

Differential Transform Method For Solving Initial Value Problems Represented By Strongly Nonlinear Ordinary Differential Equations

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Abstract: In this paper, a general framework of the differential transform method (DTM) is presented for solving strongly nonlinear initial value problems represented by ordinary differential equations. This technique doesn't require any discretization, linearization or small perturbation and therefore it reduces significantly the numerical computation. So as to show the capability, three test examples are solved as numerical examples. We show that the DTM is very simple and very effective.

Key words: Differential Transform Method • Initial Value Problem • Ordinary Differential Equation

INTRODUCTION

The Differential Transform was first introduced by Zhou [1-4] and it is applied to solve differential equation. The DTM is the method to determine the coefficients of the Taylor series of the function by solving the induced recursive equation from the given differential equation. In this paper, the Differential Transform Method has been utilized for solving initial value problems represented by strongly ordinary differential equations.

The Comparisons of the results of DTM on three applications reveal that DTM is very effective and convenient.

Differential Transform Method (DTM): The differential transform of the k th derivative of function k th is defined as follows [3-8].

$$U(k) = \frac{1}{k!} \left[\frac{d^k u}{dx^k}(x) \right]_{x=x_0} \quad (1)$$

where $u(x)$ is the original function and $U(k)$ is the transformed function.

And the differential inverse transform of $U(k)$ is defined as;

$$u(x) = \sum_{k=0}^{\infty} U(k)(x - x_0)^k \quad (2)$$

The differential transform verified The following properties [2-6];

- If $u(x) = u_1(x) \pm u_2$, then $U(k) = U_1(k) \pm U_2$
- If $u(x) = cu_1(x)$, then $U(k) = cU_1(k)$, where c is a constant.
- If $u(x) = \frac{d^n u_1(x)}{dx^n}$, then $U(k) = \frac{(k+n)!}{k!} U_1(k+n)$
- If $u(x) = u_1(x) u_2(x)$, then $U(k) = \sum_{r=0}^k U_1(r) U_2(k-r)$
- If $u(x) = u_1(x) u_2(x) \dots u_n(x)$, then

$$U(k) = \sum_{r_{n-1}=0}^k \sum_{r_{n-2}=0}^{r_{n-1}} \dots \sum_{r_2=0}^{r_3} \sum_{r_1=0}^{r_2} U_1(r_1) U_2(r_2 - r_1) \dots U_{n-1}(r_{n-1} - r_{n-2}) U_n(k - r_{n-1})$$

- If $u(x) = x^m$, then $U(k) = \delta(k-m) = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}$

- If $u(x) = u_1(x) \frac{du_2(x)}{dx}$, then

$$U(k) = \sum_{r=0}^k (k-r+1) U_1(r) U_2(k-r+1)$$

- If $u(x) = \cos(ax + b)$, then $U(k) = \frac{a^k}{k!} \cos\left(\frac{\pi k}{2} + b\right)$

Numerical Examples: In this section, some examples show the usage of DTM for solving the strongly nonlinear ordinary differential equations [9-13].

Example 1: First let us consider the following nonlinear ordinary differential equation.

$$u''(x) = \cos x e^{u(x)} \quad (3)$$

With initial conditions;

$$u(0) = 1, u'(0) = 0$$

Writing $e^{u(x)}$ by Maclaurin series, then equation (3) can be written as follows;

$$u''(x) = \cos x \left(1 + u' + \frac{(u')^2}{2} + \frac{(u')^3}{6} + \dots \right) \quad (4)$$

Applying the DTM by using properties 1, 2, 3, 5, 7 and 8 choosing $x_0 = 0$, equation (4) is transformed in the following form:

$$\begin{aligned} (k+2)(k+1)U(k+2) &= \frac{1}{k!} \cos\left(\frac{\pi k}{2}\right) U(k-m+1) \\ &+ \sum_{m=0}^k \frac{1}{m!} (k-m+1) \cos\left(\frac{\pi m}{2}\right) U(k-m+1) \\ &+ \frac{1}{2} \sum_{m=0}^k \sum_{r=0}^m \frac{1}{r!} (k-m+1)(m-r+1) \cos\left(\frac{\pi r}{2}\right) U(k-m+1)U(m-r+1) \\ &+ \frac{1}{6} \sum_{m=0}^k \sum_{r=0}^m \sum_{h=0}^r \left[\frac{1}{h!} (k-m+1)(m-r+1)(r-h+1) \cos\left(\frac{\pi h}{2}\right) \right] \\ &U(k-m+1)U(m-r+1)U(r-h+1) \end{aligned}$$

$$u(0)=1 \rightarrow U(0)=1$$

$$u'(0)=0 \rightarrow U(1)=0$$

$$k=0 \Rightarrow U(2) = \frac{1}{2} \left(1 + U(1) + \frac{1}{2} U^2(1) + \frac{1}{6} U^3(1) \right) = \frac{1}{2}$$

$$k=1 \Rightarrow U(3) = \frac{1}{6} \left(2U(2) + 2U(1)U(2) + U^2(1)U(2) \right) = \frac{1}{6}$$

$$k=2 \Rightarrow U(4) = \frac{1}{12} \left(\frac{1}{2} + 3U(3) - \frac{U(1)}{2} + 3U(1)U(3) + 2U^2(2) - \frac{1}{4} U^2(1) + \frac{3}{2} U^2(1)U(3) + 2U(1)U^2(2) - \frac{1}{2} U^3(1) \right) = \frac{1}{24}$$

It leads to the solution of equation (3)

$$u(x) = \sum_{k=0}^{\infty} U(k) x^k = 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$$

Table 1: Results numerical for example 1

x	Error of DTM n=4	Error of DTM n=5
0	0.0000000000	0.0000000000
0.1	5.1616830 e -03	3.4589200 e -04
0.2	2.1238157 e -02	2.8675600 e -03
0.3	4.8925493 e -02	1.0009335 e -02
0.4	8.8503095 e -02	2.4465443 e -02
0.5	1.3954007 e -01	4.9055445 e -02

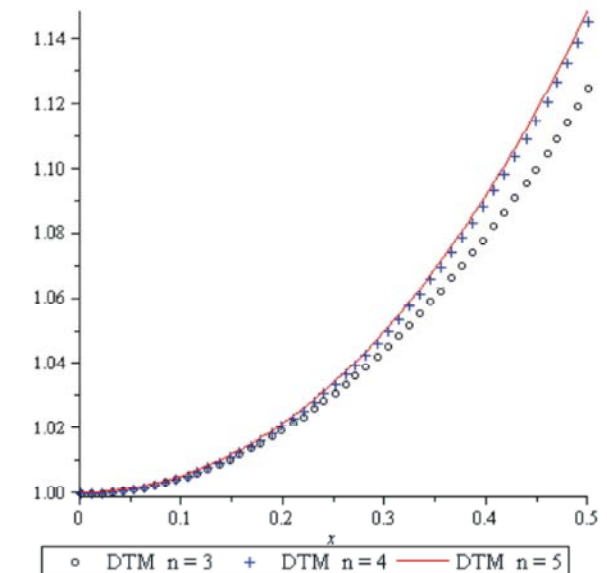


Fig. 1: The approximate solution by using DTM for example 1

Example 2: Consider the following nonlinear ordinary differential equation.

$$u^{(3)}(x) = \cos(u'' + xu) \quad (5)$$

With initial conditions,

$$u(0) = 0, u'(0) = 1, u''(0) = 0$$

Writing $\cos(u'' + xu)$ by Maclaurin series, then equation (5) can be written as follows;

$$u'''(x) = 1 - \frac{1}{2} (u'' + xu)^2 + \frac{1}{24} (u'' + xu)^4 + \dots \quad (6)$$

Applying the DTM by using properties 1, 2, 3, 5, 6 and 7 choosing $x_0 = 0$, equation (6) is transformed in the following form:

$$(k+3)(k+2)(k+1)U(k+3)=\delta(k)$$

$$-\frac{1}{2}\sum_{m=0}^k(m+2)(m+1)(k-m+2)(k-m+1)U(m+2)U(k-m+2)$$

$$-\sum_{m=0}^k\sum_{r=0}^m\delta(r-1)(k-m+2)(k-m+1)U(k-m+2)U(m-r)$$

$$-\frac{1}{2}\sum_{m=0}^k\sum_{r=0}^m\delta(r-2)U(k-m)U(m-r)$$

$$u(0)=0\rightarrow U(0)=0$$

$$u'(0)=1\rightarrow U(1)=1$$

$$u''(0)=0\rightarrow U(2)=0$$

$$k=0\Rightarrow U(3)=\frac{1}{6}$$

$$k=1\Rightarrow U(4)=0$$

$$k=2\Rightarrow U(5)=-\frac{1}{120}$$

$$k=3\Rightarrow U(6)=\frac{1}{120}$$

It leads to the solution of equation (5);

$$u(x)=\sum_{k=0}^{\infty}U(k)x^k=x+\frac{x^3}{6}-\frac{x^5}{120}+\frac{x^6}{120}+\dots$$

Table 2: Results numerical for example 2

x	Error of DTM n=6	Error of DTM n=7
0	0.000000000000	0.000000000000
0.1	9.6919330 e -04	3.3049400 e -05
0.2	7.4808446 e -03	5.8231310 e -04
0.3	2.4232431 e -02	3.1797533 e -03
0.4	5.4771613 e -02	1.0673718 e -02
0.5	1.0114121 e -01	2.7355443 e -02

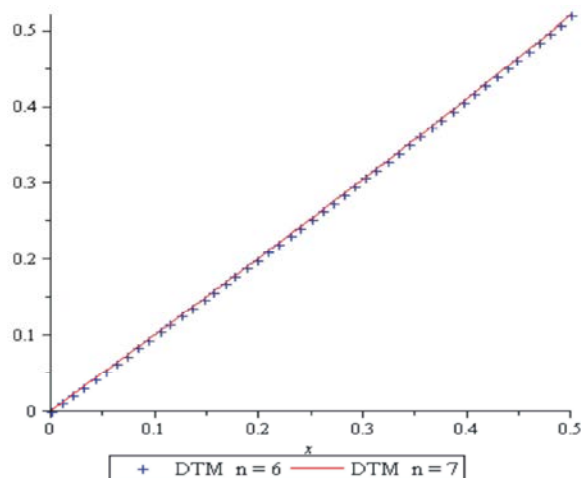


Fig. 2: The approximate solution by using DTM for example 2

Example 3: Consider the following nonlinear ordinary differential equation.

$$u^{(4)}(x)=x(\sinh(uu'))+1 \quad (7)$$

With initial conditions,

$$u(0)=1, \quad u'(0)=0, \quad u''(0)=1, \quad u^{(3)}(0)=1$$

Writing $\sinh(uu')$ by Maclaurin series, then equation (7) can be written as follows,

$$u^{(4)}(x)=1+xuu'+\frac{1}{6}x^3u'^3+\dots \quad (8)$$

Applying the DTM by using properties 1, 2, 3, 5, 6 and 7 choosing $x_0=0$, equation (8) is transformed in the following form:

$$(k+4)(k+3)(k+2)(k+1)U(k+4)=\delta(k)$$

$$+\sum_{m=0}^k\sum_{r=0}^m\delta(r-1)(k-m+1)U(k-m+1)U(m-r)$$

$$+\frac{1}{6}\sum_{m=0}^k\sum_{r=0}^m\sum_{h=0}^r\sum_{g=0}^h\sum_{p=0}^g\sum_{q=0}^p\left[\frac{\delta(q-1)(r-h+1)(m-r+1)(k-m+1)}{U(p-q)U(g-p)U(h-g)U(r-h+1)}\right]$$

$$\left[\frac{U(m-r+1)U(k-m+1)}{U(m-r+1)U(k-m+1)}\right]$$

$$u(0)=1\rightarrow U(0)=1$$

$$u'(0)=0\rightarrow U(1)=0$$

$$u''(0)=1\rightarrow U(2)=\frac{1}{2}$$

$$u^{(3)}(0)=1\rightarrow U(3)=\frac{1}{6}$$

$$k=0\Rightarrow U(4)=\frac{1}{24}$$

$$k=1\Rightarrow U(5)=0$$

$$k=2\Rightarrow U(6)=\frac{1}{360}$$

It leads to the solution of equation (7),

$$u(x)=\sum_{k=0}^{\infty}U(k)x^k=1+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^6}{360}+\dots$$

Table 3: Results numerical for example 3

x	Error of DTM n=5	Error of DTM n=7
0	0.000000000000	0.000000000000
0.1	1.0590745 e -02	5.9076255 e -04
0.2	4.5600088 e -02	5.6012137 e -03
0.3	1.1256203 e -01	2.2575883 e -02
0.4	2.2466640 e -01	6.4754448 e -02
0.5	4.0579903 e -01	1.5620234 e -01

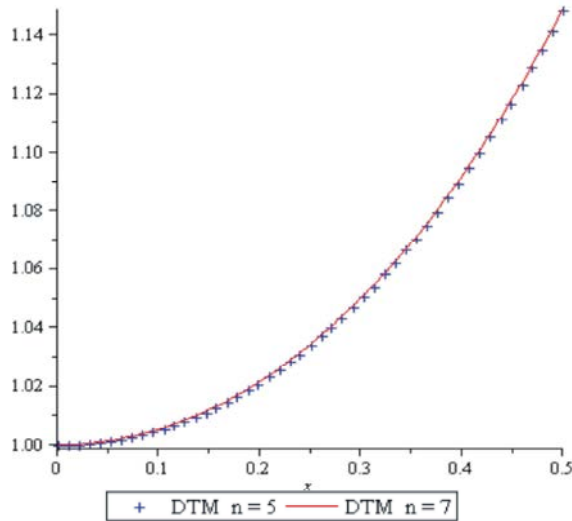


Fig. 3: The approximate solution by using DTM for example 3

CONCLUSION

In this paper, we successfully apply the DTM to find approximate solutions for strongly nonlinear ordinary differential equations. It is observed that DTM is an effective and reliable tool for the solution of nonlinear differential equations. The method gives rapidly converging series solutions. The accuracy of the obtained solution can be improved by taking more terms in the solution. Several examples were tested by applying the DTM and the results have shown remarkable performance.

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