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Within Differential Operators P-Valent Uniformly Convex Functions with Negative Coefficients

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Abstract: In the open disc $u_w = \{z : |z - w| < 1\}$ centered at w with radius 1 some generalizations of subclasses of analytic functions f within the differential operator $\nabla_w^k f(z) = f^{(k-1)}(z) + \beta\,(z-w)\,f^{(k)}(z)$ are introduced. Coefficient bounds of w-p-valent uniformly subclasses within the linear operator ∇_w^k are calculated. Some properties of the growth and distortion are discussed. Results of Hadmard product and generalized Hadmard product are obtained.

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INTRODUCTION

Let $A(p)$ be the class of all analytic functions which are defined in the unit disc $u = \{z : |z| < 1\}$ and can be written in the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}\, z^{n+p}, \quad a_{n+p} \geq 0 \ \ and \ \ p, n \in \mathbb{N} = \{1,2,3,4,...\}$$

Among the last century several authors have studied many valuable and interesting results of univalent functions, even with various generalizations as appeared in many literatures and articles. For a fixed point w in the unit disc u, Kanas and Ronning [1] introduced a more generalization form of analytic functions in the unit disc u of the form

$$f(z) = (z-w) + \sum_{k=2}^{\infty} a_k (z-w)^k, \qquad a_k \in \mathcal{C}$$

which are denoted by $\Gamma(w)$ $ST(w)$ and $CV(w)$ and they obtained some results related to the other univalent functions. Acu and Owa [2] introduced bounds for classes of ω-close-to-convex functions, ω-α-convex functions and other further studies of these classes. Al-Kasasbeh and Darus [3, 4, 5] introduced classes of analytic univalent functions that are defined in the open disc $u_w = \{z : |z - w| < 1\}$ and they proved corresponding results to these classes. The concept of uniformly convexness for analytic and univalent functions was introduced by Goodman [6, 7] and investigated by several authors (e.g see Ronning [8], Bharati *et al*. [9] and Ma and Minda [10]. The classes of uniformly convex functions are defined by

$$\mathcal{CD}(\delta,\gamma) = \{f \in A(p) : \mathrm{Re}\left(\frac{(zf'(z))'}{f'(z)}\right) > \delta\left|\frac{(zf'(z))'}{f'(z)} - 1\right| + \gamma, z \in \mathcal{U}, \delta \geq 0, \gamma \in [0,1)\},$$

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and lately, Nishiwaki and Owa [12] studied the classes of p-valent uniformly starlike and univalent functions which are defined by

$$\mathcal{CD}_p(\delta,\gamma)=\{f\in A(p):\mathrm{Re}\left(\frac{(zf'(z))'}{f(z)}\right)>\delta\left|\frac{(zf'(z))'}{f'(z)}-p\right|+\gamma, z\in\mathcal{U},\delta\geq 0,\gamma\in[0,p)\}.$$

In the recent article, for any complex number w in the open disc $u_w=\{z:|z-w|<1\}$, the class $A_w(p)$ is defined to be w-p-valent analytic functions

$$f(z)=(z-w)^p+\sum_{n=1}^{\infty}a_{n+p}\,(z-w)^{n+p},\quad a_{n+p}\in\mathbb{C}\ \ \text{and}\ \ p,n\in\mathbb{N},$$

and denote $T_w(p)$ subclass of $A_w(p)$ of all functions with the negative coefficients

$$f(z)=(z-w)^p-\sum_{n=1}^{\infty}a_{n+p}\,(z-w)^{n+p},\quad a_{n+p}\geq 0\ \ \text{and}\ \ p,n\in\mathbb{N}.$$

Also in the open disc u_w, for $\delta\geq 0$ and $0\leq\gamma<p$ the subclasses $\mathcal{CD}_p^w(\delta,\gamma)$ of $T_w(p)$ are defined to be all w-p-valently uniformly convex functions

$$\mathcal{CD}_p^w(\delta,\gamma)=\{f\in\mathcal{T}_w(p):\mathrm{Re}\left(\frac{((z-w)f'(z))'}{f'(z)}\right)>\delta\left|\frac{((z-w)f'(z))'}{f'(z)}-p\right|+\gamma\ \}.$$

For a nonnegative parameter β and $k\in\mathbb{R}$, Al-Kasasbeh and Darus [3, 4] defined the linear differential operator ∇_w^k for an analytic function f in the open disc $u_w=\{z:|z-w|<1\}$ to be $\nabla_w^k f(z)=f^{(k-1)}(z)+\beta\,(z-w)\,f^{(k)}(z)$. The differential operator ∇_w^k within the class of w-p-valently uniformly convex functions is introduced as follows.

Definition 1.1:. *Let* $(z)\in T_w(p)$. *Then* $f(z)\in\mathcal{CD}\nabla_w^k(p,\gamma,\delta,\beta)$ *if and only if*

$$\mathrm{Re}\left(1+\frac{(z-w)\nabla_w^{k+1}f(z)}{\nabla_w^k f(z)-f^{k-1}(z)}-\frac{1}{\beta}\right)>\delta\left|1+\frac{(z-w)\nabla_w^{k+1}f(z)}{\nabla_w^k f(z)-f^{k-1}(z)}-\frac{1}{\beta}-p\right|+\gamma,$$

for $u\in u_w$, $\delta\geq 0$ and $0\leq\gamma<p$.

The class $\mathcal{CD}\nabla_w^k(p,\gamma,\delta,\beta)$ is a generalization of various subclasses of univalent functions, it is easy to note that if $k=1$, $w=0$ and $p=1$, then $\mathcal{CD}\nabla_0^1(1,\gamma,\delta,\beta)\equiv\mathcal{CD}(\delta,\gamma)$ due to Shams *et al*. [11] in the unit disc u_0. Also, if $k=1$, $w=0$ and $p\in\mathbb{N}$, then $\mathcal{CD}\nabla_0^1(p,\gamma,\delta,\beta)\equiv\mathcal{CD}_p(\delta,\gamma)$ due to Nishiwaki and Owa [12] in the unit disc u_0.

The Main Results: The coefficient bounds for a function f in the class $\mathcal{CD}\nabla_w^k(p,\gamma,\delta,\beta)$ which is analytic in the open disc u_w are estimated.

Theorem 2.1: *A function* $f(z)\in T_w(p)$ *is belonging to the class* $\mathcal{CD}\nabla_w^k(p,\gamma,\delta,\beta)$ *if and only if*

$$\sum_{n=1}^{\infty}a_{n+p}\leq\frac{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}. \tag{2.1}$$

Proof: Since $f(z)\in \mathcal{CD}\nabla_w^k(p,\gamma,\delta,\beta)$, for $\delta\geq 0$, $0\leq\gamma<p$ and $z\in u$ then

$$\operatorname{Re}\left(1+\frac{(z-w)\nabla_w^{k+1}f(z)}{\nabla_w^k f(z)-f^{k-1}(z)}-\frac{1}{\beta}\right)-\delta\left|1+\frac{(z-w)\nabla_w^{k+1}f(z)}{\nabla_w^k f(z)-f^{k-1}(z)}-\frac{1}{\beta}-p\right|>\gamma,$$

and

$$\left|f^{(k)}(z)+(z-w)f^{(k+1)}(z)\right|-\delta\left|(1-p)f^{(k)}(z)+(z-w)\ f^{(k+1)}(z)\right|>\gamma\,|f^{(k)}(z)|\,.$$

Assume that $(z-w)$ approaches to 1, since it is observed that $|z-w|<1$ in the disc u_w, then

$$\sum_{n=1}^{\infty}a_{n+p}\prod_{i=1}^{k}(p+n+1-i)[n+p+1-\gamma-\delta(n+2)]\leq[p-\gamma-2(\delta-1)]\prod_{i=1}^{k}(p+1-i).$$

So that for $0\leq\gamma<p$ and $\delta\geq 0$

$$a_{n+p}\leq\frac{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}$$

Conversely, in the view of definition,

$$\delta\left|1+\frac{(z-w)\nabla_w^{k+1}f(z)}{\nabla_w^k f(z)-f^{k-1}(z)}-\frac{1}{\beta}-p\right|-\operatorname{Re}\left(1+\frac{(z-w)\nabla_w^{k+1}f(z)}{\nabla_w^k f(z)-f^{k-1}(z)}-\frac{1}{\beta}\right)\leq-\gamma,$$

which is equivalent to

$$\delta\left|1+\frac{(z-w)\nabla_w^{k+1}f(z)}{\nabla_w^k f(z)-f^{k-1}(z)}-\frac{1}{\beta}-p\right|-\operatorname{Re}\left(1+\frac{(z-w)\nabla_w^{k+1}f(z)}{\nabla_w^k f(z)-f^{k-1}(z)}-\frac{1}{\beta}-1\right)\leq 1-\gamma.$$

Therefor

$$\delta\left|1+\frac{(z-w)\nabla_w^{k+1}f(z)}{\nabla_w^k f(z)-f^{k-1}(z)}-\frac{1}{\beta}-p\right|-\operatorname{Re}\left(1+\frac{(z-w)\nabla_w^{k+1}f(z)}{\nabla_w^k f(z)-f^{k-1}(z)}-\frac{1}{\beta}-1\right)$$

$$\leq(\delta+1)\left(1+\frac{(z-w)\nabla_w^{k+1}f(z)}{\nabla_w^k f(z)-f^{k-1}(z)}-\frac{1}{\beta}-p\right)$$

$$\leq(\delta+1))\left(\frac{\prod_{i=1}^{k}(p+1-i)(p(1-2\delta)+2)-\sum_{n=1}^{\infty}|a_{n+p}|\prod_{i=1}^{k}(p+n+1-i)(1+\delta(n+2))}{\prod_{i=1}^{k}(p+1-i)-\sum_{n=1}^{\infty}|a_{n+p}|\prod_{i=1}^{k}(p+n+1-i)}\right)$$

$$\leq (\delta+1))\left(\frac{\prod_{i=1}^{k}(p+1-i)(p(1-2\delta)+2)-\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)](1+\delta(n+2))}{[p+n+1-\gamma-\delta(n+2)]}}{\prod_{i=1}^{k}(p+1-i)-\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}{[p+n+1-\gamma-\delta(n+2)]}}\right).$$

Since there is $\gamma \in [0, p]$ and $\delta \geq 0$ such that

$$\left(\frac{\prod_{i=1}^{k}(p+1-i)(p(1-2\delta)+2)-\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)](1+\delta(n+2))}{[p+n+1-\gamma-\delta(n+2)]}}{\prod_{i=1}^{k}(p+1-i)-\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}{[p+n+1-\gamma-\delta(n+2)]}}\right)\leq\left(\frac{1-\gamma}{\delta+1}\right).$$

Then

$$\delta\left|1+\frac{(z-w)\nabla_w^{k+1}f(z)}{\nabla_w^k f(z)-f^{k-1}(z)}-\frac{1}{\beta}-p\right|-\mathrm{Re}\left(1+\frac{(z-w)\nabla_w^{k+1}f(z)}{\nabla_w^k f(z)-f^{k-1}(z)}-\frac{1}{\beta}-1\right)$$

$$\leq(\delta+1)\left(\frac{1-\gamma}{\delta+1}\right)=1-\gamma.$$

Which is equivalent to

$$\delta\left|\frac{(z-w)\nabla_w^{k}f(z)}{\nabla_w^{k-1} f(z)-f^{(k-2)}(z)}-\frac{1}{\beta}-p\right|-\mathrm{Re}\left(\frac{(z-w)\nabla_w^{k}f(z)}{\nabla_w^{k-1} f(z)-f^{(k-2)}(z)}-\frac{1}{\beta}\right)\leq-\gamma.$$

Thus $f(z)\in\mathcal{SD}\nabla_w^k(p,\gamma,\delta,\beta)$ and the result is sharp for

$$f(z)=(z-w)^p-\frac{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}(z-w)^{p+n}.\square$$

Next corollary discusses the distortion and growth properties.

Theorem 2.2: *Let* $f(z)\in\mathcal{CD}\nabla_w^k(p,\gamma,\delta,\beta)$ *, for* $z \in u_w = \{z : r{=}|z-w|{<}1\}$. *Then*

$$r^p-\left(\frac{3-(2\delta+\gamma)}{2(3-(3\delta+\gamma))}\right)r^{p+1}\leq|f(z)|\leq r^p+\left(\frac{3-(2\delta+\gamma)}{2(3-(3\delta+\gamma))}\right)r^{p+1}$$

and

$$pr^{p-1}-\left(\frac{3-(2\delta+\gamma)}{2(3-(3\delta+\gamma))}\right)(p+1)r^{p}\le\left|f'(z)\right|\le pr^{p-1}+\left(\frac{3-(2\delta+\gamma)}{2(3-(3\delta+\gamma))}\right)(p+1)r^{p}$$

with equality for

$$f(z)=(z-w)^{p}-\frac{3-(2\delta+\gamma)}{2(3-(3\delta+\gamma))}(z-w)^{p+1}.$$

Proof: For $p = n = 1$ in (2.1), we have

$$\sum_{n=1}^{\infty}a_{n+p}\le\frac{3-(2\delta+\gamma)}{2(3-(3\delta+\gamma))}\tag{2.2}$$

Thus,

$$\left|f(z)\right|\le r^{p}+\sum_{n=1}^{\infty}a_{n+p}\,r^{p+1}\le r^{p}+r^{p+1}\sum_{n=1}^{\infty}a_{n+p}\le r^{p}+\frac{3-(2\delta+\gamma)}{2(3-(3\delta+\gamma))}r^{p+1},$$

and

$$\left|f(z)\right|\ge r^{p}-\sum_{n=1}^{\infty}a_{n+p}\,r^{p+1}\ge r^{p}-r^{p+1}\sum_{n=1}^{\infty}a_{n+p}\ge r^{p}-\frac{3-(2\delta+\gamma)}{2(3-(3\delta+\gamma))}r^{p+1}.$$

Also, from (2.2) and Theorem 2.1, it follows that

$$\sum_{n=1}^{\infty}(p+n)a_{n+p}\le\frac{3-(2\delta+\gamma)}{2(3-(3\delta+\gamma))}.$$

For $r = |z-w| < 1$, we have

$$\left|f'(z)\right|\le p|z-w|^{p-1}+\sum_{n=1}^{\infty}(p+1)a_{n+p}|z-w|^{p}\le pr^{p-1}+(p+1)r^{p+1}\sum_{n=1}^{\infty}a_{n+p}$$
$$\le pr^{p-1}+\frac{3-(2\delta+\gamma)}{2(3-(3\delta+\gamma))}(p+1)r^{p},$$

and

$$\left|f'(z)\right|\ge p|z-w|^{p-1}-\sum_{n=1}^{\infty}(p+1)\,a_{n+p}|z-w|^{p}\ge p\,r^{p-1}-(p+1)\,r^{p}\sum_{n=1}^{\infty}a_{n+p}$$
$$\ge p\,r^{p-1}-\frac{3-(2\delta+\gamma)}{2(3-(3\delta+\gamma))}(p+1)\,r^{p}.$$

This complete the proof of the theorem. □

Theorem 2.3: *Suppose that* $f_1(z) = (z-w)^p$ *and for each positive integer* $n \geq 0$

$$f_n(z) = (z-w)^p - \frac{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}(z-w)^{n+p}, \quad for\ z \in \mathcal{U}_w.$$

Then $f(z) \in \mathcal{CD}\nabla_w^k(p,\gamma,\delta,\beta)$ if and only if f can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty}\mu_{n+1} f_{n+1}(z) \text{ where } \mu_n \geq 0 \text{ and } \sum_{n=0}^{\infty}\mu_{n+1} = 1.$$

Proof: Assume that $f(z) = \sum_{n=0}^{\infty}\mu_{n+1} f_{n+1}(z)$

$$= (z-w)^p - \sum_{n=1}^{\infty}\mu_{n+1} a_{n+p}(z-w)^{n+p}$$

Since

$$\sum_{n=1}^{\infty}\mu_{n+1}\left(\frac{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}\right)\left(\frac{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}\right)$$

$$\leq \sum_{n=1}^{\infty} \mu_{n+1} \leq 1-\mu_1 \leq 1.$$

By Theorem 2.1 $f \in \mathcal{CD}\nabla_w^k(p,\gamma,\delta,\beta)$.
Conversely, let $f \in \mathcal{CD}\nabla_w^k(p,\gamma,\delta,\beta)$. Then

$$a_{n+p} \leq \frac{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}, \quad for \quad n+p \geq 2.$$

Without any loss of generality, assume that

$$\mu_{n+1} = \frac{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}a_{n+p}, \quad for\ n+p \geq 2,$$

and $\mu_1 = 1 - \sum_{n=1}^{\infty} \mu_{n+1}$. Then

$$f(z) = (z-w)^p - \sum_{n=1}^{\infty} \mu_{n+1} a_{n+p} (z-w)^{n+p}$$

$$= (z-w)^p - \sum_{n=1}^{\infty} \mu_{n+1} \frac{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}(z-w)^{n+p}$$

$$= (z-w)^p - \sum_{n=1}^{\infty} \mu_{n+1}[(z-w)^{n+p} - f_{n+1}(z)]$$

$$= (1 - \sum_{n=1}^{\infty} \mu_{n+1})(z-w)^{n+p} + \sum_{n=1}^{\infty} \mu_{n+1} f_{n+1}(z)$$

$$= \mu_1 (z-w)^{n+p} + \sum_{n=1}^{\infty} \mu_{n+1} f_{n+1}(z)$$

$$= \mu_1 f_1(z) + \sum_{n=1}^{\infty} \mu_{n+1} f_{n+1}(z) = \sum_{n=1}^{\infty} \mu_n f_n(z). \square$$

Let $f_i(z) \in T_w(p)$ for $i = 1,2,...,m$ be given by

$$f_i(z) = (z-w)^p - \sum_{n=1}^{\infty} a_{n+p,i}\ (z-w)^{n+p}.$$

Then the Hadamard product is defined by

$$f_1 * f_1(z) * f_2(z) * ...... * f_m(z) = (f_1 * f_1 * f_2 * ...... * f_m)(z)$$
$$= (z-w)^p - \sum_{n=1}^{\infty} (\prod_{i=1}^{m} (a_{n+p,i}))\ (z-w)^{n+p},$$

and the generalized Hadamard product is defined by

$$(f_1 \bullet f_1 \bullet f_2 \bullet ...... \bullet f_m)(z) = (z-w)^p - \sum_{n=1}^{\infty} (\prod_{i=1}^{m} (a_{n+p,i})^{\frac{1}{q_i}})\ (z-w)^{n+p}$$

where $\sum_{n=1}^{m} \frac{1}{q_i} = 1,$ and $q_i > 1$ for $i = 1,2,...,m$.

Next the Hadmard product results are presented as follows.

Theorem 2.4: *Suppose that* $f_i(z) \in \mathcal{CD}\nabla_w^k(p,\gamma,\delta,\beta)$ *, for each* $i = 1,2,...,m$ *. Then*

$(f_1 * f_2 * ... * f_m)(z) \in \mathcal{CD}\nabla_w^k(\hat{p},\gamma,\delta,\beta)$,

where $\hat{p} = max((\prod_{i=1}^{k}(\frac{p+n+1-i}{p+1-i}))^{-\frac{1}{2}})$.

Proof: To show the result is true for the positive integer m+1, assume the result is true for the positive integer m and $f_i(z) \in \mathcal{CD}\nabla_w^k(\hat{p},\gamma,\delta,\beta)$, fo1r each i = 1,2,...,m. Then

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1)[p+n+1-\gamma-\delta(n+2)]}{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}(\prod_{i=1}^{m}a_{n+p,i}) \le 1 \tag{2.1}$$

So that

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}(\prod_{i=1}^{m+1}a_{n+p,i}) \le$$

$$max\{\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}a_{n+p,1},....,\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}a_{n+p,m}$$

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}a_{n+p,m+1}\} \le$$

$$max\{\sum_{n=1}^{\infty}\frac{\hat{p}[p+n+1-\gamma-\delta(n+2)]}{[p-\gamma-2(\delta-1)]}a_{n+p,i},\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}a_{n+p,m+1}\} \le 1$$

Our assumption is completed by $\hat{p} \ge p$ such that $\hat{p} = max((\prod_{i=1}^{k}\frac{p+n+1-i}{p+1-i})^{-\frac{1}{2}})$ and apply the Cauchy-Schwarz inequality on (2.1) to get

$$\sum_{n=1}^{\infty}\hat{p}\frac{[p+n+1-\gamma-\delta(n+2)]}{[p-\gamma-2(\delta-1)]}\sqrt{(\prod_{i=1}^{m}a_{n+p,i})}\le 1 \text{ if and only if } \hat{p}\frac{[p+n+1-\gamma-\delta(n+2)]}{[p-\gamma-2(\delta-1)]}\le 1.$$

It is observed that $\theta(n)=\frac{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}$ is a decreasing quantity, since

$$\theta(n)=\frac{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}\le\varphi(2).$$

Therefore, by mathematical induction, the result is true for any positive integer m. Hence $(f_1 * f_2 .... * f_m)(z)\in \mathcal{CD}\nabla_w^k(\hat{p},\gamma,\delta,\beta)$.□

Theorem 2.5: *Suppose that* $f_i(z)\in\mathcal{CD}\nabla_w^k(p,\gamma,\delta,\beta)$ *, for each* $i = 1,2,...,m$*. Then*

$(f_1\bullet f_1\bullet f_2\bullet ......\bullet f_m)(z)\in\mathcal{CD}\nabla_w^k(\dot{p},\gamma,\delta,\beta),$

where $\dot{p}=max((\prod_{i=1}^{k}(\frac{p+n+1-i}{p+1-i}))^{-\frac{1}{q_i}})$.

Proof: Assume that $f_i(z)\in\mathcal{CD}\nabla_w^k(p,\gamma,\delta,\beta)$, for each $i = 1,2,3,...,m$. Then

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}a_{n+p,i}\le 1.$$

And

$$\prod_{i=1}^{m}\left(\sum_{n=1}^{\infty}\left[(\frac{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]})^{\frac{1}{q_i}}(a_{n+p,i})^{\frac{1}{q_i}}\right]^{q_i}\right)^{\frac{1}{q_i}}\le 1$$

By using the Holder inequality, we have

$$\sum_{n=1}^{\infty}\left(\prod_{i=1}^{m}\left[\frac{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}\right]^{\frac{1}{q_i}}(a_{n+p,i})^{\frac{1}{q_i}}\right)$$

$$\leq \prod_{i=1}^{m}\left(\sum_{n=1}^{\infty}\left[(\frac{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]})^{\frac{1}{q_i}}(a_{n+p,i})^{\frac{1}{q_i}}\right]^{q_i}\right)^{\frac{1}{q_i}}$$

So that

$$\sum_{n=1}^{\infty}\left(\prod_{i=1}^{m}\left[\frac{\prod_{i=1}^{k}(p+n+1-i)[p+n+1-\gamma-\delta(n+2)]}{\prod_{i=1}^{k}(p+1-i)[p-\gamma-2(\delta-1)]}\right]^{\frac{1}{q_i}}(a_{n+p,i})^{\frac{1}{q_i}}\right)\leq 1.$$

Now, if $\dot{p}=max((\prod_{i=1}^{k}(\frac{p+n+1-i}{p+1-i}))^{-\frac{1}{q_i}})$, then

$$\sum_{n=1}^{\infty}\left(\left[\frac{\dot{p}[p+n+1-\gamma-\delta(n+2)]}{[p-\gamma-2(\delta-1)]}\right]\prod_{i=1}^{m}(a_{n+p,i})^{\frac{1}{q_i}}\right)\leq 1.$$

which is completed our proof.□

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