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On Orbits of $\mathbb{Q}(\sqrt{m})\setminus\mathbb{Q}$ under the Action of Hecke Group $H(\sqrt{2})$

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Abstract: We are interested in the natural action (as Möbius transformations) of hecke group $H(\sqrt{2}) = H$ on the elements of quadratic number field over the rational numbers. The objective of this study is to find the transitive H-subsets of some H-sets of $\mathbb{Q}(\sqrt{m})\setminus\mathbb{Q}$ with the help of the structure of circuits of ambiguous numbers. For p ≡ 3(mod 4), the number $o_H^{*\sim}(4p)$ of H-orbits of $\mathbb{Q}^{*\sim}(\sqrt{4p}) = \left(\mathbb{Q}^*(\sqrt{n})\setminus\mathbb{Q}^{**}(\sqrt{n})\right)\cup\mathbb{Q}^{**}(\sqrt{4n})$ has been determined for each prime p≤2011.

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INTRODUCTION

Throughout this paper we take m as a square free positive integer. Since every element of $\mathbb{Q}(\sqrt{m})\setminus\mathbb{Q}$ can be expressed uniquely as $\frac{a+\sqrt{n}}{c}$, with $n = k^2m$, k is any positive integer and a, $\frac{a^2-n}{c}$ and c are relatively prime integers and we denote it by α(a,b,c), the set

$$Q^*(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : a, c \neq 0, b = \frac{a^2-n}{c} \in Z \text{ and } (a,b,c)=1\}$$

is the set of all roots of primitive second degree equations $cx^2 + 2ax + b = 0$ with reduced discriminant $\Delta = a^2 - bc$ equal to n (an equation $cx^2 + 2ax + b = 0$ is said to be primitive if (a,b,c) = 1). If n and n′ are two distinct non square positive integers then $\mathbb{Q}^*(\sqrt{n})\cap Q^*(\sqrt{n'}) = \varnothing$ hence it is easy to see to see that $\mathbb{Q}(\sqrt{m})\setminus\mathbb{Q}$ is the disjoint union of $\mathbb{Q}^*(\sqrt{k^2m})$ for all $k\in\mathbb{N}$. If $\alpha(a,b,c)\in\mathbb{Q}^*(\sqrt{n})$ and its conjugate $\bar{\alpha}$ have opposite signs then α is called an ambiguous number [1]. The actual number of ambiguous numbers in $\mathbb{Q}^*(\sqrt{n})$ has been discussed in [2] as a function of n. The classification of the elements of $\mathbb{Q}^*(\sqrt{n})$ with respect to modulo 3^r and modulo p has been explored in [3, 4].

A non-empty set Ω with an action of the group G on it, is said to be a G-set. We say that Ω is a transitive G-set if, for any p,q in Ω there exists a g in G such that $p^g = q$. In 1936, Erich Hecke [5], introduced the groups H(λ) generated by two linear-fractional transformations $x(z)=\frac{-1}{z}$ and $y(z)=\frac{-1}{z+\lambda}$. Hecke showed that H(λ) is discrete if and only if $\lambda = \lambda_q = 2\cos(\frac{\pi}{q})$, $q\in\mathbb{N}$, q≥3 or λ≥2. Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic group of order 2 and q and it has a presentation

$$H(\lambda_q) = \langle x, y : x^2 = y^q = 1\rangle \cong C_2 * C_q$$

The first few of these groups are $H(\lambda_3)=PSL(2,\mathbb{Z})$, the modular group,

$$H(\lambda_4) = H(\sqrt{2}) = \langle x, y : x^2 = y^4 = 1\rangle$$

where

$$x(z)=\frac{-1}{2z} \text{ and } y(z)=\frac{-1}{2(z+1)}$$

$$H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$$

and

$$H(\lambda_6) = H(\sqrt{3}) = \langle x, y : x^2 = y^6 = 1\rangle$$

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An action of H and its proper subgroups on $\mathbb{Q}(\sqrt{m})\cup\{\infty\}$ has been discussed in [6-9].
In [10], the H-orbits of

$$\mathbb{Q}''(\sqrt{n}) = \mathbb{Q}^{**}(\sqrt{n}) \cup \mathbb{Q}^{*\blacklozenge}(\sqrt{4n})$$

where

$$\mathbb{Q}^{*\blacklozenge}(\sqrt{4n}) = \left(\mathbb{Q}^{*}(\sqrt{n}) \setminus \mathbb{Q}^{**}(\sqrt{n})\right) \cup \mathbb{Q}^{**}(\sqrt{4n})$$

have been found for $p \equiv 1 \pmod 4$. This paper is a continuation of paper [10] for $p \equiv 3 \pmod 4$. The main objective of this paper is to explore the structure of the circuits formed by the elements of $\mathbb{Q}_1^{*\sim}(\sqrt{4p})$, $p \equiv 3 \pmod 4$. We find some proper H-subsets of $\mathbb{Q}''(\sqrt{n})$ and define the types of H-orbits to obtain ambiguous lengths of H-orbits with various properties of these orbits of $\mathbb{Q}''(\sqrt{p})$. In Section 4, we discuss the H-orbits of $\mathbb{Q}^{*\sim}(\sqrt{4p})$ for $o_H^{*\sim}(4p) > 4$ and have been able to prove that $(\alpha)^H, (\bar{\alpha})^H, (-\alpha)^H$ and $(-\bar{\alpha})^H$ are distinct orbits of $\mathbb{Q}^{*\sim}(\sqrt{4p})$ for each $\alpha \in\sim Q^{*\sim}(\sqrt{4p}) \setminus ((\frac{\sqrt{p}}{1})^H \cup (\frac{\sqrt{p}}{-1})^H)$. The complete list of H-orbits of $\mathbb{Q}''(\sqrt{p})$, $p \equiv 3 \pmod 4$, has been determined for all prime $p \leq 2011$.

PRELIMINARIES

We quote from [2, 6, 7, 11] the following results for later reference. Let

$$\alpha = \frac{a+\sqrt{n}}{c} \text{ with } b = \frac{a^2-n}{c}$$

We tabulate the actions on $\alpha(a,b,c) \in \mathbb{Q}''(\sqrt{n})$ of x,y and their combinations y^2, xy, yx and y^2x in Table 1.

Theorem 2.1: [6] $\mathbb{Q}'(\sqrt{n}) = \{\frac{\alpha}{t} : \alpha \in \mathbb{Q}^{*}(\sqrt{n}),\ t=1,2\}$ is invariant under the action of H.

Theorem 2.2: [6] Let $n \equiv 1,2$ or $3 \pmod 4$. Then

$$\mathbb{Q}^{**}(\sqrt{n}) = \{\alpha(a,b,c) \in \mathbb{Q}^{*}(\sqrt{n}) : 2|c\}$$

is an H-subset of $Q''(\sqrt{n})$.

Lemma 2.3: [2] Let m be a square-free positive integer. Then

Table 1: The action of elements of H on $\alpha \in \mathbb{Q}''(\sqrt{n})$

$\alpha = \frac{a+\sqrt{n}}{c}$	a	b	c
$x(\alpha) = \frac{-1}{2\alpha}$	-a	c/2	2b
$y(\alpha) = \frac{-1}{2\alpha+2}$	-a-c	c/2	2(2a+b+c)
$y^2(\alpha) = \frac{-\alpha-1}{2\alpha+1}$	-3a-2b-c	2a+b+c	4a+4b+c
$xy^2(\alpha) = \frac{2\alpha+1}{2\alpha+2}$	3a+2b+c	$\frac{4a+4b+c}{2}$	2(2a+b+c)
$y^2x(\alpha) = \frac{-2\alpha+1}{2\alpha-2}$	3a-2b-c	$\frac{-4a+4b+c}{2}$	2(-2a+b+c)
$xy^3(\alpha) = \frac{\alpha}{2\alpha+1}$	a+2b	b	4a+4b+c
$(xy)^k(\alpha) = \alpha+k$	a+kc	$2ka+b+k^2c$	c
$(yx)^k(\alpha) = \frac{\alpha}{-2k\alpha+1}$	a-2kb	b	$-4ka+4k^2b+b+c$
$(y^3x)^k(\alpha) = \alpha-k$	a-kc	$2ka+b+k^2c$	c

$$|\mathbb{Q}_1^{*}(\sqrt{m})| = \tau^{*}(m) = 2\tau(m) + 4\sum_{a=1}^{\lfloor\sqrt{m}\rfloor}\tau(m-a^2)$$

Lemma 2.4: [2] Let n be square free positive integer. Then

$$|\mathbb{Q}_1^{**}(\sqrt{n})| = 2\tau'(n) + 4\sum_{a=1}^{\lfloor\sqrt{n}\rfloor}\tau'(n-a^2)$$

where $\tau'(u)$ denotes those divisors of u, which are divisible by 2.

Lemma 2.5: [6] Let $\alpha \in \mathbb{Q}''(\sqrt{n})$. Then $\alpha^H = \bar{\alpha}^H$ if and only if there exists an element β in α^H such that $x(\beta) = \bar{\beta}$.

It is well known that $G = \langle x, y : x^2 = y^3 = 1\rangle$ represents the modular group, where $x(z) = \frac{-1}{z}, y(z) = \frac{z-1}{z}$ are linear fractional transformations.

Lemma 2.6: [7] Let $n \equiv 1,2$ or $3 \pmod 4$. Let Y be any G-subset of $\mathbb{Q}^{*\sim}(\sqrt{4n})$. Then $Y \cup x(Y)$ is an H-subset of $\mathbb{Q}^{*\sim}(\sqrt{4n})$.

Lemma 2.7: [2] Let $p \equiv 3 \pmod 4$. Then $\mathbb{Q}^{*}(\sqrt{p})$ splits into at least two G-orbits, namely, $(\frac{\sqrt{p}}{1})^G$ and $(\frac{\sqrt{p}}{-1})^G$ under the action of G.

Theorem 2.8: [11] If $n \equiv 0$ or 3 (mod 4), then

$$S = \{\alpha \in Q^*(\sqrt{n}) : b \text{ or } c \equiv 1 (\text{mod } 4)\}$$

and

$$-S = \{\alpha \in Q^*(\sqrt{n}) : b \text{ or } c \equiv -1 (\text{mod } 4)\}$$

are exactly two disjoint G-subsets of $Q^*(\sqrt{n})$ depending upon classes [a,b,c] modulo 4.

Theorem 2.9: [4] Let p be an odd prime factor of n. Then both of

$$S_1^p = \{\alpha \in Q^*(\sqrt{n}) : (b/p) \text{ or } (c/p) = 1\}$$

and

$$S_2^p = \{\alpha \in Q^*(\sqrt{n}) : (b/p) \text{ or } (c/p) = -1\}$$

are G-subsets of $Q^*(\sqrt{n})$. In particular, these are the only G-subsets of $Q^*(\sqrt{n})$ depending upon classes [a,b,c] modulo p.

AMBIGUOUS LENGTHS OF THE H-ORBITS OF $\mathbb{Q}^{*}(\sqrt{p})$

We start this section with the following definition.

Definition 3.1: By a circuit, we shall mean closed path of edges and squares in the coset diagram for H-orbit α^H where $\alpha \in \mathbb{Q}^*(\sqrt{n})$.

If $n_1, n_2, n_3, n_4, \ldots, n_k$ is a sequence of positive integers and

$$i_j = 0,1,2, i_l \neq i_{l+1} \ (l=1,2,\ldots,k-1), i_1 \neq i_k \qquad (1)$$

Then by a circuit of the type $(n_{1i_1}, n_{2i_2}, n_{3i_3}, n_{4i_4}, \ldots, n_{ki_k})$ we shall mean the circuit (counter clockwise) in which $n_j, j=1,2,3,\ldots,k$ squares have i_j vertices outside the circuit.

Remarks 3.2

1. Since it is immaterial with which ambiguous number of α^H the circuit begins, we can express type (1) by any of the following k-equivalent forms

$$(n_{1i_1}, n_{2i_2}, \ldots, n_{ki_k}) = (n_{2i_2}, n_{3i_3}, \ldots, n_{ki_k}, n_{1i_1})$$
$$= \ldots (n_{ki_k}, n_{1i_1}, \ldots, n_{k-1i_{k-1}})$$

2. The type $(n_{1i_1}, n_{2i_2}, n_{3i_3}, n_{4i_4}, \ldots, n_{ki_k})$ can be described by the equations (1) or more briefly by

$$i_j = 0,1,2, i_t \neq i_{t+1(\text{mod } k)} \qquad (2)$$

3. This circuit induces an element $g = (xy^{i_k+1})^{n_k} \ldots (xy^{i_2+1})^{n_2}(xy^{i_1+1})^{n_1}$ of H and fixes a particular vertex of a square lying on the circuit and hence the ambiguous length of this circuit is given by $2(n_1 + n_2 + n_3 + \ldots + n_k)$
4. All of the $2(n_1 + n_2 + \ldots + n_k)$ numbers lies in the same orbit and hence each of them has the same type.
   For example, by the circuit of the type $(6_0, 1_2, 1_1, 2_0, 1_1, 1_2)$ we mean the circuit (Fig. 1) induces an element

$$h = (xy)^3(xy)^3(xy)^2(xy)^2(xy)^2(xy)^3(xy)^3$$

of H which fixes vertex $\frac{\sqrt{11}}{1}$ as follows. Let $k_1 = \frac{\sqrt{11}}{1}$.

$$(xy)^3(k_1) = \frac{3+\sqrt{11}}{1} = k_2, \ (xy^3)(k_2) = \frac{-1+\sqrt{11}}{5} = k_3$$

$$(xy^2)(k_3) = \frac{-2+\sqrt{11}}{2} = k_4, \ (xy)^2(k_4) = \frac{2+\sqrt{11}}{2} = k_5$$

$$(xy^2)(k_5) = \frac{1+\sqrt{11}}{5} = k_6, \ (xy^3)(k_6) = \frac{-3+\sqrt{11}}{1} = k_7$$

and $(xy)^3(k_7) = k_1$. The ambiguous length of this circuit is 2(6+1+1+2+1+1).

We now find the H-subsets of

$$\mathbb{Q}^{*}(\sqrt{n}) = \mathbb{Q}^{**}(\sqrt{n}) \cup \mathbb{Q}^{*\sim}(\sqrt{4n})$$

where

$$\mathbb{Q}^{*\sim}(\sqrt{4n}) = \left(\mathbb{Q}^*(\sqrt{n}) \setminus \mathbb{Q}^{**}(\sqrt{n})\right) \cup \mathbb{Q}^{**}(\sqrt{4n})$$

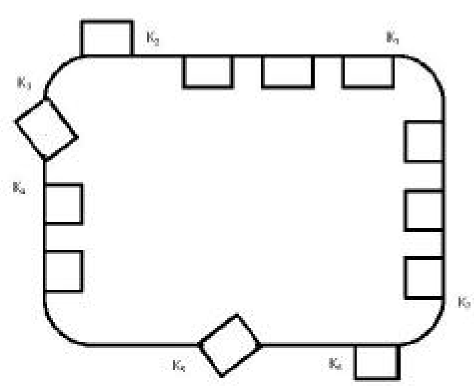

Fig. 1: Orbit of $k_1 = \frac{\sqrt{11}}{1}$ and $h(k_1) = k_1$

Also we explore the action of H by finding H-orbits of $\mathbb{Q}^{*\sim}(\sqrt{4n})$ and discuss the ambiguous cardinalities of these H-orbits.

Theorem 3.3 Let p ≡ 3 (mod 8). Then $\mathbb{Q}''(\sqrt{n})$ splits into three H-subsets. In particular $(\frac{\sqrt{n}}{1})^H$, $(\frac{\sqrt{n}}{-1})^H$ and $(\frac{1+\sqrt{n}}{2})^H$ are at least three H-orbits of $\mathbb{Q}''(\sqrt{n})$.

Before giving the proof we need the following crucial remark.

Remark 3.4: Let p ≡ 3 (mod 4). Then $a^2 - n \not\equiv 0 (\text{mod } 4)$ for all $a \in \{0,1,2...,\lfloor\sqrt{n}\rfloor\}$.

Proof of Theorem 3.3 Since p ≡ 3 (mod 8) and $\mathbb{Q}''(\sqrt{n}) = \mathbb{Q}^{**}(\sqrt{n}) \cup \mathbb{Q}^{*\sim}(\sqrt{4n})$, by Theorem 2.2, $\mathbb{Q}^{**}(\sqrt{n})$ is an H-subset of $\mathbb{Q}''(\sqrt{n})$, by Theorem 2.8 and Lemma 2.6, $\mathbb{Q}^{*\sim}(\sqrt{4n})$ splits into two H-subsets. Thus, by Remark 3.2, it is clear that $\mathbb{Q}''(\sqrt{n})$ splits into three H-subsets. Clearly $(\frac{\sqrt{n}}{1})^H$, $(\frac{\sqrt{n}}{-1})^H$ and $(\frac{1+\sqrt{n}}{2})^H$ are at least three H-orbits of $\mathbb{Q}''(\sqrt{n})$. This completes the proof.

Theorem 3.5: Let p ≡ 7 (mod 8). Then $\mathbb{Q}''(\sqrt{n})$ splits into three H-subsets. In particular $(\frac{\sqrt{n}}{1})^H$, $(\frac{\sqrt{n}}{-1})^H$, $(\frac{1+\sqrt{n}}{2})^H$ and $(\frac{1+\sqrt{n}}{-2})^H$ are at least four H-orbits of $\mathbb{Q}''(\sqrt{n})$.

We now determine the ambiguous lengths of these H-orbits as a function of p, which help us to determine the remaining H-orbits of $\mathbb{Q}''(\sqrt{p})$ and hence to classify them as well. The following lemma is concerned with the H-orbits $(\frac{\sqrt{p}}{1})^H$ and $(\frac{\sqrt{p}}{-1})^H$ for p ≡ 3 (mod 8). Before this we have the following

Remark 3.6: Let $\alpha(a,b,c) \in \mathbb{Q}^{*}(\sqrt{n})$ and $k \in \mathbb{N}$. Then

1. $(xy)^k(\alpha) = \alpha + k = (y^3x)^{-k}(\alpha)$
2. $(yx)^k(\alpha) = \frac{\alpha}{1-2k\alpha} = (xy^3)^{-k}(\alpha)$
3. $h^k(\alpha) = \alpha_1 \in \alpha^H \Rightarrow h^{-k}(\alpha) = -\overline{\alpha_1} \in \alpha^H$

Proof: These results can be verified by Table 1.

From [12], if p ≡ 3 (mod 8) then the set $\{\frac{\pm1+\sqrt{p}}{c}, \frac{\pm1+\sqrt{p}}{-2}\}$ is contained in $(\frac{\sqrt{p}}{1})^G$ and the set $\{\frac{\pm1+\sqrt{p}}{-c}, \frac{\pm1+\sqrt{p}}{2}\}$ is contained in $(\frac{\sqrt{p}}{-1})^G$. By Theorems 3.3 and 3.4, the set $\{\frac{\pm1+\sqrt{p}}{2}, \frac{\pm1+\sqrt{p}}{-2}\}$ is contained in $(\frac{\pm1+\sqrt{p}}{2})^H$ or in $(\frac{\pm1+\sqrt{p}}{-2})^H$. So it is clear that $(\frac{\sqrt{n}}{1})^H$ and $(\frac{\sqrt{n}}{-1})^H$ are in

$$\mathbb{Q}^{*\sim}(\sqrt{4n}) = \left(\mathbb{Q}^{*}(\sqrt{n}) \setminus \mathbb{Q}^{**}(\sqrt{n})\right) \cup \mathbb{Q}^{**}(\sqrt{4n}).$$

Lemma 3.7: Let p ≡ 3 (mod 8) such that p-2 = q^2. Then the circuits of $(\frac{\sqrt{p}}{1})^H$ and $(\frac{\sqrt{p}}{-1})^H$ have type

$$(q_0, (\tfrac{q-1}{2})_2, 1_1, (q-1)_0, 1_1, (\tfrac{q-1}{2})_2, q_0)$$

or

$$(2q_0, (\tfrac{q-1}{2})_2, 1_1, (q-1)_0, 1_1, (\tfrac{q-1}{2})_2)$$

and hence

$$|(\sqrt{p})^H|_{amb} = 2(4q) = |(\tfrac{\sqrt{p}}{-1})^H|_{amb}$$

Proof: In order to prove this result, it is sufficient to find the element h∈H such that $(h)(\alpha) = \alpha$ where $\alpha = \frac{\sqrt{p}}{1}$. Using Remark 3.6(1) we obtain, $(xy)^q(\alpha) = \frac{q+\sqrt{p}}{1} = \alpha_1$ (say). Again Remark 3.6(3) gives

$$(xy)^{-q}(\alpha) = \frac{-q+\sqrt{p}}{1} = -\overline{\alpha_1}$$

Now

$$(xy^3)^{\frac{q-1}{2}}(\alpha_1) = \frac{(q-1)(q^2-p) + q + \sqrt{p}}{(q-1)[2q + (q^2-p)(q-1)]+1} = \alpha_2$$

and

$$(xy^3)^{-(\frac{q-1}{2})}(-\overline{\alpha_1}) = \frac{-(q-1)(q^2-p) + q + \sqrt{p}}{(q-1)[2q + (q^2-p)(q-1)]+1} = -\overline{\alpha_2}$$

By Table 1

$$(xy^2)(\alpha_2) = \frac{(q-1)(q^2-p+1) + \sqrt{p}}{-(q^2-p)} = \alpha_3$$

and

$$(xy^2)^{-1}(-\bar{\alpha}_2) = \frac{-(q-1)(q^2-p+1)+\sqrt{p}}{-(q^2-p)} = -\bar{\alpha}_3$$

Finally we have

$$(xy)^{-(q-1)}(-\bar{\alpha}_3) = \frac{-(q-1)(q^2-p+1)+\sqrt{p}}{-(q^2-p)} = \alpha_3$$

Thus

$$(xy)^q(xy^3)^{\frac{q-1}{2}}(xy^2)(xy)^{q-1}(xy^2)(xy^3)^{\frac{q-1}{2}}(xy)^q(\frac{\sqrt{p}}{1}) = \frac{\sqrt{p}}{1}$$

Hence the circuit of $(\frac{\sqrt{p}}{1})^H$ has type $(2q_0,(\frac{q-1}{2})_2,1_1,(q-1)_0,1_1,(\frac{q-1}{2})_2)$ and hence

$$|(\sqrt{p})^H|_{amb} = 2(4q) = |(\frac{\sqrt{p}}{-1})^H|_{amb}$$

Example 3.8: By Lemma 2.3, $|Q_1^*(\sqrt{227})|=180$ and by lemma 2.4, $|Q_1^{**}(\sqrt{227})|=60$. Hence

$$|Q_1^{*\sim}(\sqrt{4.227})| = 2(|Q_1^*(\sqrt{227})| - |Q_1^{**}(\sqrt{227})| = 2(120)$$

Thus the circuits of $(\frac{\sqrt{227}}{1})^H$ and $(\frac{\sqrt{227}}{-1})^H$ have type $(15_0,7_2,1_1,14_0,1_1,7_2,15_0)$ or $(30_0,7_2,1_1,14_0,1_1,7_2)$ and hence $|(\frac{\sqrt{227}}{1})^H|_{amb}=120=|(\frac{\sqrt{227}}{-1})^H|_{amb}$. Clearly $o_H^{*\sim}(4.227)=2$ and $o_H(227)=3$.

Lemma 3.9: Let $p \equiv 3 \pmod 4$ such that $p+1 = q^2$. Then the circuit of $(\frac{\sqrt{p}}{1})^H$ and $(\frac{\sqrt{p}}{-1})^H$ have type (q_0, q_1) and hence

$$|(\sqrt{p})^H|_{amb} = 2(2q) = |(\frac{\sqrt{p}}{-1})^H|_{amb}$$

Proof: Analogous to the proof of Lemma 3.7.

Example 3.10: By Lemma 3.3, $|Q_1^*(\sqrt{3})|=12$ and by 2.4, $|Q_1^{**}(\sqrt{3})|=4$. Thus

$$|Q_1^{*\sim}(\sqrt{4.3})| = 2(|Q_1^*(\sqrt{3})| - |Q_1^{**}(\sqrt{3})| = 16$$

The circuits of $(\frac{\sqrt{3}}{1})^H$ and $(\frac{\sqrt{3}}{-1})^H$ have type $(2_0,2_2)$. Hence $|(\frac{\sqrt{3}}{1})^H|_{amb} = 8 = |(\frac{\sqrt{3}}{-1})^H|_{amb}$. The circuit of $(\frac{1+\sqrt{3}}{2})^H$ have the type $(1_0,1_2)$ and $|(\frac{1+\sqrt{3}}{2})^H|_{amb} = 4$. So $O_H(3) = 3$.

Remark 3.11: Let $p \equiv 3 \pmod 4$. Then p+1 is a perfect square if and only if p = 3.

The following lemma is concerned with the H-orbits $(\frac{\sqrt{p}}{1})^H$ and $(\frac{\sqrt{p}}{-1})^H$ for $p \equiv 7 \pmod 8$.

Lemma 3.12: Let $p \equiv 7 \pmod 8$ such that $p+2 = q^2$. Then the circuits of $(\frac{\sqrt{p}}{1})^H$ and $(\frac{\sqrt{p}}{-1})^H$ have type

$$((q-1)_0,1_1,(\frac{q-1}{2}-1)_0,1_2,(q-1)_0,1_2,(\frac{q-1}{2}-1)_0,1_1,(q-1)_0)$$

or

$$(2(q-1)_0,1_1,(\frac{q-1}{2}-1)_0,1_2,(q-1)_0,1_2,(\frac{q-1}{2}-1)_0,1_1)$$

and hence

$$|(\frac{\sqrt{p}}{1})^H|_{amb} = 2(4q-2) = |(\frac{\sqrt{p}}{-1})^H|_{amb}.$$

Proof: To prove this result, it is enough to find an element $h \in H$ such that $(h)(\alpha) = \alpha$, where $\alpha = \frac{\sqrt{p}}{1}$. By Remark 3.6,

$$(xy)^{q-1}(\frac{\sqrt{p}}{1}) = \frac{q-1+\sqrt{p}}{1} = \alpha_1$$

(say). Then

$$(xy)^{-(q-1)}(\frac{\sqrt{p}}{1}) = \frac{-(q-1)+\sqrt{p}}{1} = -\bar{\alpha}_1$$

(say). Now by Table 1,

$$(xy^2)(\alpha_1) = \frac{2(q^2-p)-q+\sqrt{p}}{(2q^2-p)} = \alpha_2$$

and

$$(xy^2)^{-1}(\alpha_1) = \frac{-2(q^2-p)+q+\sqrt{p}}{(2q^2-p)} = -\bar{\alpha}_2$$

Again by Remark 3.6, we have

$$(xy)^{\frac{q-3}{2}}(\alpha_2) = \frac{(q^2-p)(q-1)-q+\sqrt{p}}{2(q^2-p)} = \alpha_3$$

and

$$(xy)^{-\frac{q-3}{2}}(-\bar{\alpha}_2) = \frac{-(q^2-p)(q-1)+q+\sqrt{p}}{2(q^2-p)} = -\bar{\alpha}_3$$

By Table 1

$$(xy^3)(\alpha_3) = (\frac{q^3+q(b-3)+1+\sqrt{p}}{q^2-p}) = \alpha_4$$

and

$$(xy)^{-1}(-\bar{\alpha}_3) = \frac{-q^3+q(p+3)-1+\sqrt{p}}{q^2-p} = -\bar{\alpha}_4$$

Finally

$$(xy)^{q-1}(\alpha_4) = \frac{(2q-1)(q^2-p-1)-q+\sqrt{p}}{q^2-p} = \alpha_5$$

and $\alpha_5 = -\bar{\alpha}_4$. Thus

$$(xy)^{q-1}(xy^2)(xy)^{\frac{q-1}{2}-1}(xy^3)(xy)^{q-1}$$
$$(xy^3)(xy)^{\frac{q-1}{2}-1}(xy^2)(xy)^{q-1}(\alpha) = \alpha$$

Hence the circuit of $(\frac{\sqrt{p}}{1})^H$ have type

$$(2(q-1)_0, 1_1, (\frac{q-1}{2}-1)_0, 1_2, (q-1)_0, 1_2, (\frac{q-1}{2}-1)_0, 1_1)$$

and

$$|(\frac{\sqrt{p}}{1})^H|_{amb} = 2(4q-2) = |(\frac{\sqrt{p}}{-1})^H|_{amb}$$

Example 3.13: By Lemma 2.3, $|Q_1^*(\sqrt{79})| = 204$ and by Lemma 2.4, $|Q_1^{**}(\sqrt{79})| = 64$. Hence

$$|Q_1^{*\sim}(\sqrt{4.79})| = 2(|Q_1^*(\sqrt{79})| - |Q_1^{**}(\sqrt{79})| = 2(140)$$

The circuits of $(\frac{\sqrt{79}}{1})^H$ and $(\frac{\sqrt{79}}{-1})^H$ have the type $(16_0, 1_1, 3_0, 1_2, 8_0, 1_2, 3_0, 1_1)$ and hence

$$|(\sqrt{79})^H|_{amb} = 68 = |(\frac{\sqrt{79}}{-1})^H|_{amb}$$

Clearly

$$\left|Q^{*\sim}(\sqrt{4.79})\right| > |(\frac{\sqrt{79}}{1})^H|_{amb} + |(\frac{\sqrt{79}}{-1})^H|_{amb}$$

and hence $o_H^{*\sim}(4.79) > 2$. The remaining H-orbits of $Q^{*\sim}(\sqrt{4.79})$ will be discussed in the next section.

Example 3.14 $o_H^{*\sim}(4.167) = 2$ and $O_H(167) = 3$. The circuits of $(\frac{\sqrt{167}}{1})^H$ and $(\frac{\sqrt{167}}{-1})^H$ have type $(12_0, 1_1, 5_0, 1_2, 12_0, 1_2, 5_0, 1_1, 12_0)$ or $(24_0, 1_1, 5_0, 1_2, 12_0, 1_2, 5_0, 1_1)$. Hence

$$|(\frac{\sqrt{167}}{1})^H|_{amb} = 100 = |(\frac{\sqrt{167}}{-1})^H|_{amb}$$

and

$$|(\frac{1+\sqrt{167}}{2})^H|_{amb} = 48$$

Since by Lemma 2.3, $|Q_1^*(\sqrt{167})| = 148$ and by Lemma 2.4, $|Q_1^{**}(\sqrt{167})| = 48$. So

$$|Q_1^{*\sim}(\sqrt{4.167})| = 2(|Q_1^*(\sqrt{167})| - |Q_1^{**}(\sqrt{167})| = 2(100)$$

H-Orbits of $\mathbb{Q}^{*\sim}(\sqrt{p})$, p ° 3(mod 4)

Recall that $H = \langle x, y : x^2 = y^4 = 1\rangle$ is a Möbius group with $x(z) = \frac{-1}{2z}$ and $y(z) = \frac{-1}{2(z+1)}$. One of the main objectives is to determine the complete list of H-orbits (transitive H-subsets) of $\mathbb{Q}^{*}(\sqrt{p})$ with $p \equiv 3 \pmod 4$ and $p \le 2011$. We concentrate on the distribution of the elements of $\mathbb{Q}^{*\sim}(\sqrt{4p})$ in H-orbits and prove that if $p \equiv 3 \pmod 4$ then the number $o_H^{*\sim}(4p) \equiv 2 \pmod 4$. If

$$\left|\mathbb{Q}^{*\sim}(\sqrt{4p})\right| = |(\frac{\sqrt{p}}{1})^H|_{amb} + |(\frac{\sqrt{p}}{-1})^H|_{amb}$$

then clearly $o_H^{*\sim}(4p) = 2$. However if

$$\left|\mathbb{Q}^{*\sim}(\sqrt{4p})\right| > |(\frac{\sqrt{p}}{1})^H|_{amb} + |(\frac{\sqrt{p}}{-1})^H|_{amb}$$

(for example 3.13) then we have the following lemma which helps us to find the remaining H-orbits of $\mathbb{Q}^{*\sim}(\sqrt{4p})$.

Lemma 4.1: Let $p \equiv 3 \pmod 4$. Then

1. $(\alpha)^H \cap (-\alpha)^H = \varnothing$ for all $\alpha \in Q^{*\sim}(\sqrt{4p}) \setminus (\frac{\sqrt{p}}{1})^H \cup (\frac{\sqrt{p}}{-1})^H)$.

2. $(\alpha)^H \cap (\bar{\alpha})^H = \varnothing$ for all $\alpha \in Q^{*\sim}(\sqrt{4p}) \setminus ((\frac{\sqrt{p}}{1})^H \cup (\frac{\sqrt{p}}{-1})^H)$.

Proof: First part follows by Theorem 2.8 and Lemma 2.6. By [12], we know that $\frac{\pm 1+\sqrt{p}}{c}, \frac{\pm 1+\sqrt{p}}{-c}$ are contained in $(\frac{\sqrt{p}}{1})^H$ or $(\frac{\sqrt{p}}{-1})^H$ where $c \not\equiv 0 (\text{mod } 2)$. Hence by Lemma 2.5 we have $(\alpha)^H \cap (\bar{\alpha})^H = \varnothing$ for all $\alpha \in Q^{*\sim}(\sqrt{4p}) \setminus ((\sqrt{p})^H \cup (\frac{\sqrt{p}}{-1})^H)$.

Here we discuss examples with $O_H(p) = 12$.

Example 4.2: We explore the H-orbits of $\mathbb{Q}^{''}(\sqrt{79})$ in the following algorithm.

Step I: First we write $79\text{-}a^2$, $1 \le a \le \lfloor \sqrt{79} \rfloor$ into its prime decomposition in order to find the positive divisors of $79\text{-}a^2$:

$$79-1=78=2.3.13,\ 79-4=75=3.5^2,\ 79-9=70=2.5.7$$
$$79-16=63=3^2.7,\ 79-25=54=2.3^3,\ 79-36=43,$$
$$79-49=30=2.3.5 \text{ and } 79-64=15=3.5$$

Step-II: By Remark 3.6(1) and Lemma 3.12,

$$(\sqrt{79})^H_{amb} \cup (\frac{\sqrt{79}}{-1})^H_{amb} = \{\frac{\pm a+\sqrt{79}}{\pm 1}$$
$$\frac{\pm a+\sqrt{79}}{\pm(p-a^2)}, \frac{\pm a+\sqrt{79}}{\pm 4}, \frac{\pm a+\sqrt{79}}{\pm 2(p-a^2)}, \text{where } 0 \le a \le 8\}$$

Step-III: In the remaining divisors of $p-a^2, 1\le a\le 8$, the smallest odd prime divisors of $p-a^2, 1\le a\le 8$ is 3. So we take

$$\frac{1+\sqrt{79}}{3} \in Q^{*\sim}(\sqrt{4p}) \setminus ((\sqrt{p})^H \cup (\frac{\sqrt{p}}{-1})^H)$$

and then by Lemma 4.1, we have four more H-orbits of $Q^*(\sqrt{79})$ namely, $(\frac{1+\sqrt{79}}{3})^H$, $(\frac{-1+\sqrt{79}}{-3})^G$, $(\frac{1+\sqrt{79}}{-3})^H$ and $(\frac{-1+\sqrt{79}}{3})^H$.

Step-IV: Now

$$\frac{\pm a+\sqrt{79}}{\pm 3}, \frac{\pm a+\sqrt{79}}{\pm(c=\frac{p-a^2}{3})}, \ a=1,2,3,4,5,7,8$$

are contained in

$$A = (\frac{1+\sqrt{79}}{3})^H \cup (\frac{-1+\sqrt{79}}{-3})^H \cup (\frac{1+\sqrt{79}}{-3})^H \cup (\frac{-1+\sqrt{79}}{3})^H$$

where c = 3,5,7,9,13,15,18,21,26,25. Since

$$\frac{\pm 4+\sqrt{79}}{\pm 3}, \frac{\pm 7+\sqrt{79}}{\pm 3}, \frac{\pm 7+\sqrt{79}}{\pm 5}, \frac{\pm 8+\sqrt{79}}{\pm 5}$$
$$\frac{\pm 8+\sqrt{79}}{\pm 3}, \frac{\pm 2+\sqrt{79}}{\pm 5}, \frac{\pm 2+\sqrt{79}}{\pm 3}, \frac{\pm 2+\sqrt{79}}{\pm 25}$$
$$\frac{\pm 3+\sqrt{79}}{\pm 5}, \frac{\pm 2+\sqrt{79}}{\pm 15}, \frac{\pm 1+\sqrt{79}}{\pm 13} \in A$$

Also

$$\frac{\pm 5+\sqrt{79}}{\pm 9}, \frac{\pm 4+\sqrt{79}}{\pm 9}, \frac{\pm 4+\sqrt{79}}{\pm 7}$$
$$\frac{\pm 4+\sqrt{79}}{\pm 21}, \frac{\pm 3+\sqrt{79}}{\pm 7} \text{ and } \frac{\pm 3+\sqrt{79}}{\pm 14} \in A$$

Hence by step (I) and Lemma 2.3, we have

$$Q_1^{*\sim}(\sqrt{4.79}) \setminus ((\sqrt{79})^H \cup (\frac{\sqrt{79}}{-1})^H \cup A) = \varnothing$$

This implies that $o_H^{*\sim}(4p) = 6$.

Example 4.3 $Q^{''}(\sqrt{223})$ splits into twelve H-orbits, namely

$$(\frac{\sqrt{223}}{1})^H, (\frac{\sqrt{223}}{-1})^H, (\frac{1+\sqrt{223}}{3})^H, (\frac{1+\sqrt{223}}{-3})^H$$
$$(\frac{-1+\sqrt{223}}{3})^H, (\frac{-1+\sqrt{223}}{-3})^H, (\frac{1+\sqrt{223}}{2})^H$$
$$(\frac{1+\sqrt{223}}{-2})^H, (\frac{1+\sqrt{223}}{6})^H, (\frac{1+\sqrt{223}}{-6})^H$$
$$(\frac{-1+\sqrt{223}}{6})^H \text{ and } (\frac{-1+\sqrt{223}}{-6})$$

The first six orbits are lying in

$$(Q^*(\sqrt{233}) \setminus Q^{**}(\sqrt{233})) \cup Q^{**}(\sqrt{4.233})$$

whereas the last six orbits are in $Q^{**}(\sqrt{233})$. Since

$$(1/223)=(2/223)=(-6/223)=(-3/233)=1$$

and

$$(3/223)=(6/223)=(-1/223)=(-2/223)=-1$$

so

$$(\frac{\sqrt{223}}{1})^H, (\frac{1+\sqrt{223}}{2})^H, (\frac{\pm 1+\sqrt{223}}{-6})$$

and

$(\frac{\pm 1+\sqrt{223}}{-3})^H$ are in $S_1^{223} \cup x(S_1^{223})$ and the rest of these are in $S_2^{223} \cup x(S_2^{223})$.

Now we state the main theorem of this paper.

Theorem 4.3: Let p ≡ 3 (mod 4). Then the number $o_H^{*\sim}(4p)$ is congruent to 2 modulo 4.

Proof: By Theorems 3.3 and 3.5, if p ≡ 3 (mod 4), then $Q^{*\sim}(\sqrt{4p})$ splits into at least two H-orbits, namely $(\frac{\sqrt{p}}{1})^H$ and $(\frac{\sqrt{p}}{-1})^H$. So

$$B = Q^{*\sim}(\sqrt{4p}) \backslash ((\sqrt{p})^H \cup (\frac{\sqrt{p}}{-1})^H)$$

may or may not be empty. If $B \neq \varnothing$, then $o_H^{*\sim}(4p) = 2$. However if $B \neq \varnothing$, then by Lemma 4.1, we get four more H-orbits, namely $(\alpha)^H, (\bar{\alpha})^H, (-\alpha)^H$ and $(-\bar{\alpha})^H$ for some $\alpha \in B$ with the same ambiguous lengths. Again if $B \backslash ((\alpha)^H \cup (\bar{\alpha})^H \cup (-\alpha)^H \cup (-\bar{\alpha})^H) = \varnothing$, then $o_H^{*\sim}(4p) = 6$, otherwise we continue this process of forming the orbits, each time adding four more orbits in the previous number of orbits. Since ambiguous numbers are finite, so after a finite number of steps all the ambiguous numbers are exhausted in the closed paths of the H-orbits. We get a finite number of H-orbits of $Q^{*\sim}(\sqrt{4p})$. Clearly this number is congruent to 2 modulo 4.

We conclude this paper with the following remark.

Remark 4.4: It has been proved in [11] that if p ≡ 1 (mod 4) then the number $o_H^{*\sim}(4p) \equiv 2 (\text{mod } 4)$ and also Theorem 4.4 gives $o_H^{*\sim}(4p) \equiv 2 (\text{mod } 4)$ for p ≡ 3 (mod 4). Hence the number $o_H^{*\sim}(4p)$ is even for each odd prime p.

Note: The smallest prime p ≡ 3 (mod 4) such that $o_H(p) = 28$ is 1087 and $o_H(p) = 4$ for all primes p ≡ 3 (mod 4) and p≤2011 other than listed in Example 4.6, Table 2-4.

Table 2: Primes p ≡ 3 (mod 8) such that p≤2011, $o_H(p)$ = 9

p	$\tau^*(p)$	p-1	H-orbits α^H of $Q^{\sim}(\sqrt{p})$ with $\lvert\alpha^H\rvert_{amb}$
443	388	2(13)(17)	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=168$, $\lvert(\frac{1+\sqrt{p}}{2})^H\rvert_{amb}=84$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm13})^H\rvert_{amb}=48$, $\lvert(\frac{\pm1+\sqrt{p}}{26})^H\rvert_{amb}=20$
659	556	2(7)(47)	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=216$, $\lvert(\frac{1+\sqrt{p}}{2})^H\rvert_{amb}=108$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm7})^H\rvert_{amb}=80$, $\lvert(\frac{\pm1+\sqrt{p}}{14})^H\rvert_{amb}=36$
1091	612	2(5)(109)	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=264$, $\lvert(\frac{1+\sqrt{p}}{2})^H\rvert_{amb}=132$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm5})^H\rvert_{amb}=72$, $\lvert(\frac{\pm1+\sqrt{p}}{10})^H\rvert_{amb}=36$
1171	1356	$2(3^2)(5)(13)$	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=392$, $\lvert(\frac{1+\sqrt{p}}{2})^H\rvert_{amb}=96$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm3})^H\rvert_{amb}=264$, $\lvert(\frac{\pm1+\sqrt{p}}{6})^H\rvert_{amb}=124$
1627	924	2(3)(271)	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=336$, $\lvert(\frac{1+\sqrt{p}}{2})^H\rvert_{amb}=164$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm3})^H\rvert_{amb}=144$, $\lvert(\frac{\pm1+\sqrt{p}}{6})^H\rvert_{amb}=68$
1787	876	2(19)(47)	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=368$, $\lvert(\frac{1+\sqrt{p}}{2})^H\rvert_{amb}=172$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm19})^H\rvert_{amb}=112$, $\lvert(\frac{\pm1+\sqrt{p}}{38})^H\rvert_{amb}=52$
1811	908	2(5)(181)	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=368$, $\lvert(\frac{1+\sqrt{p}}{2})^H\rvert_{amb}=180$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm5})^H\rvert_{amb}=120$, $\lvert(\frac{\pm1+\sqrt{p}}{10})^H\rvert_{amb}=60$
1987	1164	2(3)(331)	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=400$, $\lvert(\frac{1+\sqrt{p}}{2})^H\rvert_{amb}=196$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm3})^H\rvert_{amb}=192$, $\lvert(\frac{\pm1+\sqrt{p}}{6})^H\rvert_{amb}=92$
1523	684	$7^2(31)$	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=312$, $\lvert(\frac{1+\sqrt{p}}{2})^H\rvert_{amb}=156$, $\lvert(\frac{\pm2+\sqrt{p}}{\pm7})^H\rvert_{amb}=72$, $\lvert(\frac{\pm5+\sqrt{p}}{\pm14})^H\rvert_{amb}=36$
1907	772	11(173)	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=360$, $\lvert(\frac{1+\sqrt{p}}{2})^H\rvert_{amb}=180$, $\lvert(\frac{\pm2+\sqrt{p}}{\pm11})^H\rvert_{amb}=80$, $\lvert(\frac{\pm3+\sqrt{p}}{26})^H\rvert_{amb}=36$

Table 3: Primes $p \equiv 7 \pmod 8$ such that $p \leq 2011$, $o_H(p) = 12$

p	$\tau^*(p)$	p-1	H-orbits α^H of $Q''(\sqrt{p})$ with $\lvert\alpha^H\rvert_{amb}$
79	204	2(3)(13)	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=68$, $\lvert(\frac{1+\sqrt{p}}{\pm2})^H\rvert_{amb}=16$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm3})^H\rvert_{amb}=36$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm6})^H\rvert_{amb}=8$
223	324	2(3)(37)	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=116$, $\lvert(\frac{1+\sqrt{p}}{\pm2})^H\rvert_{amb}=28$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm3})^H\rvert_{amb}=52$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm6})^H\rvert_{amb}=12$
1223	564	2(13)(47)	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=276$, $\lvert(\frac{1+\sqrt{p}}{\pm2})^H\rvert_{amb}=68$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm13})^H\rvert_{amb}=52$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm26})^H\rvert_{amb}=12$
1567	1076	$2(3^3)(29)$	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=364$, $\lvert(\frac{1+\sqrt{p}}{\pm2})^H\rvert_{amb}=88$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm3})^H\rvert_{amb}=80$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm6})^H\rvert_{amb}=44$
1847	676	2(13)(71)	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=340$, $\lvert(\frac{1+\sqrt{p}}{\pm2})^H\rvert_{amb}=84$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm13})^H\rvert_{amb}=60$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm26})^H\rvert_{amb}=12$
359	372	5(71)	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=148$, $\lvert(\frac{1+\sqrt{p}}{\pm2})^H\rvert_{amb}=36$, $\lvert(\frac{\pm2+\sqrt{p}}{\pm5})^H\rvert_{amb}=52$, $\lvert(\frac{\pm3+\sqrt{p}}{\pm10})^H\rvert_{amb}=12$
839	540	5(167)	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=228$, $\lvert(\frac{1+\sqrt{p}}{\pm2})^H\rvert_{amb}=56$, $\lvert(\frac{\pm2+\sqrt{p}}{\pm5})^H\rvert_{amb}=68$, $\lvert(\frac{\pm3+\sqrt{p}}{\pm10})^H\rvert_{amb}=16$
1367	636	29(47)	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=292$, $\lvert(\frac{1+\sqrt{p}}{\pm2})^H\rvert_{amb}=72$, $\lvert(\frac{\pm2+\sqrt{p}}{\pm29})^H\rvert_{amb}=68$, $\lvert(\frac{\pm3+\sqrt{p}}{\pm14})^H\rvert_{amb}=16$

Table 4: Primes $p \equiv 7 \pmod 8$ such that $p \leq 2011$, $o_H(p) = 20$

p	$\tau^*(p)$	$p-a^2$, a = 1,2	H-orbits α^H of $Q''(\sqrt{p})$ with $\lvert\alpha^H\rvert_{amb}$
439	596	438 = 2.3.73	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=164$, $\lvert(\frac{1+\sqrt{p}}{\pm2})^H\rvert_{amb}=40$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm3})^H\rvert_{amb}=68$,
		435 = 3.5.29	$\lvert(\frac{\pm1+\sqrt{p}}{\pm6})^H\rvert_{amb}=16$, $\lvert(\frac{\pm2+\sqrt{p}}{\pm5})^H\rvert_{amb}=52$, $\lvert(\frac{\pm3+\sqrt{p}}{\pm10})^H\rvert_{amb}=12$
499	748	498 = 2.3.83	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=192$, $\lvert(\frac{1+\sqrt{p}}{2})^H\rvert_{amb}=92$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm3})^H\rvert_{amb}=96$,
		$495 = 3^2.5.11$	$\lvert(\frac{\pm1+\sqrt{p}}{6})^H\rvert_{amb}=44$, $\lvert(\frac{\pm2+\sqrt{p}}{\pm9})^H\rvert_{amb}=64$, $\lvert(\frac{\pm3+\sqrt{p}}{14})^H\rvert_{amb}=28$
727	716	$2.3.11^2$	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=212$, $\lvert(\frac{1+\sqrt{p}}{\pm2})^H\rvert_{amb}=52$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm3})^H\rvert_{amb}=84$,
			$\lvert(\frac{\pm1+\sqrt{p}}{\pm6})^H\rvert_{amb}=20$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm11})^H\rvert_{amb}=52$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm22})^H\rvert_{amb}=12$
1327	1156	1326 = 2.3.13.17	$\lvert(\frac{\sqrt{p}}{\pm1})^H\rvert_{amb}=$, $\lvert(\frac{1+\sqrt{p}}{\pm2})^H\rvert_{amb}=316$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm3})^H\rvert_{amb}=148$,
			$\lvert(\frac{\pm1+\sqrt{p}}{\pm6})^H\rvert_{amb}=36$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm13})^H\rvert_{amb}=84$, $\lvert(\frac{\pm1+\sqrt{p}}{\pm26})^H\rvert_{amb}=20$

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