics for Pseudo-Exponential distribution is given as:

$$f_{[r:n]}(y_1, y_2) = \frac{\alpha n!}{(r-1)!(n-r)!}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h}\int_0^{\infty} x^2 y_1 e^{-x\{y_1(1+y_2)+\alpha(n-r+h+1)\}}dx$$
$$= \frac{\alpha n!}{(r-1)!(n-r)!}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h} y_1\int_0^{\infty} x^2 e^{-x\{y_1(1+y_2)+\alpha(n-r+h+1)\}}dx$$

Making the transformation $x\{y_1(1+y_2)+\alpha(n-r+h+1)\} = w$, we have:

$$f_{[r:n]}(y_1, y_2) = \frac{\alpha n!}{(r-1)!(n-r)!}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h} y_1$$
$$\int_0^{\infty}\left\{\frac{w}{y_1(1+y_2)+\alpha(n-r+h+1)}\right\}^2 e^{-w}\frac{dw}{y_1(1+y_2)+\alpha(n-r+h+1)}$$
$$= \frac{\alpha n!}{(r-1)!(n-r)!}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h}\frac{y_1}{\left[y_1(1+y_2)+\alpha(n-r+h+1)\right]^3}\int_0^{\infty} e^{-w}w^2 dw$$

or

$$f_{[r:n]}(y_1, y_2) = \frac{\alpha n!}{(r-1)!(n-r)!}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h}\frac{2y_1}{\left[y_1(1+y_2)+\alpha(n-r+h+1)\right]^3}; \qquad (3.5)$$

The p-th moment of distribution (3.5) is given as:

$$E(\mathbf{y}^p) = \int_0^{\infty}\int_0^{\infty} y_1^p y_2^p f_{[r:n]}(y_1, y_2)\, dy_1 dy_2$$
$$= \frac{\alpha n!}{(r-1)!(n-r)!}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h}\int_0^{\infty}\int_0^{\infty}\frac{2y_1^{p+1}y_2^p}{\left[y_1(1+y_2)+\alpha(n-r+h+1)\right]^3}dy_1 dy_2;$$

Now $\int_0^{\infty}\frac{2y_1^{p+1}y_2^p}{\left[y_1(1+y_2)+\alpha(n-r+h+1)\right]^3}dy_1 = \frac{(p+1)\Gamma(p+1)\Gamma(1-p)}{\{\alpha(n-r+h+1)\}^{1-p}}(y_2+1)^{-(p+2)}$

So

$$\mu_k' = \frac{\alpha n!}{(r-1)!(n-r)!}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h}\frac{(p+1)\Gamma(p+1)\Gamma(1-p)}{\{\alpha(n-r+h+1)\}^{1-p}}\int_0^{\infty}(y_2+1)^{-(p+2)}dy_2$$

Solving integral by using Gradshteyn and Ryzhik [5], p-th moment of (3.5) is given as:

$$\mu_k' = \frac{\alpha n!\,\Gamma(p+1)\Gamma(1-p)}{(r-1)!(n-r)!}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h}\{\alpha(n-r+h+1)\}^{p-1}. \qquad (3.6)$$

We now find the distribution of two vectors of concomitants for trivariate Pseudo-Exponential distribution by using (2.2). The distribution (2.2) requires the joint distribution of two order statistics and this distribution for random variable X is given as:

$$f_{r,s:n}(x_1, x_2) = C_{r,s:n}\sum_{j=0}^{r-1}\sum_{h=0}^{s-r-1}(-1)^{j+h}\binom{r-1}{j}\binom{s-r-1}{h}e^{-\alpha(s-r-h+j)x_1}e^{-\alpha(n-s+h+1)x_2} \qquad (3.7)$$

Using (3.3) and (3.7) in (2.2), the distribution of two pairs of concomitants is given as:

$$f_{[r,s:n]}(\mathbf{y}_1, \mathbf{y}_2) = C_{r,s:n}\sum_{j=0}^{r-1}\sum_{h=0}^{s-r-1}(-1)^{h+j}\binom{r-1}{j}\binom{s-r-1}{h}\int_0^{\infty}\int_0^{\infty} x_1^2 y_{1r} exp[-x_1 y_{1r}(1+y_{2r})]$$
$$\times x_2^2 y_{1s} exp[-x_2 y_{1s}(1+y_{2s})]e^{-\alpha(s-r-h+j)x_1}e^{-\alpha(n-s+h+1)x_2}dx_1 dx_2$$

After simplification, the joint distribution is given as:

$$f_{[r,s:n]}(\mathbf{y}_1, \mathbf{y}_2) = C_{r,s:n}\sum_{j=0}^{r-1}\sum_{h=0}^{s-r-1}(-1)^{h+j}\binom{r-1}{j}\binom{s-r-1}{h}\times$$
$$\left[\frac{24\{y_{1r}(1+y_{2r})+\alpha(s-r-h+j)\}^2}{\{y_{1r}+y_{1s}+y_{1r}y_{2r}+y_{1s}y_{2s}+\alpha(n-r+j+1)\}^5}+\right.$$
$$\frac{12\{y_{1r}(1+y_{2r})+\alpha(s-r-h+j)\}}{\{y_{1r}+y_{1s}+y_{1r}y_{2r}+y_{1s}y_{2s}+\alpha(n-r+j+1)\}^4}+$$
$$\left.\frac{4}{\{y_{1r}+y_{1s}+y_{1r}y_{2r}+y_{1s}y_{2s}+\alpha(n-r+j+1)\}^3}-\frac{4}{\{y_{1s}(1+y_{2s})+\alpha(n-r+j+1)\}}\right]$$

(3.8)

The product moments can be found from (3.8).

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