Fixed Point Type Theorem in S-Metric Spaces

Javad Mojaradi Afra

Institute of Mathematics, National Academy of Sciences of RA

Abstract: A variant of fixed point theorem is proved in the setting of S-metric spaces.

Key words: S-metric spaces • Coupled coincidence fixed point • K-contraction condition

INTRODUCTION

There are different type of generalization of metric spaces in several ways. For example, concepts of 2-metric spaces and D-metric spaces introduced by [1] and [2], respectively. The idea of partial metric was introduced by [3] or the notion of G-metric spaces announced by [4]. Some authors have proved fixed point type theorems in these spaces [5, 6]. Impression of D-metric space and also S-metric spaces was initiated by Sedghi, [7, 8].

In this paper, we find some new results on S-metric spaces and prove fixed point type theorem for k-contraction condition on S-metric space and offer some examples.

Basic Concepts of S-metric Spaces: In this section we offer some concepts introduced S. Sedghi, et al. [8] and results [9, 8]. We modify them for our purposes and present some new considerations.

Definition 2.1: Let \( X \) be a nonempty set. We call S-metric on \( X \) is a function \( S: X^3 \rightarrow [0, \infty] \) which satisfies the following conditions for each \( x, y, z, a \in X \)

(i) \( S(x, y, z) \geq 0 \)

(ii) \( S(x, y, z) = 0 \) if and only if \( x = y = z \),

(iii) \( S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a) \)

The set \( X \) in which S-metric is defined is called S-metric space.

The standard examples of such S-metric spaces are:

- Let \( X \) be any normed space, then \( S(x, y, z) = \|y + z - 2x\| + \|y - z\| \) is a S-metric on \( X \).
- Let \( (X, d) \) be a metric space, then \( S(x, y, z) = d(x, z) + d(y, z) \) is a S-metric on \( X \). This S-metric is called the usual S-metric on \( X \).
- Another S-metric on \((X, d)\) is \( S(x, y, z) = d(x, y) + d(x, z) + d(y, z) \) which is symmetric with respect to the argument.

In the paper we will often use a following important relation.

Lemma 2.1: (See[8]). In a S-metric space for \( x, y \in X \).

Lemma 2.2: Let \((X, S)\) be a S-metric space. If there exists sequences \( \{x_n\} \) and \( \{y_n\} \) such that \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} y_n = y \), then \( \lim_{n \to \infty} S(x_n, y_n, y_n) = S(x, y, y) \).

There exists a natural topology on a S-metric spaces. At first let us remind a notion of (open) ball.

Definition 2.2: Let \((X, S)\) be a S-metric space. For \( r > 0 \) and \( x \in X \) we define a ball with the center \( x \) and radius \( r \) as follows:

\[ B_r(x) = \{ y \in X : S(x, y, y) < r \}. \]

This is quite different concept of ball in a usual metric space which shows the following example:
Example 2.1: Let $X = \mathbb{R}$. Let $S(x, y, z)$ be a usual $S$-metric on $\mathbb{R}$ for all $x, y, z \in \mathbb{R}$. Therefore

$$B_s(x_0, 2) = \{y \in X : S(y, y, x_0) < 2\} = \{y \in \mathbb{R} : 2d(y, x_0) < 2\} = \{y \in \mathbb{R} : d(y, x_0) < 1\} = B_d(x_0, 1).$$

By using the notion of ball we can introduce the standard topology on $S$-metric space.

Remark 2.1: Any ball is open set in this topology and $x_n \to x$ means that $S(x_n, x_n, x) \to 0$ and $\{x_n\}$ is cauchy sequence if for every $\epsilon > 0$ there exists a positive integer $N$, if $n, m > N$ then $x_n \in B_s(x_m, \epsilon)$ (which is the same as $x_n \in B_d(x_m, \epsilon)$).

We prove the following very important result:

Lemma 2.3: Any $S$-metric space is a Hausdorff space.

Proof: Let $(X, S)$ be a $S$-metric space. Suppose $x \neq y$ and put $r = \frac{1}{3}S(x, x, y)$. Let us show that $B_S(x, r) \cap B_S(y, r) = \emptyset$, for $x, y \in S$. Suppose this is not true then there exists $z \in X$ such that $z \in B_S(x, r) \cap B_S(y, r)$, therefore by definition of ball we have $S(z, z, x) < r$ and $S(z, z, y) < r$. By Lemma 2.1 and (iii), we get

$$3r = S(x, x, y) \leq 2S(z, z, x) + S(z, z, y) = 2S(x, x, z) + S(y, y, z) < 3r$$

which is a contradiction.

The following concepts which will be used in our consideration was introduced in [9, 10].

Definition 2.3: (See[10]). An element $(x, y) \in X \times X$ is called a coupled fixed point(c.f.p) of a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Remark 2.2: An element $(x, y)$ is a coupled coincidence point of $F : X \times X \to X$ if and only if it is usual fixed point for mapping $F : X \times X \to X \times X$ given by $F(x, y) = (F(x, y), F(y, x))$.

Definition 2.4: (See[9]). An element $(x, y) \in X \times X$ is called a coupled coincidence point(c.c.p) of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 2.5: Let $X$ be a nonempty set. We say the mappings $F : X \times X \to X$ and $g : X \to X$ satisfy the $L$-condition if $gF(x, y) = F(gx, gy)$, for all $x, y \in X$.

The next notion is modification of usual contraction condition.

Definition 2.6: Let $(X, S)$ be a $S$-metric space. We say the mappings $F : X \times X \to X$ and $g : X \to X$ satisfy the $L$-contraction if

$$S(F(x, y), F(x, y), F(z, w)) \leq k(S(gx, gx, gz) + S(gy, gy, gw)),$$  \hspace{1cm} (1)

for all $x, y, z, w, u, v \in X$. As in classical case this condition is quite important for our results.

Main Result: The following crucial lemma help us to prove c.c.p theorem on $S$-metric space. The results such kind can be found e.g. in [6].

Lemma 3.1: Let $(X, S)$ be a $S$-metric space and $F : X \times X \to X$ and $g : X \to X$ be two mappings satisfying $k$-contraction for $k \in (0, \frac{1}{2})$. If $(x, y)$ is a c.c.p of the mappings $F$ and $g$, then $F(x, y) = gx = gy = F(y, x)$.

Proof: Suppose $(x, y)$ is a c.c.p of the mappings $F$ and $g$, we have $gx = F(x, y)$ and $gy = F(y, x)$. Suppose $gx \neq gy$. Then by (1) and Lemma 2.1, we get

$$S(F(x, y), F(x, y), F(z, w)) \leq k(S(gx, gx, gz) + S(gy, gy, gw)),$$  \hspace{1cm} (1)
Since \( gx \neq gy \) by (ii) we have \( S(gx, gx, gy) \neq 0 \). Hence \( 2k \geq 1 \) which is a contradiction. So \( gx = gy \) and therefore \( F(x, y) = gx = gy = F(y, x) \).

**Theorem 3.1:** Let \((X, S)\) be a S-metric space and \( F : X \times X \to X \) and \( g : X \to X \) be two mappings satisfying \( k\)-contraction for \( k \in (0, \frac{1}{2}) \) and \( L\)-condition. If \( g(X) \) is continuous with closed range such that \( F(X \times X) \subseteq g(X) \), then there is a unique \( x \) in \( X \) such that \( gx = F(x, x) = x \).

**Proof:** Let \( x_0, y_0 \in X \). Since \( F(X \times X) \subseteq g(X) \), we can choose \( x, y \in X \) such that \( gx = F(x, y) \) and \( gy = F(y, x) \). Then starting from the pair \((x_0, y_0)\), we can choose \( x_1, y_1 \in X \) such that \( gx_1 = F(x_0, y_1) \) and \( gy_1 = F(y_0, y_1) \). Then there exists sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that \( gx_{n+1} = F(x_n, y_n) \) and \( gy_{n+1} = F(y_n, y_n) \). For \( n \in \mathbb{N} \), from \( k\)-contraction condition, we have

\[
S(gx_n, gx_{n+1}, gy_{n+1}) \leq k(S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n)).
\]

From

\[
S(gx_{n-1}, gx_{n-1}, gx_n) \leq k(S(gx_{n-2}, gx_{n-2}, gx_{n-1}) + S(gy_{n-2}, gy_{n-2}, gy_{n-1})),
\]

since the similar inequality is correct for \( S(gy_{n-1}, gy_{n-1}, gy_{n}) \), we have

\[
S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_{n}) \leq 2k(S(gx_{n-2}, gx_{n-2}, gx_{n-1}) + S(gy_{n-2}, gy_{n-2}, gy_{n-1})),
\]

holds for all \( n \in \mathbb{N} \). By repeating this procedure enough time, we obtain for each \( n \in \mathbb{N} \)

\[
S(gx_n, gx_n, gx_{n+1}) \leq \frac{1}{2}(2k)^n(S(gx_0, gx_0, gx_1) + S(gx_0, gx_0, gy_1)).
\]

Let \( m, n \in \mathbb{N} \) with \( m > n + 2 \). By (iii) and Lemma (2.1), we have

\[
S(gx_n, gx_m, gx_m) \leq 2S(gx_n, gx_n, gx_{n+1}) + S(gx_m, gx_{m}, gx_{m+1}) = 2S(gx_n, gx_n, gx_{n+1}) + S(gx_{n+1}, gx_{n+1}, gx_m) \leq 2S(gx_n, gx_n, gx_{n+1}) + 2S(gx_{n+1}, gx_{n+1}, gx_{n+2}) + S(gx_m, gx_{m}, gx_{m+2}) \leq 2 \sum_{i=n}^{m-2} S(gx_i, gx_i, gx_{i+1}) + S(gx_{m-1}, gx_{m-1}, gx_{m}).
\]

By (2) we will have,
Letting $n, m \to \infty$, we have

$$\lim_{n,m \to \infty} S(gx_n, gx_m, gx_m) = 0.$$
for all \( x, y, u, v \in X \) and \( k \in \left( 0, \frac{1}{2} \right) \). Then there is a unique \( x \in X \) such that \( F(x, x) = x \).

Now we present some examples.

**Example 3.1:** Let \( X = [0,1] \). Suppose \( S(x, y, z) \) be usual \( S \)-metric on \( X \), for all \( x, y, z \in X \). Then \( (X, S) \) is a complete \( S \)-metric space. Now we define a map \( F : X \times X \to X \) by \( F(x, y) = \frac{1}{6} xy \) for \( x, y \in X \). Also, define \( g : X \to X \) by \( g(x) = x \) for \( x \in X \). Since

\[
| xy - uv | \leq | x - u | + | y - v |
\]

holds for all \( x, y, u, v \in X \), we have

\[
S(F(x, y), F(x, y), F(z, w)) = 2 \left| \frac{1}{6} xy - \frac{1}{6} zw \right| \\
\frac{1}{6} \left( | x - z | + 2 | y - w | \right) = \\
\frac{1}{6} (S(gx, gu, gz) + S(gy, gv, gw))
\]

holds for all \( x, y, u, v, z, w \in X \). It’s clear that \( F \) and \( g \) satisfy all the hypothesis of Theorem 3.1. Therefore \( F \) and \( g \) have a unique common fixed point. Here \( F(0,0) = g(0) = 0 \).

**Example 3.2:** Let \( X = [0,1] \). Suppose \( S(x, y, z) \) be usual \( S \)-metric on \( X \), for all \( x, y \in X \). Then \( (X, S) \) is a complete \( S \)-metric space. Define a map \( F : X \times X \to X \) by \( F(x, y) = \frac{1}{6} (x + y) \) for \( x, y \in X \). Also,

\[
S(F(x, y), F(u, v), F(z, w)) = | F(x, y) - F(z, w) | + | F(u, v) - F(z, w) | = \\
\frac{1}{6} | z - x + w - y | + \frac{1}{6} | z - u + v - w | \\
\frac{1}{6} (| x - z | + | u - z |) + \frac{1}{6} (| y - w | + | v - w |) = \\
\frac{1}{6} (S(x, u, z) + S(y, v, w)).
\]

Then by Theorem 3.2, \( F \) has a unique fixed point. Here \( x = \frac{1}{4} \) is the unique fixed point of \( F \), that is \( F(x, x) = x \).

**REFERENCES**
