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Non-nil Ideal Radical and Non-nil Noetherian R[x]

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Abstract: In the part two of this paper we investigate Non-Nil ideal radical. In the part three we show that if R non-nil noetherian ring, Nil(R) divided prime ideal and $nil(R[x]) \subseteq (f) \forall f \in R[x] \backslash nil(R[x])$. Then R[x] Non-nil noetherian Ring.

Key words: Non-nil radical ideal . non-nil noetherian . divided prime ideal . minimal prime ideal

INTRODUCTION

We suppose that all of the rings in this article are commutative with $1 \neq 0$.

Let R be a ring. Then nil(R) denotes the set of nilpotent elements of R.I is called non-nil ideal of R if $I \not\subseteq nil(R)$ and $\sqrt{I} = \{r \in R | \exists n \in N, r^n \in I\}$. J is a non-nil ideal radical of R if $\exists$ non-nil ideal $I \ni J = \sqrt{I}$. P is divided prime ideal if $p \subseteq (x) \forall x \in R \backslash P$. Min(R) is the Set of minimal prime ideals of R. If I be an ideal of R then min(I) is the set of minimal prime ideals contains I.R is non-nil noetherian ring if every non-nil ideal of R be finitely generated.

NON-NIL IDEAL RADICAL

Theorem 2.1: Let R be a ring then this following are equivalent:

1) R non-nil noetherian.
2) R satisfies in ascending chain condition on non-nil ideals of R.
3) Every non-empty set of non-nil ideals of R has maximal element.

Proof: (1)? (2) Let $I_1 \subseteq I_2 \subseteq I_3$? ……… be arbitrary ascending chain of non-nil ideal of R. $\cup_{i \in N} I_i$ is a non-nil ideal of R.thus $\cup_{i \in N} I_i = (a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_k})$ suppose that j = max{$i_1, i_2, \ldots i_k$} we have $\cup_{i \in N} I_i \subseteq I_j$ thus $I_k = I_j \ \forall_k \geq j$.

(2)? (3) Let F be non-empty set of non-nil ideal of R and $I_1 \in F$ If I_1 is not maximal element of F there exists I_2 belong to $F \ni I_1$? I_2. So if F does not have maximal element then there will exist an infinite chain of non-nil ideal of R and this is A contradiction.

(3)? (1) Suppose I be a non-nil ideal of R and F = {$(\alpha_1, \alpha_2, \ldots \alpha_m) | m \in N, \alpha_i \in I, (\alpha_1, \ldots \alpha_m)$ non-nil ideal}.

Since I is non-nil thus $\exists a \in I$ such that $a \notin nil(R)$, $(a) \in F$, $F \neq \emptyset$. So according to (3) F has maximal element $(\alpha_1,..,\alpha_m)$ we will show that $I = (\alpha_1,..,\alpha_m)$.

If $I \neq (\alpha_1,..,\alpha_m)$ then $\exists \alpha \in I \backslash (\alpha_1,..,\alpha_m)$ thus $(\alpha_1,..,\alpha_m) \subsetneq (\alpha_1, \ldots.., \alpha_m, a)$ and $(\alpha_1,..,\alpha_m, a) \in F$ we get to a contradiction.

Theorem 2.2: Let R and S be two rings and $f: R \to S$ an onto ring homomorphism. If R non-nil noetherian ring then S will be non-nil noetherian.

Proof: Assume that I_1? I_2? I_3? ……. be an increasing chain f non-nil ideals of S. $f^{-1}(I_1)$? $f^{-1}(I_2)$? ……is a chain of non-nil ideals of R and so $\exists k \ni \forall k \geq n\ f^{-1}(I_k) = f^{-1}(I_n)$ and thus $\forall k \geq n I_k = f(f^{-1}(I_k)) = f(f^{-1}(I_n)) = I_n$.

Corollary 2.3: Let R be a ring and I be an ideal of R. If R non-nil noetherian then R/I is non-nil noetherian.

Theorem 2.4: Let R be non-nil noetherian ring and I be non-nil ideal of R. There exists $P_1, P_2, \ldots, P_n \in$ min (R) s.t. $P_1 P_2, \ldots P_n \subseteq I$.

Proof: Let F be the set of non-nil ideals of R which does not contain finite product of minimal prime ideal of R and too $F \neq \emptyset$. According to theorem 2.1. F has maximal element as P.P is not prime. Thus there exists ideals A,B of R s. t. $AB \subseteq P$ and A? P, B? P. Since P? A+P, P? B+P so $A+P \notin F$, $B+P \notin F$ and so $\exists P_1, \ldots, P_n$, $Q_1 \ldots Q_m \in$ min (R) s.t.$P_1 \ldots P_n$?A+P, $Q_1 Q_2 \ldots Q_m$? B+P. Thus $P_1 \ldots P_n Q_1, \ldots Q_m$? (A+P)(B+P)? P is a contradiction.

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Theorem 2.5: If R be non-nil noetherian ring then every non-nil ideal of R has primary decomposition.

Proof: Suppose that S be the set of all non-nil ideal of R which does not have primary decomposition. We will show that $S = \varnothing$. If $S \neq \varnothing$ by theorem 2.1. S has maximal element I.I is not primary and thus $\exists a,b \in R$ s.t.$\alpha \notin l$, $\forall n \in N$ $b^n \notin l$. Let $\forall k \in N$ $l_k = \{x \in R | b^k x \in l\}$ it is clear that $\forall k \in N$ l_k is non-nil ideal and $l_1 ?\ l_2 ? \ldots$ increasing chain of non-nil ideal of R. by.

Theorem 2.6: $\exists m \in N$ s.t. $\forall_i \geq m$ $l_i = l_m$.

We define $E = \{b^m y + c | y \in R,\ c \in l\}$ E is a non-nil ideal of R and $I ?\ l_m$, $I ?\ E, I = E \cap l_m$ because $a \in l_m$, $a \notin I \& b^m \in E \& b^m \notin I. \forall x \in E \cap l_m x = b^m y + c$ s.t. $y \in R$, $c \in l \& b^m x \in l$. Thus $b^{2m} y \in l$ and so $y \in l_{2m} = l_m$ hence $b^m y \in I$ & thus $x \in I$. E and l_m have primary decomposition so I has primary decomposition and this is a contradiction.

Corollary 2.7: Let R be a non-nil noetherian ring and I be a non-nil ideal of R.

There exist prime ideals $P_1, \ldots\ldots, P_n$ such that $\sqrt{I} = P_1 \cap \ldots \cap P_n$.

Proof: Since R non-nil noetherian by theorem 2.5. I has primary decomposition so there exists primary ideals $Q_1 \ldots Q_n$ such that $I = Q_1 \cap \ldots \cap Q_n$ and for every i,j,1

$$< i < n, 1 < j < n, i \neq j; \sqrt{Q_i} \neq \sqrt{Q_j}$$

and

$$Q_1 \cap Q_2 \cap \ldots Q_i \cap Q_{i+1} \cap \ldots Q_n ?\ Q_i$$

thus

$$\sqrt{I} = \sqrt{Q_1} \cap \sqrt{Q_2} \cap \ldots \cap \sqrt{Q_n}.$$

Corollary 2.8: Let R be a non-nil ideal of R. There exists $P_1, \ldots, P_n \in Min(I)$ such that $\sqrt{I} = P_1 \cap \ldots \cap P_n$.

Corollary 2.9: If R be non-nil noetherian ring and I non-nil ideal of R. Then Min(I) is a finite set.

Proof: By corollary 2.7. there exists $P_1, \ldots, P_n \in$ min (I) s.t. $\sqrt{I} = P_1 \cap \ldots \cap P_n$ suppose that $P \in Min(I)$ since $\sqrt{I} = \cap_{p \in Min(I)} P$ thus $P_1 \cap \ldots \cap P_n ?\ P$ hence $P = P_i \exists 1 \leq i \leq n$.

NON-NIL NOETHERIAN R[X] WITH NON-NIL NOETHERIAN RING R

In this section we will show that if nil(R) divided prime ideal of R and $\forall f \in R[x] \backslash nil(R[x]) nil(R[x] \subseteq)(f)$ and R non-nil noetherian ring then R[x] is non-nil noetherian.

Lemma 3.1: Let R be a ring and f: $R \to R[x]$ denote the natural Ring homomorphism then $nil(R[x]) = (nil(R))^e = nil(R)R[x]$.

Proof: By R.Y. Sharp (1990,exercise 1.36,page 24)

$Nil(R[X]) = \{c_0 + c_1 x + \ldots\ldots + c_0 x^n | n \in N_0,\ c_i \in nil(R)$ for all $i = 0, \ldots., n\}$ and by R. Y. sharp (1990,exercise 2.47,page 44) $(nil(R))^e = \{r_0 + r_1 x + \ldots + r_n x^n \in R[x] | n \in N_0$, $r_i \in nil(R)$ for all $i = 0, \ldots.., n\}$ thus we have $nil(R[x]) = (nil(R))^e$.

Lemma 3.2: Let R be a ring and f:$R \to R[x]$ denote natural Ring homomorphism then

$$\frac{R[x]}{nil(R[x])} = \frac{R[x]}{(nil(R))^e} = \frac{R[x]}{nil(R)R[x]} \cong \frac{R}{nil(R)}[x]$$

Proof: By R. Y. Sharp (1990, exercise 2.47, page 44)

$$\frac{R[x]}{(nil(R))^e} = \frac{R[x]}{nil(R)R[x]} \cong \frac{R}{nil(R)}[x]$$

and by lemma 3.1. nil(R[x])=nil(R)R[x] thus

$$\frac{R[x]}{nil(R[x])} \cong \frac{R}{nil(R)}[x]$$

Theorem 3.3: (R.Y. Sharp, 1990, theorem 8.7) Let R be a noetherian ring and let x be an indeterminate. Then the ring R[x] of polynomials is again a noetherian ring.

Theorem 3.4: (Badawi, A., 2003, theorem 2.2.) Let R be a ring and nil(R) divided prime ideal of R.then R is a non-nil noetherian ring if and only if $\frac{R}{nil(R)}$ is a noetherian domain.

Theorem 3.5: Let R be a ring and nil(R) divided prime ideal of R and $\forall f \in R[x] nil(R[x]) \subseteq (f)$. Then R non-nil noetherian ring if and only if R[x] non-nil noetherian ring.

Proof: Let R[x] be non-nil noetherian since τ: $R[x] \to R$, $\tau(r_0 + r_1 x + \ldots + r_n x^n) = r_0$ onto homomorphism by theorem 2.2. R is non-nil noetherian.

Converse: Let R be non-nil noetherian since nil(R) prime ideal by lemma 3.1. nil(R[x]) prime ideal. Thus nil(R[x]) divided prime ideal by lemma 3.1. and theorems 3.3. and 3.4. $\frac{R[x]}{nil(R[x])}$ noetherian ring and so R[x] is non-nil noetherian.

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