

## An Accurate Analytic Solution for Differential and Integral Equations by Modified Homotopy Perturbation Method

<sup>1</sup>M.S.H. Chowdhury, <sup>1</sup>Nur Isnida Razali, <sup>1</sup>Selami Ali and <sup>2</sup>M.M. Rahman

<sup>1</sup>Department of Science in Engineering, Kuliyah of Engineering,  
International Islamic University Malaysia, P.O. Box 10, 50728 Kuala Lumpur, Malaysia

<sup>2</sup>Department of Physics, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor Darul Ehsan, Malaysia

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**Abstract:** In this paper, the new modified Homotopy Perturbation Method (HPM) is applied for analytical treatment of differential equations and integral equations. The new modified HPM yields an analytical solution in terms of a rapidly convergent infinite power series with easily computable terms. The efficiency of the new modified technique is examined by several illustrative examples. In all cases of differential and integral equations, the new modified HPM yields the exact solutions in minimal iterations only.

**Key words:** Homotopy-perturbation method . modified HPM . differential equations . integral equations

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### INTRODUCTION

Euler is known as the father of perturbation methods since J. Euler (1707-1783) and J.L. Lagrange (1736-1813) were the first to apply perturbation methods in celestial mechanics and Perturbation methods belong to perhaps, the most romantic area of modern mathematics. Various perturbation methods have been widely applied to solve nonlinear problems in mechanics, physics and other exact sciences. The coupling of the perturbation method and the homotopy method is called the Homotopy Perturbation Method (HPM). The HPM is the most effective and convenient method for both ODEs and PDEs without mathematical difficulties. The application of the Homotopy-Perturbation Method (HPM) in nonlinear problems which has been devoted by scientists and engineers, because this method is to continuously deform a difficult problem into a set of problems which is easier to solve. The method yields rapid convergent series solutions in most cases.

In recent years, much attention has been devoted to the study of the Homotopy-Perturbation Method (HPM) [1-8] for solving a wide range of problems whose mathematical models yield differential equation or system of differential equations. HPM deforms a difficult problem into a set of problems which are easier to solve without any need to transform nonlinear terms. The applications of HPM in nonlinear problems have been demonstrated by many researchers, cf. [9-12]. Recently, HPM was employed for solving singular second-order differential equations [13], nonlinear population dynamics models [14] and time-dependent Emden-Fowler type equations [15], the Klein-Gordon and sine-Gordon equations [16]. Very recently, Chowdhury *et al.* [17] were the first to successfully apply the multistage Homotopy-Perturbation Method (MHPM) to the chaotic Lorenz system and Odibat [18] propose a new modification of the HPM for linear and nonlinear operators. Chowdhury *et al.* [19] proposed modified HPM for solving differential and integral equations.

Wazwaz [20, 21] proposed a new modification of the Adomian Decomposition Method (ADM) to handle the linear and nonlinear operators.

In this work, we will present an alternative approach called new modified HPM for finding series solutions to linear and nonlinear differential and integral equations. The efficiency and accuracy of modified HPM and new modified HPM are demonstrated through several test examples.

### METHODOLOGY

Homotopy-Perturbation Method (HPM) is a reliable and effective method for solving various nonlinear problems. In this section, first we shall present a review of the standard HPM and the modified HPM for

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**Corresponding Author:** M.S.H. Chowdhury, Department of Science in Engineering, Kuliyah of Engineering, International Islamic University Malaysia, P.O. Box: 10, 50728, Kuala Lumpur, Malaysia

convenience of the readers and finally we will introduce a new modification of HPM to handle linear and nonlinear inhomogeneous differential equations.

To do so, we consider the following general nonlinear differential equation

$$Lu + Ru + Nu = g(x) \quad (1)$$

where  $L$  is the highest order derivative which is assumed to be easily invertible,  $R$  the linear differential operator of order less than  $L$ ,  $Nu$  represents the nonlinear terms and  $g$  is the source term.

According to the HPM, we construct a homotopy of Eq. (1) which satisfies

$$H(u, p) = L(u) - L(v_0) + pL(v_0) + p[R(u) + N(u) - g(x)] = 0 \quad (2)$$

where  $p \in [0, 1]$  is an embedding parameter and  $u_0 = v_0$  is an initial approximation which satisfies boundary conditions. When we put  $p = 0$  and  $p = 1$  in Eq. (2), we get

$$H(u, 0) = L(u) - L(v_0) = 0 \text{ and } H(u, 1) = Lu + Ru + Nu - g(x) = 0 \quad (3)$$

which are the linear and nonlinear original equations respectively. In topology this called deformation and  $L(u) - L(u_0)$  and  $Lu + Ru + Nu - g(x)$  are called homotopic. Supposing the solution of (1) can be expressed as

$$u(x) = \sum_{n=0}^{\infty} p^n u_n = u_0(x) + pu_1(x) + p^2 u_2(x) + p^3 u_3(x) + \dots \quad (4)$$

According to HPM, the approximate solution of Eq. (4) can be expressed as a series of the power of  $p$ , i.e.,

$$u = \lim_{p \rightarrow 1} u = u_0 + u_1 + u_2 + u_3 \dots \quad (5)$$

The series (5) is convergent in most of the cases. However the rate of convergence depends on  $L(u)$ .

Now we substitute (4) into (2) and equating the like terms of  $p$ , we obtain

$$u_0(x) = L^{-1}(g(x)) + \phi(x) = f(x)$$

$$p^{k+1}: u_{k+1}(x) = -L^{-1}(Ru_k) - L^{-1}(Nu_k) = -L^{-1}(Ru_k) - L^{-1}(H_k), k \geq 0 \quad (6)$$

where the function  $f(x)$  represents the terms arising from integrating the source term  $g(x)$  and from using the given conditions,  $\phi(x)$ , all of which are assumed to be prescribed.

The nonlinear term  $Nu_k = F(u)$  is usually represented by an infinite series of the so-called He's polynomials [22],

$$F(u) = \sum_{k=0}^{\infty} H_k$$

The polynomials  $H_k$  are generated for all kinds of nonlinearity so that  $A_0$  depends only on  $u_0$ ,  $A_1$  depends on  $u_0$  and  $u_1$  and so on. The He's polynomial  $A_k(u_0, u_1, u_2, \dots, u_k)$  [22], is given by,

$$H_k = \frac{1}{k!} \frac{d^k}{dp^k} \left[ N \left( \sum_{i=0}^k p^i u_i \right) \right]_{p=0} \quad (A)$$

**The modified HPM:** In our earlier works [19] we have introduced an alternative of choosing the initial approximations is

$$v_0 = L^{-1}(g(x)) + \phi(x) = f(x) \quad (7)$$

The modified form is based on the assumption that the initial approximation  $v_0$  given in Eq. (7) can be decomposed into two parts, namely  $f_0$  and  $f_1$  such that  $f = f_0 + f_1$ .

The MHPM have suggested slight variation in the standard HPM on the components  $u_0$  and  $u_1$ . The suggestion is that only the part  $f_0$  be combined with the component  $u_0$  and  $f_1$  be added with  $u_1$ . Under this assumption Eqn. (6) become as follows

$$\begin{aligned} u_0(x) &= f_0(x) \\ p^1: u_1 &= f_1 - L^{-1}(Ru_0) - L^{-1}(Nu_0) \\ p^{k+2}: u_{k+2}(x) &= -L^{-1}(Ru_{k+1}) - L^{-1}(Nu_{k+1}) = -L^{-1}(Ru_{k+1}) - L^{-1}(H_{k+1}), k \geq 0 \end{aligned} \quad (8)$$

The zeroth component  $u_0$  in the recursive scheme of the standard HPM (6) is defined by the total function  $f(x)$ , but in recursive scheme (8) of the modified HPM the zeroth component  $u_0$  is defined only by a part  $f_0(x)$  of  $f(x)$ . And the remaining part  $f_1(x)$  of  $f(x)$  is added to the component  $u_1$  in (8). The small difference of reducing the number of terms of  $u_0$  could reduce the computational work. Furthermore, because of the dependence of the He's polynomials on the initial component  $u_0$  in the nonlinear equations, the reduction of terms in  $u_0$  could reduce calculations. Additional, this small difference in the components  $u_0$  and  $u_1$  may give the exact solution by using two iterations only. However, the success of the MHPM depends completely on the correct selection of the function  $f_0$  and  $f_1$ , here the trials are the only technique that can be used.

**The new modified HPM:** Inspired by Wazwaz and Sayed [20], we propose the  $f(x)$  can be expand by a series of infinite components. Under this idea we can expressed  $f(x)$  in Taylor series as

$$f(x) = \sum_{n=0}^{\infty} f_n(x)$$

Under this hypothesis Eqn. (6) can be written as follows

$$\begin{aligned} u_0(x) &= f_0(x) \\ p^{k+1}: u_{k+1} &= f_{k+1}(x) - L^{-1}(Ru_k) - L^{-1}(Nu_k) = f_{k+1}(x) - L^{-1}(Ru_k) - L^{-1}(H_k), k \geq 0 \end{aligned} \quad (9)$$

If  $f(x)$  consists of one term only then the scheme (9) convert to (6) and if  $f(x)$  consists of two terms then the scheme (9) convert to (8). It is obvious that the algorithm (9) of the new modification of HPM reduces the number of terms involved in every component and also minimized the size of calculations compare to the standard HPM. Moreover, this reduction of terms in each element facilitates the construction of He's polynomials.

## APPLICATIONS TO DIFFERENTIAL EQUATIONS

To present a clear impression of this works, we choose two numerical examples of linear and nonlinear differential equations. The first example will be examined by both modified HPM and our proposed new modified HPM and the second example of nonlinear differential equations will be tested using the new modification of HPM presented above.

**Example 1:** First we consider the linear partial differential equation [20]

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + u = 0, \quad u(x, 0) = 1 + \sin x, \quad \frac{\partial u}{\partial t}(x, 0) = 0 \quad (10)$$

Now we rewrite Eq. (10) in operator form as follows

$$Lu + u_{xx} + u = 0 \quad (11)$$

where  $L = \frac{\partial^2}{\partial t^2}$  and  $L^{-1} = \int_0^t \int_0^t (\bullet) ds ds$  is a two-fold integral operator.

We construct a homotopy of the Eqn. (11) which satisfies the following relation

$$L(u) - L(v_0) + pL(v_0) + p[u_{xx} + u] = 0 \quad (12)$$

**The modified HPM:** To apply the modified HPM, let us take the initial approximation,

$$v_0 = L^{-1}(0) + (1 + \sin x) = f(x) = f_0 + f_1$$

where  $f_0 = 1$  and  $f_1 = \sin x$ .

The iterative formula based on (8) we obtain,

$$\begin{aligned} u_0(x, t) &= 1 \\ p^1: u_1(x, t) &= \sin x - L^{-1}(u_0 + (u_0)_{xx}) = \sin x - \int_0^t \int_0^t (1 + 0) dt dt = \sin x - \frac{1}{2}t^2 \\ p^2: u_2(x, t) &= -L^{-1}(u_1 + (u_1)_{xx}) = -L^{-1}\left(\sin x - \frac{1}{2}t^2 - \sin x\right) = \frac{1}{4!}t^4 \\ p^3: u_3(x, t) &= -L^{-1}(u_2 + (u_2)_{xx}) = -\frac{1}{6!}t^6 \\ p^4: u_4(x, t) &= -L^{-1}(u_3 + (u_3)_{xx}) = \frac{1}{8!}t^8 \\ &\vdots \end{aligned}$$

Hence the series solution is given by

$$u(x, t) = \sin x + \left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \frac{1}{8!}t^8 + \dots\right)$$

and this will be needed more terms to yield the close-form solution  $u(x, t) = \sin x + \cos t$ .

**The new modified HPM:** The Taylor expansion of  $f(x) = 1 + \sin x$  is

$$f(x) = 1 + x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \quad (13)$$

The iterative formula based on (9) we obtain

$$\begin{aligned} u_0(x, t) &= 1 \\ p^1: u_1(x, t) &= x - L^{-1}(u_0 + (u_0)_{xx}) = x - \frac{1}{2!}t^2 \\ p^2: u_2(x, t) &= -\frac{1}{3!}x^3 - L^{-1}(u_1 + (u_1)_{xx}) = -\frac{1}{3!}x^3 - \frac{1}{2}xt^2 + \frac{1}{4!}t^4 \\ p^3: u_3(x, t) &= \frac{1}{5!}x^5 - L^{-1}(u_2 + (u_2)_{xx}) = \frac{1}{5!}x^5 - \frac{1}{6!}t^6 + \frac{1}{24}xt^4 + \frac{1}{2}xt^2 + \frac{1}{12}x^3t^2 \\ p^4: u_4(x, t) &= -\frac{1}{7!}x^7 - L^{-1}(u_3 + (u_3)_{xx}) = -\frac{1}{7!}x^7 - \frac{1}{8!}t^8 - \frac{1}{12}xt^6 - \frac{1}{12}x^3t^4 - \frac{1}{240}x^5t^2 - \frac{1}{720}xt^6 - \frac{1}{144}x^3t^4 \end{aligned}$$

and so on. Hence the series solution is given by

$$u(x,t) = \left( x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots \right) + \left( 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \dots \right)$$

Therefore proceeding in this way by cancelling these noise terms from series solution and this will in the limit of infinitely many terms gives close-form solution

$$u(x,t) = \sin x + \cos t$$

**Example 2:** Now consider the nonlinear differential equation [20]

$$\frac{d^2u}{dx^2} + u \frac{du}{dx} + u = \frac{1}{2} \sin 2x, u(0) = 0, \quad \frac{du}{dx}(0) = 1 \quad (14)$$

We construct a homotopy which satisfies the following relation

$$\frac{d^2u}{dx^2} - \frac{d^2v_0}{dx^2} + p \left( \frac{d^2v_0}{dx^2} + u \frac{du}{dx} + u - \frac{1}{2} \sin 2x \right) = 0$$

Let us take the initial approximation

$$v_0 = L^{-1} \left( \frac{1}{2} \sin 2x \right) + \frac{5}{4}x = \frac{5}{4}x - \frac{1}{8} \sin 2x = f(x)$$

The Taylor expansion of  $f(x) = \frac{5}{4}x - \frac{1}{8} \sin 2x$  is

$$f(x) = x + \frac{1}{6}x^3 - \frac{1}{30}x^5 + \frac{1}{315}x^7 - \frac{1}{5670}x^9 + \dots$$

The iterative formula based on (9) we obtain,

$$\begin{aligned} u_0(x) &= x \\ u_1(x) &= \frac{1}{6}x^3 - L^{-1}(u_0 + H_0) = -\frac{1}{3!}x^3 \\ u_2(x) &= -\frac{1}{30}x^5 - L^{-1}(u_1 + H_1) = \frac{1}{5!}x^5 \\ u_3(x) &= \frac{1}{315}x^7 - L^{-1}(u_2 + H_2) = -\frac{1}{7!}x^7 \end{aligned}$$

where  $H_n$ ,  $n \geq 0$  are He's polynomials were determined by using the formula given in Eq. (A).

Hence the series solution is given by

$$u(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

Therefore proceeding in this way in the limit of infinitely many terms gives close-form solution

$$u(x) = \sin x$$

### APPLICATIONS TO INTEGRAL EQUATIONS

**Example 3:** Consider the linear integral equation [20]

$$u(x) = 1 + \sinh x - \cosh x + \int_0^x u(t) dt \quad (15)$$

We construct a homotopy of the Eqn. (14) which satisfies the following relation

$$u - v_0 + p \left[ v_0 - 1 - \sinh x + \cosh x - \int_0^x u(t) dt \right] = 0 \quad (16)$$

**The modified HPM:** To apply the modified HPM, let us take the initial approximation,

$$v_0 = 1 + \sinh x - \cosh x = f(x) = f_0 + f_1$$

where  $f_0 = \sinh x$  and  $f_1 = 1 - \cosh x$

The iterative formula based on (8) we obtain,

$$\begin{aligned} u_0(x) &= f_0 = \sinh x \\ p^1: u_1(x) &= f_1 + \int_0^x u_0(t) dt = 1 - \cosh x + \int_0^x \sinh t dt = 0 \\ p^{k+1}: u_{k+1}(x) &= \int_0^x u_k(t) dt = 0, k \geq 1 \end{aligned}$$

Hence, by using only two iterations the exact solution is reached,  $u(x) = \sinh x$ . However, the success of this MHPM depends completely on the correct selection of the function  $f_0$  and  $f_1$ , here the trials are the only technique that can be used.

**The new modified HPM:** The Taylor expansion of  $1 + \sinh x - \cosh x = f(x)$  is

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n!} = x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \frac{1}{5!}x^5 - \frac{1}{6!}x^6 + \dots$$

The iterative formula based on (9) we obtain,

$$\begin{aligned} u_0(x) &= x \\ p^1: u_1(x) &= -\frac{1}{2!}x^2 + \int_0^x u_0(t) dt = -\frac{1}{2!}x^2 + \int_0^x t dt = -\frac{1}{2!}x^2 + \frac{1}{2}x^2 = 0 \\ p^2: u_2(x) &= \frac{1}{3!}x^3 + \int_0^x u_1(t) dt = \frac{1}{3!}x^3 + \int_0^x 0 dt = \frac{1}{3!}x^3 \\ p^3: u_3(x) &= -\frac{1}{4!}x^4 + \int_0^x u_2(t) dt = -\frac{1}{4!}x^4 + \int_0^x \frac{1}{3!}t^3 dt = 0 \\ p^4: u_4(x) &= \frac{1}{5!}x^5 + \int_0^x u_3(t) dt = \frac{1}{5!}x^5 + \int_0^x 0 dt = \frac{1}{5!}x^5 \\ p^5: u_5(x) &= -\frac{1}{6!}x^6 + \int_0^x u_4(t) dt = -\frac{1}{6!}x^6 + \int_0^x \frac{1}{5!}t^5 dt = 0 \\ p^6: u_6(x) &= \frac{1}{7!}x^7 + \int_0^x u_5(t) dt = \frac{1}{7!}x^7 + \int_0^x 0 dt = \frac{1}{7!}x^7 \end{aligned}$$

Hence the series solution can be written as:

$$u(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \dots$$

Therefore proceeding in this way by cancelling noise terms from series solution and this will in the limit of infinitely many terms gives close-form solution

$$u(x) = \sinh x$$

**Example 4:** Finally, we consider the nonlinear integral equation [20]

$$u(x) = \sec x + \tan x - \int_0^x u^2(t) dt \quad (17)$$

Now we construct a homotopy which satisfies the following relation

$$u - v_0 + p \left[ v_0 - \sec x - \tan x + \int_0^x u^2(x) dx \right] = 0 \quad (18)$$

**The modified HPM:** To apply the modified HPM, let us take the initial approximation,

$$v_0 = \sec x + \tan x = f(x) = f_0 + f_1$$

where  $f_0 = \sec x$  and  $f_1 = \tan x$ .

The iterative formula based on (8) we obtain

$$u_0(x) = f_0 = \sec x$$

$$p^1: u(x) = \tan x - \int_0^x H_0(x) dx = \tan x - \int_0^x u_0^2(x) dx = \tan x - \int_0^x \sec^2 x dx = 0$$

$$p^{k+1}: u_{k+1}(x) = \int_0^x H_k(x) dx = 0, \quad k \geq 1$$

where  $H_n, n \geq 0$  are He's polynomial were determined by using the formula give in Eq. (A). Therefore, we get exact solution immediately  $u(x) = \sec x$ , which completely depends on the correct selection of the function  $f_0$  and  $f_1$ .

**The new modified HPM:** The Taylor expansion of  $\sec x + \tan x = f(x)$  is

$$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{5}{24}x^4 + \frac{2}{15}x^5 + \frac{61}{720}x^6 + \frac{17}{315}x^7 + \dots$$

Following the iterative formula (9) we obtain

$$u_0(x) = 1$$

$$p^1: u(x) = x - \int_0^x u_0^2(x) dx = x - x = 0$$

$$p^2: u_2(x) = \frac{1}{2}x^2 - \int_0^x H_1 dx = \frac{1}{2}x^2 - \int_0^x 2u_0u_1 dx = \frac{1}{2}x^2$$

$$p^3: u_3(x) = \frac{1}{3}x^3 - \int_0^x H_2 dx = \frac{1}{3}x^3 - \int_0^x (2u_0u_2 + u_1^2) dx = \frac{1}{3}x^3 - \frac{1}{3}x^3 = 0$$

$$p^4: u_4(x) = \frac{5}{24}x^4 - \int_0^x H_3 dx = \frac{5}{24}x^4 - \int_0^x (2u_0u_3 + 2u_1u_2) dx = \frac{5}{24}x^4$$

$$p^5: u_5(x) = \frac{2}{15}x^5 - \int_0^x H_4 dx = \frac{2}{15}x^5 - \int_0^x (2u_0u_4 + 2u_1u_3 + u_2^2) dx = \frac{2}{15}x^5 - \int_0^x \left( \frac{2}{15}x^4 + \frac{1}{4}x^4 \right) dx = \frac{2}{15}x^5 - \frac{2}{15}x^5 = 0$$

⋮

where  $H_n, n \geq 0$  are He's polynomials were determined by using the formula given in Eq. (A). Hence the series solution can be written as

$$u(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \dots$$

this will in the limit of infinitely many terms give a closed-form solution

$$u(x) = \sec x$$

## CONCLUSION

In this paper, we first proposed a reliable modification to the homotopy-perturbation method (HPM) by introducing a new technique to choose initial component that already reduced the computational work and accelerates the rapid convergence of the HPM series solution. We have chosen two examples from differential equations and two examples from integral equations. From the test examples, we see that both the modified HPM and new modification of the HPM provided exact solution by using minimal terms in series solution. However the success of the modified HPM depends completely on the correct selection of the function  $f_0$  and  $f_1$ . The new modification overcomes the difficulties expanding by a series of infinite components. It can be concluded that the new modification of HPM is a promising tool for solving linear-nonlinear differential and integral equations.

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## REFERENCES

1. Cveticanin, L., 2005. The homotopy-perturbation method applied for solving complex-valued differential equations with strong cubic nonlinearity. *J. Sound Vibration*, 285: 1171-1179.
2. Debnath, L., 1997. *Nonlinear Partial Differential Equations for Scientists and Engineers*, Birkhauser, Boston.
3. Ganji, D.D. and A. Rajabi, 2006. Assessment of homotopy-perturbation and perturbation methods in heat radiation equations. *Int. Commun. Heat Mass Transfer*, 33: 391-400.
4. Ganji, D.D. and A. Sadighi, 2007. Application of homotopy-perturbation and variational iteration methods to nonlinear heat transfer and porous media equations. *J. Comput. Appl. Math.*, 207: 23-24.
5. He, J.H., 1999. Homotopy perturbation technique. *Comput. Methods Appl. Mech. Engrg.*, 178: 257-262.
6. He, J.H., 2000. A coupling method of homotopy technique and perturbation technique for nonlinear problems. *Int. J. Non-Linear Mech.*, 35: 37-43.
7. He, J.H., 2003. Homotopy perturbation method: A new nonlinear analytical technique. *Appl. Math. Comput.*, 135: 73-79.
8. He, J.H., 2005. Homotopy perturbation method for bifurcation of nonlinear problems. *Int. J. Nonlinear Sci. Numer. Simul.*, 6: 207-208.
9. He, J.H., 2005. Application of homotopy perturbation method to nonlinear wave equations, *Chaos Solitons Fractals*, 26: 695-700.
10. He, J.H., 2006. Homotopy perturbation method for solving boundary value problems. *Phys. Lett. A*, 350: 87-88.
11. He, J.H., 2006. *Non-Perturbative Methods for Strongly Nonlinear Problems*. Die Deutsche bibliothek, Germany.
12. He, J.H., 2006. Some asymptotic methods for strongly nonlinear equations. *Int. J. Modern Phys. B*, 20: 1141-1199.
13. Chowdhury, M.S.H. and I. Hashim, 2007. Solutions of a class of singular second-order IVPs by homotopy-perturbation method. *Phys. Lett. A*, 365: 439-447.
14. Chowdhury, M.S.H., 2007. I. Hashim and O. Abdulaziz, Application of homotopy-perturbation method to nonlinear population dynamics models. *Phys. Lett. A*, 368: 258-261.



15. Chowdhury, M.S.H. and I. Hashim, 2007. Solutions of time-dependent Emden-Fowler type equations by homotopy-perturbation method. *Phys. Lett. A*, 368: 305-313.
16. Chowdhury, M.S.H. and I. Hashim, 2009. Application of homotopy-perturbation method to Klein-Gordon and sine-Gordon equations. *Chaos Solitons Fractals*, 39: 1928-1935.
17. Chowdhury, M.S.H., I. Hashim and S. Momani, 2009. The multistage homotopy-perturbation method: A powerful scheme for handling the Lorenz system. *Chaos Solitons Fractals*, 40: 1929-1937.
18. Odibat, Z.M., 2007. A new modification of the homotopy perturbation method for linear and nonlinear operators. *Appl. Math. Comput.*, 189: 746-753.
19. Chowdhury, M.S.H., I. Hashim and M.M Rahman, 2012. A reliable modification of the homotopy perturbation method for differential and integral equations. In the Proceedings of the 2<sup>nd</sup> Regional Conference on Applied and Engineering Mathematics (RCAEM-II).
20. Wazwaz, A.M. and S.M. El-Sayed, 2001. A new modification of the Adomian decomposition method for linear and nonlinear operators. *Appl. Math. Comput.*, 122: 393-405.
21. Wazwaz, A., 1999. A reliable modification of Adomian decomposition method. *Appl. Math. Comput.*, 102: 77-86.
22. Ghorbani, A., 2009. Beyond Adomian polynomials: He polynomials. *Chaos solitons Fractals*, 39 (3): 1486-1492.