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Impulsive Delay Reaction-Diffusion Cohen-Grossberg Neural Networks with Zero Dirichlet Boundary Conditions

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Abstract: An impulsive Cohen-Grossberg neural network with time-varying and S-type distributed delays and reaction-diffusion terms is considered. By using Hardy-Sobolev inequality, under suitable conditions in terms of *M*-matrices which involve the reaction-diffusion coefficients and the dimension of the spatial domain, it is proved that for the system with zero Dirichlet boundary conditions the equilibrium point is globally exponentially stable. Examples are given.

Key words: Cohen-Grossberg neural networks, S-type delays, impulses, reaction-diffusion, Hardy-Sobolev inequality

INTRODUCTION

Since Cohen-Grossberg neural networks [7] were proposed in 1983, extensive work has been done on this subject due to their extensive applications in classification of patterns, associative memories, image processing, quadratic optimization and other areas. In implementation of neural networks, however, time delays inevitably occur due to the finite switching speed of neurons and amplifiers.

Most widely studied and used neural networks can be classified as either continuous or discrete. Recently, there has been a somewhat new category of neural networks which are neither purely continuous-time nor purely discrete-time. This third category of neural networks called impulsive neural networks displays a combination of characteristics of both the continuous and discrete systems [9].

It is well known that diffusion effect cannot be avoided in the neural networks when electrons are moving in asymmetric electromagnetic fields [15], so the activations must be considered to vary in space as well as in time. The papers [13, 14] are devoted to the exponential stability of impulsive Cohen-Grossberg neural networks with, respectively, time-varying and distributed delays and reaction-diffusion terms. In the above cited papers and many others as well as in our recent paper [4] the stability conditions were independent of the diffusion. On the other hand, in [17, 21, 22] the estimate of the exponential convergence rate depends on the reaction-diffusion.

In the present paper we consider an impulsive Cohen-Grossberg neural network with both time-varying and S-type distributed delays [5, 10, 12, 20] and reaction-diffusion terms as in [18, 21, 22] which are of a form more general than in [13, 14] and zero Dirichlet boundary conditions. By using Hardy-Sobolev inequality as in [21], under suitable conditions in terms of M-matrices which involve the reaction-diffusion coefficients and the dimension of the spatial domain, it is proved that for the system with zero Dirichlet boundary conditions the equilibrium point is globally exponentially stable. More precise results can be obtained by using Hardy-Poincaré inequality as in [22]. Examples are given.

MODEL DESCRIPTION AND PRELIMINARIES

We consider the impulsive Cohen-Grossberg neural network with time-varying and S-type distributed delays and reaction-diffusion terms and zero Dirichlet boundary conditions:

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$$\frac{\partial u_i(t,x)}{\partial t} = \sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu}\left(D_{i\nu}(t,x,u)\frac{\partial u_i(t,x)}{\partial x_\nu}\right) + \alpha_i\big(u_i(t,x)\big)\Bigg[-\beta_i\big(u_i(t,x)\big) + \sum_{j=1}^{m} a_{ij} f_j\left(u_j(t,x)\right)$$
$$+\sum_{j=1}^{m} b_{ij} g_j\left(u_j\big(t-\tau_{ij}(t),x\big)\right) + \sum_{j=1}^{m} c_{ij}\int_{-\infty}^{0} h_j\left(u_j(t+\theta,x)\right)\, d\eta_{ij}(\theta) + J_i\Bigg], \qquad t>0, \quad t\neq t_k, \tag{1}$$

$$\Delta u_i(t_k,x) = -B_{ik}u_i(t_k,x) + \int_{t_{k-1}-t_k}^{0} u_i(t_k+\theta)\, d\zeta_{ik}(\theta),\ \ k\in\mathbb{N},$$

$$u_i|_{\partial\Omega} = 0,\ \ u_i(s,x) = \phi_i(s,x), \quad s\le 0,\ \ x\in\Omega,\ \ i=\overline{1,m},$$

where $m\geq 2$ is the number of neurons in the network; $\Omega\subset\mathbb{R}^n$ is a bounded open set containing the origin, with smooth boundary $\partial\Omega$ and mes $\Omega>0$; $D_{i\nu}(t,x,u)>0$ are smooth functions corresponding to the transmission diffusion operator along the *i*-th neuron; $\alpha_i(u_i)$ represent amplification functions; $\beta_i(u_i)$ are appropriately behaving functions which support the stabilizing feedback term $-\alpha_i(u_i)\beta_i(u_i)$ of the *i*-th neuron; a_{ij}, b_{ij}, c_{ij} denote the connection weights (or strengths) of the synaptic connections between the *j*-the neuron and the *i*-th neuron; $f_j(u_j)$, $g_j(u_j), h_j(u_j)$ denote the activation functions of the *j*-th neuron; J_i denotes external input to the *i*-th neuron; $\tau_{ij}(t)$ correspond to the transmission delays; the past effect of the *j*-the neuron on the *i*-th neuron is given by the Lebesgue-Stieltjes integral $\int_{-\infty}^{0} h_j\left(u_j(t+\theta,x)\right) d\eta_{ij}(\theta)$; $\Delta u_i(t_k,x) = u_i(t_k+0,x) - u_i(t_k-0,x)$ denote impulsive state displacements at fixed moments (instants) of time t_k, $k\in\mathbb{N}$, involving Lebesgue-Stieltjes integrals. Here it is assumed that $u_i(t_k-0,x)$ and $u_i(t_k+0,x)$ denote respectively the left-hand and right-hand limit at t_k and the sequence of times $\{t_k\}_{k=1}^{\infty}$ satisfies $0=t_0<t_1<t_2<\cdots<t_k\to\infty$ as $k\to\infty$. The initial data $\phi(s,x) = \big(\phi_1(s,x),\cdots,\phi_m(s,x)\big)^T$ is such that $\sup_{s\le 0}\sum_{i=1}^{m}\int_{\Omega}\phi_i^2(s,x)\,dx<\infty$.

As usual in the theory of impulsive differential equations [13, 14], at the points of discontinuity t_k of the solution $t\mapsto u(t,x)$ we assume that $u_i(t_k,x)\equiv u_i(t_k-0,x)$. It is clear that, in general, the derivatives $\frac{\partial u_i}{\partial t}(t_k,x)$ do not exist. On the other hand, according to the first equality of Eq. 1, there do exist the limits $\frac{\partial u_i}{\partial t}(t_k\mp 0,x)$. According to the above convention, we assume $\frac{\partial u_i}{\partial t}(t_k,x)\equiv\frac{\partial u_i}{\partial t}(t_k-0,x)$.

Throughout the paper we assume that:

A1: $n\geq 3$ and the constant $\omega>0$ is such that for $x=(x_1,\dots,x_n)^T\in\Omega\subset\mathbb{R}^n$ we have $|x|^2=\sum_{\nu=1}^{n}x_\nu^2<\omega^2$.

A2: There exist constants $\underline{D_i}>0$ $\left(i=\overline{1,m}\right)$ such that $D_{i\nu}(t,x,u)\geq\underline{D_i}$ for $\nu=\overline{1,n}$, $t\geq 0$, $x\in\Omega$ and $u\in\mathbb{R}^m$.

A3: The amplification functions $\alpha_i\colon\mathbb{R}\to(0,+\infty)$ are continuous and bounded in the sense that $0<\underline{\alpha_i}\leq\alpha_i(u)\leq\overline{\alpha}_i$ for $u\in\mathbb{R}$, $i=\overline{1,m}$.

A4: The stabilizing functions $\beta_i\colon\mathbb{R}\to\mathbb{R}$ are continuous and monotone increasing, namely, $0<\underline{\beta_i}\leq\frac{\beta_i(u)-\beta_i(v)}{u-v}$ for $u,v\in\mathbb{R}$, $u\neq v$, $i=\overline{1,m}$.

A5: For the activation functions $f_i(u)$, $g_i(u)$, $h_i(u)$ there exist positive constants F_i, G_i, H_i such that

$$F_i=\sup_{u\neq v}\left|\frac{f_i(u)-f_i(v)}{u-v}\right|,\quad G_i=\sup_{u\neq v}\left|\frac{g_i(u)-g_i(v)}{u-v}\right|,\quad H_i=\sup_{u\neq v}\left|\frac{h_i(u)-h_i(v)}{u-v}\right|$$

for all $u,v\in\mathbb{R}$, $u\neq v$, $i=\overline{1,m}$.

A6: $\tau_{ij}(t)$ satisfy $0\leq\tau_{ij}(t)\leq\tau_{ij}$, $0\leq\dot{\tau}_{ij}(t)\leq\mu_{ij}<1$ $\left(i,j=\overline{1,m}\right)$.

A7: $\eta_{ij}(\theta)$ $\left(i,j=\overline{1,m}\right)$, $\zeta_{ik}(\theta)$ $\left(i=\overline{1,m},\ \ k\in\mathbb{N}\right)$ are nondecreasing bounded variation functions on $(-\infty,0]$ and $[t_{k-1}-t_k,0]$, respectively and $\int_{-\infty}^{0} e^{-\lambda\theta}\, d\eta_{ij}(\theta)=k_{ij}(\lambda)$ are continuous functions on $[0,\lambda_0)$ for some $\lambda_0>0$ and $k_{ij}(0)=1$ (without loss of generality).

Due to the zero Dirichlet boundary conditions Eq. 1 can have just one equilibrium point $\mathbf{0}=(0,0,\ldots,0)^T$. It is really an equilibrium point of Eq. 1 if and only if

$$-\beta_i(0)+\sum_{j=1}^{m}\left(a_{ij}f_j(0)+b_{ij}g_j(0)+c_{ij}h_j(0)\right)+J_i=0,\ \ i=\overline{1,m}.$$

For the sake of simplicity of notation, without loss of generality we assume that

A8: $\beta_i(0)=f_i(0)=g_i(0)=h_i(0)=J_i=0,\ \ i=\overline{1,m}.$

Now conditions **A4**, **A5** imply

$$\beta_i(u)u\geq\underline{\beta_i}u^2,\ \ |f_i(u)|\leq F_i|u|,\ \ \ |g_i(u)|\leq G_i|u|,\ \ |h_i(u)|\leq H_i|u|$$

for all $u\in\mathbb{R}^m$ and $i=\overline{1,m}.$

Denote

$$\|u_i(t,\cdot)\|=\left(\int_{\Omega} u_i^2(t,x)\,dx\right)^{1/2}.$$

Definition 1: The equilibrium point $u=\mathbf{0}$ of Eq. 1 is said to be *globally exponentially stable* (with Lyapunov exponent λ) if there exist constants $\lambda>0$ and $M\geq 1$ such that for any solution $u(t,x)=\left(u_1(t,x),\ldots,u_m(t,x)\right)^T$ of Eq. 1 we have

$$\sum_{i=1}^{m}\|u_i(t,\cdot)\|\leq M\sup_{s\leq 0}\sum_{i=1}^{m}\|\phi_i(s,\cdot)\|e^{-\lambda t}$$

for all $t\geq 0,\ x\in\Omega.$

Definition 2: [6] A real matrix $A=\left(a_{ij}\right)_{m\times m}$ is said to be an M-matrix if $a_{ij}\leq 0$ for $i,j=\overline{1,m},\ \ i\neq j$ and all successive principal minors of A are positive.

Lemma 1: [6]. *Let* $A=\left(a_{ij}\right)_{m\times m}$ *be a real matrix with non-positive off-diagonal elements. Then A is an M-matrix if and only if one of the following conditions holds*:

- *There exists a vector* $\xi=(\xi_1,\xi_2,\ldots,\xi_m)^T$ *with* $\xi_i>0$ *such that every component of* $\xi^T A$ *is positive−that is,* $\sum_{i=1}^{m}\xi_i a_{ij}>0,\ \ \ j=\overline{1,m}.$
- *There exists a vector* $\xi=(\xi_1,\xi_2,\ldots,\xi_m)^T$ *with* $\xi_i>0$ *such that every component of* $A\xi$ *is positive−that is,* $\sum_{j=1}^{m}a_{ij}\xi_j>0,\ \ \ i=\overline{1,m}.$

For more details about M-matrices the reader is referred to [8, 11].

Further on we will need the following lemma.

Lemma 2: (Hardy-Sobolev inequality [1]). *Let* $\Omega\subset\mathbb{R}^n$ $(n\geq 3)$ *be a bounded open set containing the origin and* $u\in H_0^1(\Omega)$. *Then*

$$\int_{\Omega} |\nabla u|^2 \, dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\Omega} \frac{u^2}{|x|^2} \, dx \,.$$

Now we introduce the following matrices: $\underline{D} = \mathrm{diag}\left(\underline{D}_1, \dots, \underline{D}_m\right)$, $\underline{\alpha} = \mathrm{diag}\left(\underline{\alpha}_1, \dots, \underline{\alpha}_m\right)$, $\overline{\alpha} = \mathrm{diag}(\overline{\alpha}_1, \dots, \overline{\alpha}_m)$, $\underline{\beta} = \mathrm{diag}\left(\underline{\beta}_1, \dots, \underline{\beta}_m\right)$, $F = \mathrm{diag}(F_1, \dots, F_m)$, $G = \mathrm{diag}(G_1, \dots, G_m)$, $H = \mathrm{diag}(H_1, \dots, H_m)$, $|A| = \left(|a_{ij}|\right)_{m\times m}$, $|B(\mu)| = \left(\frac{|b_{ij}|}{1-\mu_{ij}}\right)_{m\times m}$, $|C| = \left(|c_{ij}|\right)_{m\times m}$.

MAIN RESULTS

Theorem 1: *Let Eq.* 1 *satisfy assumptions* **A1−A8**. *If there exists a vector* $\xi = (\xi_1, \dots, \xi_m)^T$ *with* $\xi_i > 0$ *and a number* $\lambda \in (0, \lambda_0)$ *such that*

$$\sum_{i=1}^{m} \left\{ \left[\lambda - \left(\frac{n-2}{2\omega}\right)^2 \underline{D_i} - \underline{\alpha_i}\underline{\beta_i} \right] \delta_{ij} + \overline{\alpha}_i \left[|a_{ij}| F_j + |b_{ij}| G_j \frac{e^{\lambda \tau_{ij}}}{1-\mu_{ij}} + |c_{ij}| H_j k_{ij}(\lambda) \right] \right\} \xi_i < 0 \tag{2}$$

for $j = \overline{1,m}$, *where* $\delta_{ii} = 1$, $\delta_{ij} = 0$ *for* $j \neq i$, *then there exists a constant* $M \geq 1$ *such that for any solution* $u(t,x) = \left(u_1(t,x), \dots, u_m(t,x)\right)^T$ *of Eq.* 1 *we have*

$$\sum_{i=1}^{m} \|u_i(t,\cdot)\| \leq M e^{-\lambda t} \prod_{k=1}^{i(0,t)} \left(\max_{i=\overline{1,m}} |1 - B_{ik}|) + \max_{i=\overline{1,m}} \int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta} \, d\zeta_{ik}(\theta) \right) \sup_{s\leq 0} \sum_{i=1}^{m} \|u_i(s,\cdot)\| \tag{3}$$

for $t \geq 0$, *where* $i(0,t) = \max \ \{k \in \{0\} \cup \mathbb{N}\colon t_k < t\}$ *is the number of instants of impulse effect* t_k in the interval $(0,t)$.

Proof: First let us note that Eq. 2 holds if and only if

$$\mathcal{A} = \left(\frac{n-2}{2\omega}\right)^2 \underline{D} + \underline{\alpha}\underline{\beta} - \overline{\alpha}(|A|F + |B(\mu)|G + |C|H)$$

is an *M*-matrix. In fact, if $\mathcal{A}$ is an *M*-matrix, from Lemma 1 there exists a vector $\xi > 0$ such that every component of $-\xi^T \mathcal{A}$ is negative. By continuity, there exists $\lambda \in (0, \lambda_0)$ such that Eq. 2 holds. Conversely, if Eq. 2 holds for some $\lambda^* \in (0, \lambda_0)$, then it still holds for all $\lambda \in [0, \lambda^*]$. For $\lambda = 0$, from Lemma 1 we deduce that $\mathcal{A}$ is an *M*-matrix.

We multiply the *i*-th differential equation in Eq. 1 by $u_i(t,x)$ and integrate over the domain **Ω**:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} u_i^2(t,x)\, dx = \int_{\Omega} \sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu}\left(D_{i\nu}(t,x,u) \frac{\partial u_i(t,x)}{\partial x_\nu} \right) u_i(t,x)\, dx - \int_{\Omega} \alpha_i\left(u_i(t,x)\right)\beta_i\left(u_i(t,x)\right)u_i(t,x) dx$$

$$+ \int_{\Omega} \alpha_i\left(u_i(t,x)\right)u_i(t,x) \sum_{j=1}^{m} a_{ij} f_j\left(u_j(t,x)\right)\, dx + \int_{\Omega} \alpha_i\left(u_i(t,x)\right)u_i(t,x) \sum_{j=1}^{m} b_{ij} g_j\left(u_j\left(t-\tau_{ij}(t), x\right)\right) dx$$

$$+ \int_{\Omega} \alpha_i\left(u_i(t,x)\right)u_i(t,x) \sum_{j=1}^{m} c_{ij} \int_{-\infty}^{0} h_j\left(u_j(t+\theta, x)\right)\, d\eta_{ij}(\theta)\, dx.$$

By using Green's formula, the zero Dirichlet boundary conditions, Lemma 2 and assumptions **A1**, **A2** we have

$$\int_\Omega \sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu}\left(D_{i\nu}(t,x,u)\frac{\partial u_i(t,x)}{\partial x_\nu}\right)u_i(t,x)\,dx = -\int_\Omega \sum_{\nu=1}^{n} D_{i\nu}(t,x,u)\left(\frac{\partial u_i(t,x)}{\partial x_\nu}\right)^2 dx$$

$$\le -\underline{D_i}\int_\Omega \sum_{\nu=1}^{n}\left(\frac{\partial u_i(t,x)}{\partial x_\nu}\right)^2 dx = -\underline{D_i}\int_\Omega |\nabla u_i(t,x)|^2\,dx \le -\left(\frac{n-2}{2}\right)^2 \underline{D_i}\int_\Omega \frac{u_i^2(t,x)}{|x|^2}\,dx$$

$$\le -\left(\frac{n-2}{2\omega}\right)^2 \underline{D_i}\int_\Omega u_i^2(t,x)\,dx = -\left(\frac{n-2}{2\omega}\right)^2 \underline{D_i}\|u_i(t,\cdot)\|^2.$$

Next we have

$$\int_\Omega \alpha_i(u_i(t,x))\beta_i(u_i(t,x))u_i(t,x)\,dx \ge \underline{\alpha_i}\underline{\beta_i}\int_\Omega u_i^2(t,x)\,dx = \underline{\alpha_i}\underline{\beta_i}\|u_i(t,\cdot)\|^2;$$

$$\int_\Omega \alpha_i(u_i(t,x))u_i(t,x)\sum_{j=1}^{m} a_{ij}f_j\left(u_j(t,x)\right)dx \le \overline{\alpha}_i\sum_{j=1}^{m}|a_{ij}|\int_\Omega |u_i(t,x)|\,F_j\,|u_j(t,x)|\,dx$$

$$\le \overline{\alpha}_i\sum_{j=1}^{m}|a_{ij}|F_j\left(\int_\Omega u_i^2(t,x)\,dx\right)^{1/2}\left(\int_\Omega u_j^2(t,x)\,dx\right)^{1/2} = \overline{\alpha}_i\sum_{j=1}^{m}|a_{ij}|F_j\|u_i(t,\cdot)\|\,\|u_j(t,\cdot)\|.$$

Similarly,

$$\int_\Omega \alpha_i(u_i(t,x))u_i(t,x)\sum_{j=1}^{m} b_{ij}g_j\left(u_j(t-\tau_{ij}(t),x)\right)dx \le \overline{\alpha}_i\sum_{j=1}^{m}|b_{ij}|G_j\|u_i(t,\cdot)\|\,\|u_j(t-\tau_{ij}(t),\cdot)\|$$

and

$$\int_\Omega \alpha_i(u_i(t,x))u_i(t,x)\sum_{j=1}^{m} c_{ij}\int_{-\infty}^{0} h_j\left(u_j(t+\theta,x)\right)d\eta_{ij}(\theta)\,dx \le \overline{\alpha}_i\sum_{j=1}^{m}|c_{ij}|H_j\|u_i(t,\cdot)\|\int_{-\infty}^{0}\|u_j(t+\theta,\cdot)\|\,d\eta_{ij}(\theta).$$

Combining the above inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_i(t,\cdot)\|^2 \le -\left[\left(\frac{n-2}{2\omega}\right)^2\underline{D_i} + \underline{\alpha_i}\underline{\beta_i}\right]\|u_i(t,\cdot)\|^2$$

$$+\overline{\alpha}_i\|u_i(t,\cdot)\|\sum_{j=1}^{m}\left\{|a_{ij}|F_j\|u_j(t,\cdot)\| + |b_{ij}|G_j\|u_j(t-\tau_{ij}(t),\cdot)\| + |c_{ij}|H_j\int_{-\infty}^{0}\|u_j(t+\theta,\cdot)\|\,d\eta_{ij}(\theta)\right\}$$

or

$$D^+\|u_i(t,\cdot)\| \le -\left[\left(\frac{n-2}{2\omega}\right)^2\underline{D_i} + \underline{\alpha_i}\underline{\beta_i}\right]\|u_i(t,\cdot)\|$$

$$+\overline{\alpha}_i\sum_{j=1}^{m}\left\{|a_{ij}|F_j\|u_j(t,\cdot)\| + |b_{ij}|G_j\|u_j(t-\tau_{ij}(t),\cdot)\| + |c_{ij}|H_j\int_{-\infty}^{0}\|u_j(t+\theta,\cdot)\|\,d\eta_{ij}(\theta)\right\} \tag{4}$$

where D^+ denotes the upper right Dini derivative.

If we denote $y_i(t) = e^{\lambda t}\|u_i(t,\cdot)\|$, then from Eq. 4 we find

$$D^+y_i(t) \le \left[\lambda - \left(\frac{n-2}{2\omega}\right)^2\underline{D_i} - \underline{\alpha_i}\underline{\beta_i}\right]y_i(t)$$

$$+\overline{\alpha}_i\sum_{j=1}^{m}\left\{|a_{ij}|F_jy_j(t) + |b_{ij}|G_jy_j\left(t-\tau_{ij}(t)\right)e^{\lambda\tau_{ij}} + |c_{ij}|H_j\int_{-\infty}^{0}e^{-\lambda\theta}y_j(t+\theta)\,d\eta_{ij}(\theta)\right\}. \tag{5}$$

Consider a Lyapunov functional

$$V(t)=\sum_{i=1}^{m}\left\{y_i(t)+\overline{\alpha}_i\sum_{j=1}^{m}\left|b_{ij}\right|G_j\frac{e^{\lambda\tau_{ij}}}{1-\mu_{ij}}\int_{t-\tau_{ij}(t)}^{t}y_j(s)\,ds+\overline{\alpha}_i\sum_{j=1}^{m}\left|c_{ij}\right|H_j\int_{-\infty}^{0}e^{-\lambda\theta}\left(\int_{t+\theta}^{t}y_j(s)\,ds\right)d\eta_{ij}(\theta)\right\}\xi_i,$$

where λ and $\xi_i,\ \ i=\overline{1,m}$, are as in Eq. 2.

We note that $V(t)\geq 0$ for $t\geq 0$ and

$$V(0)\leq M\sum_{i=1}^{m}\sup_{s\leq 0}y_i(s) \tag{6}$$

with

$$M=\max_{i=\overline{1,m}}\left\{\xi_i+G_i\sum_{j=1}^{m}\left|b_{ji}\right|\frac{\overline{\alpha}_j e^{\lambda\tau_{ji}}\tau_{ji}}{1-\mu_{ji}}\ \xi_j+H_i\sum_{j=1}^{m}\left|c_{ji}\right|\overline{\alpha}_j\int_{-\infty}^{0}e^{-\lambda\theta}(-\theta)\ d\eta_{ji}(\theta)\,\xi_j\right\}.$$

The above integral is convergent because of $\lambda<\lambda_0$.

Calculating the rate of change of $V(t)$ along the solutions of Eq. 1, by virtue of Eq. 5, Eq. 2 and **A6** we obtain

$$D^{+}V(t)\leq\sum_{i=1}^{m}y_j(t)\sum_{i=1}^{m}\left\{\left[\lambda-\left(\frac{n-2}{2\omega}\right)^2\underline{D_i}-\underline{\alpha_i}\underline{\beta_i}\right]\delta_{ij}+\overline{\alpha}_i\left[\left|a_{ij}\right|F_j+\left|b_{ij}\right|G_j\frac{e^{\lambda\tau_{ij}}}{1-\mu_{ij}}+\left|c_{ij}\right|H_jk_{ij}(\lambda)\right]\right\}\xi_i$$

$$+\sum_{i=1}^{m}\overline{\alpha}_i\xi_i\sum_{j=1}^{m}\left|b_{ij}\right|G_jy_j\left(t-\tau_{ij}(t)\right)\left(e^{\lambda\tau_{ij}(t)}-e^{\lambda\tau_{ij}}\frac{1-\dot{\tau}_{ij}(t)}{1-\mu_{ij}}\right)\leq 0.$$

This implies that $V(t)$ is nonincreasing on every interval $(t_{k-1},t_k]$, $k\in\mathbb{N}$, thus

$$V(t)\leq V(t_{k-1}+0)\quad\text{for}\quad t_{k-1}<t\leq t_k. \tag{7}$$

In particular,

$$V(t_k)\leq V(t_{k-1}+0),\quad k\in\mathbb{N}. \tag{8}$$

Further on, for $k\in\mathbb{N}$ we find successively

$$u_i(t_k+0,x)=(1-B_{ik})u_i(t_k,x)\ \ +\int_{t_{k-1}-t_k}^{0}u_i(t_k+\theta,x)\,d\zeta_{ik}(\theta),$$

$$\|u_i(t_k+0,\cdot)\|\leq|1-B_{ik}|\ \|u_i(t_k,\cdot)\|+\int_{t_{k-1}-t_k}^{0}\|u_i(t_k+\theta,\cdot)\|\,d\zeta_{ik}(\theta)$$

and

$$y_i(t_k+0)\leq|1-B_{ik}|y_i(t_k)+\int_{t_{k-1}-t_k}^{0}e^{-\lambda\theta}y_i(t_k+\theta)\,d\zeta_{ik}(\theta).$$

Making use of Eq. 7 and Eq. 8, we obtain

$$V(t_k+0)\leq\max_{i=\overline{1,m}}|1-B_{ik}|V(t_k)+\max_{i=\overline{1,m}}\int_{t_{k-1}-t_k}^{0}e^{-\lambda\theta}\,d\zeta_{ik}(\theta)V(t_{k-1}+0)$$

$$\leq \left(\underline{\max}_{i=1,m}|1-B_{ik}| + \underline{\max}_{i=1,m}\int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_{ik}(\theta)\right) V(t_{k-1}+0).$$

Combining the last estimate with Eq. 7, Eq. 8 and Eq. 6, we derive Eq. 3. □

For three sets of additional assumptions we will show that Eq. 3 implies global exponential stability of the equilibrium point 0 of the impulsive system Eq. 1.

Corollary 1: *Let all conditions of Theorem* 1 *hold and*

$$\underline{\max}_{i=1,m}|1-B_{ik}| + \underline{\max}_{i=1,m}\int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_{ik}(\theta) \leq 1 \tag{9}$$

for all sufficiently large values of $k \in \mathbb{N}$. *Then the equilibrium point* $\mathbf{0}$ *of the impulsive system Eq.* 1 *is globally exponentially stable with Lyapunov exponent* λ.

In the above corollary the global exponential stability was ensured by the rather small magnitudes of the impulse effects. Further we will show that we may have global exponential stability for quite large and even unbounded magnitudes of the impulse effects provided that these do not occur too often.

Corollary 2: *Let all conditions of Theorem* 1 *hold and*

$$\limsup_{t\to\infty}\frac{i(0,t)}{t} = p < +\infty.$$

Let there exist a positive constant B *satisfying the inequalities*

$$\underline{\max}_{i=1,m}|1-B_{ik}| + \underline{\max}_{i=1,m}\int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_{ik}(\theta) \leq B$$

and $p\ln B < \lambda$. *Then for any* $\tilde{\lambda} \in (0, \lambda - p\ln B)$ *the equilibrium point* $\mathbf{0}$ *of the impulsive system Eq.* 1 *is globally exponentially stable with Lyapunov exponent* $\tilde{\lambda}$.

Similar conditions were introduced in out previous paper [2].

Corollary 3: *Let all conditions of Theorem* 1 *hold and there exists a constant* $\kappa \in (0,\lambda)$ *satisfying the inequality*

$$\underline{\max}_{i=1,m}|1-B_{ik}| + \underline{\max}_{i=1,m}\int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_{ik}(\theta) \leq e^{\kappa(t_k-t_{k-1})} \tag{10}$$

for all sufficiently large values of $k \in \mathbb{N}$. *Then the equilibrium point* $\mathbf{0}$ *of the impulsive system Eq.* 1 *is globally exponentially stable with Lyapunov exponent* $\lambda - \kappa$.

A similar condition was introduced in the paper [16].

EXAMPLES

Denote $\varphi(t) = (|t+1| - |t-1|)/2$. Let Ω be the unit ball in $\mathbb{R}^3$: $\Omega = \{x \in \mathbb{R}^3 \,|\, |x| < 1\}$ and let ∇^2 denote the Laplacian in $\mathbb{R}^3$:

$$\nabla^2 u_i = \frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2}, \quad i = 1,2.$$

Consider the system

$$\begin{aligned}\frac{\partial u_1(t,x)}{\partial t} = {} & 16\nabla^2 u_1(t,x) + (2+\sin u_1(t,x))\Big\{-2u_1(t,x) + 0.5\arctan u_1(t,x) + 0.3\varphi\big(u_2(t,x)\big)\\ & +0.1u_1(t-0.5\arctan t, x) + 0.12\arctan u_2(t-2\varphi(t)/3, x)\\ & +0.1\int_{-\infty}^{0} u_1(t+\theta,x)\, de^{\theta} + 0.15\int_{-\infty}^{0} u_2(t+\theta,x)\, de^{\theta}\Big\},\end{aligned}$$

$$\begin{aligned}\frac{\partial u_2(t,x)}{\partial t} = {} & 20\nabla^2 u_2(t,x) + (3+\sin u_2(t,x))\Big\{-3u_2(t,x) - 0.6\varphi\big(u_1(t,x)\big) + 0.5\arctan u_2(t,x)\\ & +0.16u_1(t-1-\varphi(t)/3, x) - 0.3\arctan u_2(t-2-0.75\varphi(t), x)\\ & +0.1\int_{-\infty}^{0} u_1(t+\theta,x)\, de^{\theta} - 0.2\int_{-\infty}^{0} u_2(t+\theta,x)\, de^{\theta}\Big\},\end{aligned}$$

$$\begin{aligned}\Delta u_i(t_k,x) &= -B_{ik}u_i(t_k,x) + \int_{t_{k-1}-t_k}^{0} u_i(t_k+\theta,x)\, d\zeta_{ik}(\theta), \quad k\in\mathbb{N},\\ u_i|_{\partial\Omega} &= 0, \quad u_i(s,x) = \phi_i(s,x), \quad s\le 0, \quad x\in\Omega, \quad i=1,2. \qquad (11)\end{aligned}$$

For this system assumptions **A1**-**A8** hold with $n = 3$, $\omega = 1$,

$$\underline{D} = \begin{pmatrix}16 & 0\\ 0 & 20\end{pmatrix}, \underline{\alpha} = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}, \overline{\alpha} = \begin{pmatrix}3 & 0\\ 0 & 4\end{pmatrix}, \underline{\beta} = \begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}, \quad \tau_{11} = \pi/4, \tau_{12} = 2/3, \tau_{21} = 4/3, \tau_{22} = 11/4,$$
$$\mu_{11} = 1/2, \quad \mu_{12} = 2/3, \quad \mu_{21} = 1/3, \quad \mu_{22} = 3/4, \quad k_{ij}(\lambda) = 1/(1-\lambda), \quad i,j = 1,2, \quad \lambda_0 = 1,$$
$$F = G = H = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad |A| = \begin{pmatrix}0.5 & 0.3\\ 0.6 & 0.5\end{pmatrix}, \quad |B(\mu)| = \begin{pmatrix}0.2 & 0.36\\ 0.24 & 1.2\end{pmatrix}, \quad |C| = \begin{pmatrix}0.1 & 0.15\\ 0.1 & 0.2\end{pmatrix},$$

the matrix $\mathcal{A} = \begin{pmatrix}3.6 & -2.43\\ -3.76 & 3.4\end{pmatrix}$ is an *M*-matrix. Further on, the vector $\xi = (6,5)^T$ is such that $\xi^T\mathcal{A} = (2.8, 2.42)$ has positive components. Let us denote by $\Phi_j(\lambda)$, $j = 1,2$, the left-hand sides of inequalities Eq. 2 for the given vector ξ. Then

$$\Phi_1(\lambda) = 6\lambda + 3.6e^{\pi\lambda/4} + 4.8e^{4\lambda/3} + \frac{3.8}{1-\lambda} - 15,$$

$$\Phi_2(\lambda) = 5\lambda + 6.48e^{2\lambda/3} + 24e^{11\lambda/4} + \frac{6.7}{1-\lambda} - 39.6.$$

Since $\Phi_1(0.02849) = -2.250318157 < 0$ and $\Phi_2(0.02849) = -0.00085427 < 0$, we can take $\lambda = 0.02849$. Theorem 1 is valid for system Eq. 11.

Let us consider the impulsive conditions

$$\Delta u_1(t_k,x) = -0.5u_1(t_k,x) + 0.25\int_{-1}^{0} u_1(t_k+\theta,x)\, de^{\theta},$$

$$\Delta u_2(t_k,x) = -0.25u_2(t_k,x) + 0.25\int_{-1}^{0} u_2(t_k+\theta,x)\, de^{\theta}, \quad t_k = k, \quad k\in\mathbb{N}. \qquad (12)$$

Now

$$\max_{i=1,2}|1-B_{ik}| + \max_{i=1,2}\int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_{ik}(\theta) = \frac{3}{4} + \frac{1}{4}\int_{-1}^{0} e^{-\lambda\theta}\, de^{\theta} = \frac{3}{4} + \frac{1-e^{\lambda-1}}{4(1-\lambda)}, \quad \lambda < 1.$$

Obviously, inequalities Eq. 9 are valid for all $k \in \mathbb{N}$ and all $\lambda \in (0,1)$, in particular, for $\lambda = 0.02849$. According to Corollary 1, the equilibrium point $(0,0)^T$ of system Eq. 11 with impulsive conditions Eq. 12 is globally exponentially stable with Lyapunov exponent 0.02849.

Next consider the impulsive conditions

$$\Delta u_1(t_k,x) = -100u_1(t_k,x) + \int_{-200}^{0} u_1(t_k+\theta,x)\, de^{\theta},$$

$$\Delta u_2(t_k,x) = -\,50u_2(t_k,x) + \int_{-200}^{0} u_2(t_k+\theta,x)\, de^{\theta}, \quad t_k = 200k, \; k \in \mathbb{N}. \tag{13}$$

Now

$$\max_{i=1,2}|1-B_{ik}| + \max_{i=1,2}\int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_{ik}(\theta) = 99 + \int_{-200}^{0} e^{-\lambda\theta}\, de^{\theta} = 99 + \frac{1-e^{200(\lambda-1)}}{1-\lambda}$$

for $\lambda \in (0,1)$ and we can take $B = 100.029325485$ which is the value of the above expression for $\lambda = 0.02849$. Further on, $p = 0.005$, for $\lambda = 0.02849$ we have

$$\lambda - p\ln B \approx 0.02849 - 0.005 \times 4.605170186 = 0.005462683$$

According to Corollary 2, the equilibrium point $(0,0)^T$ of system Eq. 11 with impulsive conditions Eq. 13 is globally exponentially stable with Lyapunov exponent any $\lambda \in (0, 0.005462683)$.

Finally, let us consider the impulsive conditions

$$\Delta u_1(t_k,x) = -(k+1)u_1(t_k,x) + k\int_{-2k+1}^{0} u_1(t_k+\theta,x)\, de^{\theta},$$

$$\Delta u_2(t_k,x) = -\,(k^2+1)u_2(t_k,x) + k^2\int_{-2k+1}^{0} u_2(t_k+\theta,x)\, de^{\theta}, \quad t_k = k^2, \; k \in \mathbb{N}. \tag{14}$$

Now for $\lambda = 0.02849$ inequality Eq. 10 becomes $2.02932548k^2 \le e^{\kappa(2k-1)}$. Obviously, for any $\kappa > 0$ this inequality is valid for all natural k large enough. For instance, for $\kappa = 0.02$ inequality Eq. 10 holds for $k \ge 305$, while for $\kappa = 0.01$ it holds for $k \ge 690$. Thus, according to Corollary 3, the equilibrium point $(0,0)^T$ of system Eq. 11 with impulsive conditions Eq. 14 is globally exponentially stable with Lyapunov exponent any $\lambda = (0.02849)$.

Impulsive conditions similar to Eq. 12-Eq. 14 were given in our previous paper [3].

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