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Fixed Point Type Theorem in *S*-Metric Spaces

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Abstract: A variant of fixed point theorem is proved in the setting of *S*-metric spaces.[1]

Key words: S-metric spaces · Coupled coincidence fixed point · K-contraction condition

INTRODUCTION

There are different type of generalization of metric spaces in several ways. For example, concepts of 2-metric spaces and *D*-metric spaces introduced by [1] and [2], respectively. The idea of partial metric space was introduced by [3] or the notion of *G*-metric spaces announced by [4]. Some authors have proved fixed point type theorems in these spaces [5, 6]. Impression of D^*-metric space and also *S*-metric spaces was initiated by Sedghi, [7, 8].

In this paper,we find some new results on *S*-metric spaces and prove fixed point type theorem for *k*-contraction condition on *S*-metric space and offer some examples.

Basic Concepts of *S*-metric Spaces: In this section we offer some concepts introduced S. Sedghi, *et al*. [8] and results [9, 8]. We modify them for our purposes and present some new considerations.

Definition 2.1: Let X be a nonempty set. We call S-metric on X is a function S: $X^3 \to [0, \infty]$ which satisfies the following conditions for each $x, y, z, a, \in X$

(i) $S(x, y, z) \geq 0$

(ii) $S(x, y, z) = 0$ if and only if $x = y = z$,

(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

The set X in which *S*-metric is defined is called *S*-metric space.

The standard examples of such *S*-metric spaces are:

- Let X be any normed space, then $S(x,y,z) = \|y+z-2x\| + \|y-z\|$ is a *S*-metric on *X*.
- Let (X, d) be a metric space, then $S(x,y,z) = d(x,z) + d(y,z)$ is a *S*-metric on *X*. This S-metric is called the *usual* S-metric on *X*.
- Another *S*-metric on (X, d) is $S(x,y,z) = d(x,y) + d(x,z) + d(y,z)$ which is symmetric with respect to the argument.

In the paper we will often use a following important relation.

Lemma 2.1: (See[8]). In a *S*-metric space $S(x,x,y) = S(y,y,x)$ for $x, y \in X$.

Lemma 2.2: Let (X, S) be a *S*-metric space. If there exists sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x,x,y)$.

There exists a natural topology on a *S*-metric spaces. At first let us remind a notion of (open) ball.

Definition 2.2: Let (X, S) be a *S*-metric space. For $r > 0$ and $x \in X$ we define a ball with the center x and radius r as follows:

$$B_S(x,r) = \{y \in X : S(y,y,x) < r\}.$$

This is quite different concept of ball in a usual metric space which shows the following example:

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Example 2.1: Let $X = \mathbb{R}$. Let $S(x, y, z)$ be a usual S-metric on $\mathbb{R}$ for all $x, y, z \in \mathbb{R}$. Therefore

$$B_s(x_0, 2) = \{y \in X : S(y, y, x_0) < 2\} = \{y \in \mathbb{R} : 2d(y, x_0) < 2\} = \{y \in \mathbb{R} : d(y, x_0) < 1\} = B_d(x_0, 1).$$

By using the notion of ball we can introduce the standard topology on S-metric space.

Remark 2.1: Any ball is open set in this topology and $x_n \to x$ means that $S(x_n, x_n, x) \to 0$ and $\{x_n\}$ is cauchy sequence if for every $\varepsilon > 0$ there exsits a positive integer N, if $n,m > N$ then $x_n \in B_d(x_m, \varepsilon)$(which is the same as $x_m \in B_d(x_n, \varepsilon)$).

We prove the following very important result:

Lemma 2.3: *Any* S-*metric space is a Hausdorff space***Proof:** Let (X, S) be a S-metric space. Suppose $x \neq y$ and put $r = \frac{1}{3}S(x,x,y)$. Let us show that $B_S(x,r) \cap B_S(y,r) = \varnothing$, for $x, y \in S$. Suppose this is not true then there exists $z \in X$ such that $z \in B_S(x,r) \cap B_S(y,r)$, therefor by definition of ball we have $S(z, z, x) < r$ and $S(z, z, y) < r$. By Lemma 2.1 and (iii), we get

$$3r = S(x,x,y) \leq 2S(z,z,x) + S(z,z,y) = 2S(x,x,z) + S(y,y,z) < 3r$$

which is a contradiction.

The following concepts which will be used in our consideration was introduced in [9, 10].

Definition 2.3: (See[10]). An element $(x, y) \in X \times X$ is called a coupled fixed point(c.f.p) of a mapping $F: X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Remark 2.2: An element (x, y) is a coupled coincidence point of $F: X \times X \to X$ if and only if it is usual fixed point for mapping $\tilde{F} : X \times X \to X \times X$ given by $\tilde{F}(x,y) = (F(x,y), F(y,x))$.

Definition 2.4: (See[9]). An element $(x, y) \in X \times X$ is called a coupled coincidence point(c.c.p) of the mappings $F: X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 2.5: Let X be a nonempty set. We say the mappings $F: X \times X \to X$ and $g : X \to X$ satisfy the L-condition if $gF(x, y) = F(gx, gy)$, for all $x, y \in X$.
The next notion is modification of usual contraction condition.

Definition 2.6: Let (X, S) be a S-metric space. We say the mappings $F: X \times X \to X$ and $g : X \to X$ satisfy the L-contraction if

$$S(F(x,y), F(x,y), F(z,w)) \leq k(S(gx,gx,gz) + S(gy,gy,gw)), \tag{1}$$

for all $x, y, z, w, u, v \in Z$.
As in classical case this condition is quite important for our results.

Main Result: The following crucial lemma help us to prove c.c.p theorem on S-metric space . The results such kind can be found e.g. in [6].

Lemma 3.1: Let (X, S) be a S-metric space and $F: X \times X \to X$ and $g : X \to X$ be two mappings satisfying k-contraction for $k \in (0, \frac{1}{2})$. If (x, y) is a c.c.p of the mappings F and g, then $F(x, y) = gx = gy = F(y, x)$.

Proof: Since (x, y) is a c.c.p of the mappings F and g, we have $gx = F(x, y)$ and $gy = F(y, x)$. Suppose $gx \neq gy$. Then by (1) and Lemma 2.1, we get

$$S(gx, gx, gy) = S(F(x, y), F(x, y), F(y, x))$$
$$\leq k(S(gx, gx, gy) + S(gy, gy, gx))$$
$$= 2kS(gx, gx, gy).$$

Since $gx \neq gy$ by (ii) we have $S(gx, gx, gy) \neq 0$. Hence $2k \geq 1$ which is a contradiction. So $gx = gy$ and therefore $F(x, y) = gx = gy = F(y, x)$.

Theorem 3.1: Let (X, S) be a S-metric space and $F: X \times X \to X$ and $g : X \to X$ be two mappings satisfying k-contraction for $k \in (0, \frac{1}{2})$ and L-condition. If $g(X)$ is continuous with closed range such that $F(X \times X) \subseteq g(X)$, then there is a unique x in X such that $gx = F(x, x) = x$.

Proof: Let $x_0, y_0 \in X$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Then starting from the pair (x_1, y_1), we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(x_1, y_1)$. Then there exists sequences $\{x_n\}$ and $\{y_n\}$ in X such that $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(x_n, y_n)$. For $n \in \mathbb{N}$, from k-contraction condition, we have

$$S(gx_n, gx_n, gx_{n+1}) \leq k(S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n)).$$

From

$$S(gx_{n-1}, gx_{n-1}, gx_n) \leq k(S(gx_{n-2}, gx_{n-2}, gx_{n-1}) + S(gy_{n-2}, gy_{n-2}, gy_{n-1})),$$

since the similar inequality is correct for $S(gy_{n-1}, gy_{n-1}, gy_n)$, we have

$$S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n) \leq 2k(S(gx_{n-2}, gx_{n-2}, gx_{n-1}) +$$
$$S(gy_{n-2}, gy_{n-2}, gy_{n-1}))$$

holds for all $n \in \mathbb{N}$. By repeating this procedure enough time,we obtain for each $n \in \mathbb{N}$

$$S(gx_n, gx_n, gx_{n+1}) \leq \frac{1}{2}(2k)^n (S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)). \tag{2}$$

Let $m, n \in \mathbb{N}$ with $m > n + 2$. By (iii)and Lemma (2.1), we have

$$S(gx_n, gx_n, gx_m) \leq 2S(gx_n, gx_n, gx_{n+1}) + S(gx_m, gx_m, gx_{n+1}) =$$
$$2S(gx_n, gx_n, gx_{n+1}) + S(gx_{n+1}, gx_{n+1}, gx_m)$$
$$\leq 2S(gx_n, gx_n, gx_{n+1}) + 2S(gx_{n+1}, gx_{n+1}, gx_{n+2}) + S(gx_m, gx_m, gx_{n+2})$$
$$\cdots$$
$$\leq 2\sum_{i=n}^{m-2} S(gx_i, gx_i, gx_{i+1}) + S(gx_{m-1}, gx_{m-1}, gx_m).$$

By (2) we will have,

$$S(gx_n, gx_n, gx_m) \le 2\sum_{i=n}^{m-2} S(gx_i, gx_i, gx_{i+1}) + S(gx_{m-1}, gx_{m-1}, gx_m) \le$$
$$2\sum_{i=n}^{m-2} \frac{1}{2}(2k)^i (S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)) +$$
$$\frac{1}{2}(2k)^{m-1}(S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)) \le$$
$$(2k)^n (S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1))[1 + 2k + (2k)^2 + (2k)^3 + ...] \le$$
$$\frac{(2k)^n}{1-2k}(S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)).$$

Letting $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} S(gx_n, gx_n, gx_m) = 0.$$

Thus, $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Similarly, $\{gy_n\}$ is a Cauchy sequence. Since $g(X)$ is closed, $\{gx_n\}$ and $\{gy_n\}$ are convergent to some $x \in X$ and $y \in X$. Since g is continuous, $\{g(gx_n)\}$ is convergent to gx and $\{g(gy_n)\}$ is convergent to gy. Moreover, since F and g satisfy L-condition, we have $g(gx_{n+1}) = g(F(x_n, y_n)) = F(gx_n, gy_n)$ and $g(gy_{n+1}) = g(F(y_n, x_n)) = F(gy_n, gx_n)$. Thus

$$S(g(gx_{n+1}), g(gx_{n+1}), F(x, y)) \le k(S(g(gx_n), g(gx_n), gx) + S(g(gy_n), g(gy_n), gy)).$$

Letting $n \to \infty$ and by Lemma (2.2), we get that $S(gx, gx, F(x, y)) \le k(S(gx, gx, gx) + S(gy, gy, gy)) = 0$.

Hence $gx = F(x, y)$ and similarly, $gy = F(y, x)$. By Lemma (3.1), (x, y) is a c.c.p of the mappings F and g. So $gx = F(x, y) = F(y, x) = gy$. We have

$$S(gx_{n+1}, gx_{n+1}, gx) = S(F(x_n, y_n), F(x_n, y_n), F(x, y)) \le$$
$$k(S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy)).$$

Letting $n \to \infty$, by Lemma 2.2, we get $S(x, x, gx) \le k(S(x, x, gx) + S(y, y, gy))$.
Similarly, $S(y, y, gy) \le k(S(x, x, gx) + S(y, y, gy))$. Thus,

$$S(x, x, gx) + S(y, y, gy) \le 2k(S(x, x, gx) + S(y, y, gy)). \tag{3}$$

Since $2k > 1$, inequality (3) occur only if $S(x, x, gx) = 0$ and $S(y, y, gy) = 0$. Hence $x = gx$ and $y = gy$. Thus, we get $gx = F(x, x) = x$. To prove the uniqueness, let $z \in X$ with $z \neq x$ such that $z = gz = F(z, z)$. Then

$$S(x, x, z) \le 2kS(gx, gx, gz) = 2kS(x, x, z).$$

Since $2k > 1$ we get a contradiction.

The following result is immediate corollary from the previous theorem g being the identical mapping.

Theorem 3.2: Let (X, S) be a complete S-metric space and $F: X \times X \to X$ be a mapping satisfying following contraction condition

$$S(F(x, y), F(u, v), F(z, w)) \le k(S(x, u, z) + S(y, v, w))$$

for all $x, y, u, v \in X$ and $k \in (0,\frac{1}{2})$. Then there is a unique $x \in X$ such that $F(x, x) = x$.

Now we present some examples.

Example 3.1: Let $X = [0,1]$. Suppose $S(x, y, z)$ be usual S-metric on X, for all $x, y, z \in X$. Then (X, S) is a complete S-metric space. Now we define a map $F: X \times X \to X$ by $F(x,y) = \frac{1}{6}xy$ for $x, y \in X$. Also, define $g: X \to X$ by $g(x) = x$ for $x \in X$. Since

$$|xy - uv| \leq |x-u| + |y-v|$$

holds for all $x, y, u, v \in X$, we have

$$S(F(x,y),F(x,y),F(z,w)) = 2\left|\frac{1}{6}xy - \frac{1}{6}zw\right| \leq$$
$$\frac{1}{6}(2|x-z| + 2|y-w|) =$$
$$\frac{1}{6}(S(gx,gu,gz) + S(gy,gv,gw))$$

holds for all $x, y, u, v, z, w \in X$. It's clear that F and g satisfy all the hypothesis of Theorem 3.1. Therefore F and g have a unique common fixed point. Here $F(0,0) = g(0) = 0$.

Example 3.2: Let $X = [0,1]$. Suppose $S(x, y, z)$ be usual S-metric on X, for all $x, y \in X$. Then (X, S) is a complete S-metric space. Define a map $F: X \times X \to X$ by $F(x,y) = 1 - \frac{1}{6}(x+y)$ for $x, y \in X$. Also,

$$S(F(x,y),F(u,v),F(z,w)) = |F(x,y) - F(z,w)| + |F(u,v) - F(z,w)| =$$
$$\frac{1}{6}|z - x + w - y| + \frac{1}{6}|z - u + v - w| \leq$$
$$\frac{1}{6}(|x-z| + |u-z|) + \frac{1}{6}(|y-w| + |v-w|) =$$
$$\frac{1}{6}(S(x,u,z) + S(y,v,w)).$$

Then by Theorem 3.2, F has a unique fixed point. Here $x = \frac{3}{4}$ is the unique fixed point of F, that is $F(x, x) = x$.

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