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Isologic and Isoclinic Extensions of Finite Groups

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Abstract: Let G be a finite p-group of exponent p^e. In this paper we present a new bound for the exponent of the Schur multiplier of G, when G is of class 3, 4 or 5 and e satisfies in some conditions.

Key words: Exponent · Finite group · Schur multiplier · p-group · Isologic

INTRODUCTION

It has been conjectured that the exponent of the Schur multiplier of a finite p-group is a divisor of the exponent of the group itself. I.D. Macdonld, J.W. Wamsley and others have constructed an example of a group of exponent 4 whereas its Schur multiplier has exponent 8, namely, the conjecture is not true in general. On the other hand M.R. Jones has shown in [3] that the conjecture is true for p-groups of class 2 and emphasized that it is true for some p-groups of class 3, but the did not characterize in which conditions it may be true. He has also proved that if G is a p-group of class c$\geq$2 and e(G)= p^e, then e(M(G))$\leq p^{e(c-1)}$ (see [3, Corollary 2.7]), in which e(X) denotes the exponent of a group X. A result of G.Ellis [2, Theorem B(i)] shows that with these assumptions we have $e(M(G))\leq p^{[c/2]e}$, where [c/2] denotes the smallest integer n such that $n \geq c/2$. Clearly the recent bound sharpens the bound obtained by M.R. Jones.

In this paper we show that the conjecture is true for p-groups of class 4 and 5, when e is odd or $p \not\equiv -1$ modulo 3 and 4

(Theorems 2.2, 2.5 and Remark 2.8).it is also shown that for such e or p, we have, $e(M(G))\leq p^e$ (Corollaries 2.4, 2.7 and Remark 2.8). This sharpens (under some assumptions) the above result of M.R. Jones [3] and also the results of J. Burns and G. Ellis [1] and G. Ellis [2] for c=3, 4 and 5.

Notation and Preliminaries: Let x and y be two elements of a finite group G, then $[x,y]$, the commutator of x and y and x^y denote the elements $x^{-1}y^{-1}xy$ and $y^{-1}xy$, respectively. The commutator of higher weight is defined inductively as follows; $[x_1,x_2,...,x_{n-1},x_n]=[[x_1,x_2,...,x_{n-1}],x_n]$ (n>2). The lower central series of a group G is denoted by

$\gamma_1(G)=G\supseteq\gamma_2(G)=G'\supseteq\gamma_3(G)\supseteq...\supseteq\gamma_n(G)\supseteq\gamma_{n+1}(G)\supseteq...$ in which
$\gamma_n(G)=[G,_{n-1}G]=[G,G,...,G]$,

(G is repeated n times), (n>2). Finally the minimum number of generators of a group X is denoted by $d(X)$. Other notations, where not explained, will be standard.

Theorem 1.1: (Schur 1907) Let G be a finite group and $1\to R\to F\to G\to 1$ be a free presentation for G. Then

$$M(G)\cong(R\cap F')/(R,F).$$

Definition 1.2: Let G be a finitegroup. An exact sequence $1\to A\to G^*\to G\to 1(*)$, where $A\subset Z(G^*)\cap G^{*\prime}$ is called a stem-extension of G. if furthermore $A\cong M(G)$, the (*) is called a stem-cover of G and in this case G* is said to be a covering group of G.

It is known that such a G^* always exists although need not be unique.

The Bound Obtained

Lemma 2.1: Let X be a group and $x,y\in X$. then for every positive integer n, we have the following identity modulo

$$[x,y^n]=\prod_{r=1}^{5}[x,_r y]^{\binom{n}{r}}\prod_{s=0}^{1}[x,y,[y,[x,y]],_s y]^{\binom{n}{s+2}}$$

inwhich $[a,_k b]$ is [a,b,b,…,b] (b is repeated k times).

Proof: The proof can be done using induction on n. □

Theorem 2.2: Let G be a finite p-group (p>3) of class 4 and e(G)=p^e in which e is an odd number. If G* is a covering group of G,

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then $e(G^*)|e(G)$

Proof: Suppose that $G=<g_1,g_2,...g_{d(G)}>$ and $f:G^*\to G$ is the epimorphism satisfied in the definition of the covering group. Let $u_i\in G^*$ be such that $f(u_i)= g_i$ for $1\leq I \leq d(G)$. Since $\ker f\subseteq \Phi(G^*)$, then $G^*= <u_1,u_2,...u_{d(G)}>$.

For each $x_1\in G^*$ we have $\left[u_i,u_j,x_1\right]^{p^e}\in Z(G^*),(1\leq i,j\leq d(G))$, therefore $[[u_i,u_j,x_1]^{p^e},x_2]=1$ for all $x_2\in G^*$. Now using Lemma 2.1, one can easily check that

$$[u_i,u_j,x_1,x_2]^{p^e}=1;\ \forall x_1,x_2\in G^*. \quad (1)$$

A similar argument shows that:

$$[u_i,u_j,x_1,x_2,x_3]^{p^e}=1; \qquad \forall x_1,x_2,x_3\in G^*. \quad (2)$$

On the other hand for all $x\in G^*;[u_i,u_j,x]^{p^e}=1,(1\leq i,j\leq d(G))$. Byapplying Lemma 2.1, it is concluded that

$$[u_i,u_j,x]^{p^e}[u_i,u_j,x,x]^{p^e(p^e-1)/2}[u_i,u_j,x,x,x]^{p^e(p^e-1)(p^e-2)/6}=1. \quad (3)$$

Clearly $(p^e-1)/2\in Z$, hence $[u_i,u_j,x,x]^{p^e(p^e-1)/2}=1$, by (1). Also if $p\equiv 1 \pmod 3$, then $3\mid p^e-1$ and so $6|p^e-1$ and if $p\equiv -1 \pmod 3$, then $3\mid p^e-2$ (since e is odd) and hence $6\mid (p^e-1)(p^e-2)$. Therefore in any case we have $[u_i,u_j,x,x,x]^{p^e(p^e-1)(p^e-2)/6}=1$, by (2). Now (3) follows that

$$[u_i,u_j,x]^{p^e}=1, \qquad \forall x\in G^*. \quad (4)$$

In the following we intend to prove that $[u_i,u_j]^{p^e}=1$.

The property of covering group implies that $u_j^{p^e}\in Z(G^*)$ and so $[u_i,u_j^{p^e}]=1$. therefore by Lemma 2.1 and the above comments it is enough to show that

$$[u_{i,4}u_j]^{p^e(p^e-1)(p^e-2)(p^e-3)/24}=1,$$

and by (2), we must illustrate that $24\mid(p^e-1)(p^e-2)(p^e-3)$.

(Note that:

$$[u_i,u_j,[u_j,[u_i,u_j]]]^{p^e}=1,$$
$$[u_i,u_j,[u_j,[u_i,u_j]],u_j]=1.)$$

We consider two cases:

Case 1: $p\equiv 1 \pmod 4$. If $p\equiv 1 \pmod 3$, then $12|(p^e-1)$ and since p^e-3 is even, hence $24\mid (p^e-1)(p^e-3)$. If $p\equiv -1$ (mod 3), then $3\mid p^e-1$.
But $4\mid p^e-1$. and $2\mid p^e-3$. whence we are done.

Case 2: $p\equiv -1 \pmod 4$. It follows that $4\mid p^e-3$. if $p\equiv 1$ (mod 3), than $6\mid p^e-1$ and so $24\mid (p^e-1)(p^e-3)$. Let $p\equiv 1 \pmod 3$, then $3\mid p^e-2$ and also $2\mid p^e-1$, $4\mid p^e-3$ They therefore follow that $24\mid (p^e-1)(p^e-2)(p^e-3)$, as required.

(Recall that p is odd and so $p\neq 2 \pmod 4$). □

The argument which is done in the proof of Theorem 2.2, shows that in some cases, in can be omitted the extra condition to be odd for e. In other words we have;

Corollary 2.3: Let G be a finite p-group ($p>3$) of class 4 and $p\neq -1$ modulo 3 and 4. Suppose also that G^* is a covering group of G, then $e(G^*)|e(G)$

As it is mentioned in the introduction, the following corollary sharpens the bound of M.R. Jones [3, Corollary 2.7] and J.Burns and G.Ellis[1, Theorem 6] and G. Ellis [2, Theorem B (i)] on the exponent of the Schur multiplier of some prime-power groups.

Corollary 2.4: Let G be a finite p-group ($p>3$) of class 4 and $e(G)=p^e$ then

$$e(M(G))\leq p^e$$

When one of the following conditions hold:

- e is an odd number.
- $p\neq 1$ modulo 3 and 4.
  (Note that $\leq$ can be taken to mean "divides").

In the next theorem we show that our above results can be extended to p-groups of one class more.

Theorem 2.5: Let $p>5$ be a prime and G be a finite p-group of class 5 with $e(G)=p^e$, in which e is odd. If G^* is a covering group of G, then $e(G^*)|\ e$.

Proof: Wekeep all the notations used in the proof of Theorem 2.2. By a similar argument which is applied in the proof of Theorem 2.2 and repeated use of Lemma 2.1, one can prove that;

$$[u_{it}, u_{j}] = 1 \ (2 \leq t \leq 5), \qquad (5)$$

$$[u_i, u_j, [u_j, [u_i, u_j]]]^{p^e} = 1, \qquad (6)$$

$$[u_i, u_j, [u_j, [u_i, u_j]], u_j]^{p^e} = 1. \qquad (7)$$

(Note that in Theorem 2.2, $\gamma_6(G^*) = 1$ whereas our recent assumption implies that $\gamma_7(G^*) = 1$.)

On the other hand, since $[u_i, u_j^{p^e}] = 1, (1 \leq i, j \leq d(G)),$ then by Lemma 2.1 we deduce that;

$$\prod_{r=1}^{5} [u_i, {}_r u_j]^{\binom{p^e}{r}} \prod_{s=0}^{1} [u_i, u_j, [u_j, [u_i, u_j]], {}_s u_j]^{\binom{p^e}{s+2}} = 1 \qquad (8)$$

Since $\binom{p^e}{s+2} / p^e \in Z, (s = 0,1)$, it is immediately follows, from the relations (6) and (7) that both terms of the second product is identity.

We have also shown in Theorem 2.2 that $\binom{p^e}{r} / p^e \in Z$ for r=2,3 and 4. Whence by (5) it is concluded that:

$$[u_i, {}_r u_j]^{\binom{p^e}{r}} = 1 \qquad (2 \leq r \leq 4).$$

We claim that $\binom{p^e}{5} / p^e \in Z.$

Case 1: $p \equiv 1$ (mod 5). Hence $5 | p^e - 1$. We know from the previous that $24 | (p^e - 1)(p^e - 2)(p^e - 3)$. Consequently $120 | (p^e - 1)(p^e - 2)(p^e - 3)(p^e - 4)$.

Case 2: $p \equiv 2$ (mod 5). If $e \equiv 1$ (mod 4), then we can write $e = 4k + 1$, for some $k \in Z$. Hence by Fermat theorem $p^e \equiv p$ (mod 5) and therefore $5 | p^e - 2$. Now similar previous, it is concluded that $5 | p^e - 3$ and so again similar to case 1, the required assertion follows.

Case 3: $p \equiv 3$ (mod 5). similar to case 2, $e \equiv 1$ (mod 4) implies that $5 | p^e - 3$ and $e \equiv 3$ (mod 4) implies that $5 | p^e - 3$. In each case we are in a position like case 1.

Case 4: $p \equiv 3$ (mod 5). Then $5 | p^e - 4$ and the assertion follows immediately.

Now clearly we have $[u_i, {}_5 u_j]^{\binom{p^e}{5}} = 1,$ by (5).

It therefore follows from (8) that $[u_i, u_j]^{p^e} = 1,$ as required.

The above process shows that the condition of being odd for e can be replaced with the other condition, as follows:

Corollary 2.6: Suppose that G is a finite *p*-group (p>5) of class 5 and $p \not\equiv -1$ modulo 3 and 4. Then for every covering group G^* of G, we have □.

Proof: Let p^e be the exponent of G. By the assumption $24 | (p^e - 1)(p^e - 2)(p^e - 3)$. Now if in a addition to cased considered in the proof of the Theorem 2.5, we pay attention to the two cases $e \equiv 0$ or 2 modulo 4 (when e is even), then with a similar argument to Theorem 2.5, the required assertion follows.

Corollary 2.7: If G is a finite *p*-group (p>5) of class 5 and exponent p^e. Then

$$e(M(G)) \leq p^e,$$

When one of the following conditions hold:

- e is an odd number.
- $p \not\equiv -1$ modulo 3 and 4.

Corollary 2.7 shows that the bounds p^{4e} and p^{3e} which are obtained for the exponent of the Schur multiplier of *p*-finite groups of class 5 in [3] and [2] respectively, can be reduced to p^e for some *p*-finite groups.

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