

## On the Localization Eigenfunction Expansions Associated with the Schrodinger Operator

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**Abstract:** In this paper we study eigenfunction expansions associated with the Schrodinger operator with a singular potential. In the paper it is obtained sufficient conditions for localization and uniformly convergence of the regularizations of the corresponding expansions

**Key words:** Eigenfunction expansion . Schrodinger operator . convergence . localization

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### INTRODUCTION

A theory of distribution has many applications in engineering and sciences [1]. Methods of modern mathematical physics and engineering require studying applicability for the solution of problems in engineering. For example, any simple wave generated by pointed instant source of energy can be expressed in terms of distribution's expansions by eigenfunctions associated with the partial differential operator. These expansions can be studied in generalized sense or even in classical sense in the domains where a distribution coincides with locally integrable function.

In quantum mechanics it is important for the Hamiltonian operator to be self-adjoint in corresponding Hilbert space. According Von Neuman's spectral theorem a self-adjoint operator in Hilbert space can be represented through its spectrum. Spectral expansions associated with a self-adjoint operator always convergence in the norm. Thus, potential of the Hamiltonian must be such that the perturbed operator should "safe" self-adjointness property of the free Hamiltonian. Then a solution of the time dependent Schrödinger's equation can be represented as spectral expansions associated with corresponding Hamiltonian operator.

However, spectral expansions of the given function from the Hilbert space may not converge in sense that needed in application for engineering sciences. Convergence or divergence of spectral expansions depends on expanding function and the topology [2, 3, 5, 8]. In case of divergence regularizations of the spectral expansions are required [2-7, 9, 10]. For the spectral expansions the Riesz regularization usually is applied [4].

In this paper we study regularization of spectral expansions associated with Schrodinger's operator. In case of the free Hamiltonian regularization of spectral expansions studied by many scientists [2-10]. Eigenfunction expansions connected with the singular Schrodinger operator studied by A.R. Khalmukhamedov [5] in the spaces of regular functions.

For the first time regularization of the spectral expansions in the spaces of singular functionals (distributions) studied by Sh.A. Alimov in [2]. In this work the author considered spectral expansions associated with the free Hamiltonian. In present work we consider the same problems in the spaces of distributions for the Hamiltonian with singular potential.

Consideration of the spaces of distribution for the study is very important. Because many phenomenon studying in science and engineering, as a rule, described by singular functionals. For instance, the Dirac delta function, which describes integral characteristics of the phenomenon, occurs in many problems of quantum physics, geophysics, chemical physics and biophysics. For the solution of Integra-differential equation for electrical circuits (equation for the current force) one has differentiate it, means differentiate the Heaviside function and get the Dirac delta function in the equation. In these cases one has to study possibility of representation of the Dirac delta function by spectral expansions associated with differential operators. This problem has some difficulties due to singularity of the Dirac delta function.

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**Spaces of functions and functionals:** Let  $\Omega$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 2$  with smooth boundary  $\partial\Omega$ . Denote by  $C^\infty(\Omega)$  space of infinite time differentiable on  $\Omega$  functions. By  $\Sigma(\Omega)$  denote topological space  $C^\infty(\Omega)$  equipped with topology of uniform convergence in compact subsets of the domain of the sequences of functions and their derivatives of arbitrary orders. Then the space of all linear and continuous functionals on  $\Sigma(\Omega)$  we denote by  $\Sigma'(\Omega)$  and it is well known that space  $\Sigma'(\Omega)$  consists of distributions with compact supports.

In the paper we study spectral expansions of distributions from the Sobolev spaces  $H^l(\Omega)$ . First we define these spaces for the nonnegative integer numbers  $l$ . Then the Sobolev space  $H^l(\Omega)$  consists of those functions  $u$ , which belongs to the Hilbert space  $L_2(\Omega)$  and has generalized derivatives upto  $\alpha$ ,  $|\alpha| \leq l$ , that also belong to  $L_2(\Omega)$ . The space  $H^l(\Omega)$  is normed space and norm of each function  $u$  from this space defined by:

$$\|u\|_l = \left( \int_{\Omega} \sum_{|\alpha| \leq l} |D^\alpha u|^2 dx \right)^{\frac{1}{2}}$$

Denote by  $J$  the space of infinite differentiable in  $\mathbb{R}^N$  functions  $u$ , such that for any  $\alpha$  and  $\beta$ :

$$\sup_x |x^\beta \cdot D^\alpha u(x)| < \infty$$

This space is known as the Schwartz space. The space of linear and continuous functionals on  $J$  denote by  $J'$ . Space  $J'$  is called space of tempered distributions. Note that Fourier transformations  $F$  and  $F^{-1}$  are one to one continuous mappings of  $J'$  to  $J'$ . Using this one can define the Sobolev spaces  $H^l(\mathbb{R}^N)$  for any real number  $l$ . Define  $H^l(\mathbb{R}^N)$ , as subspace of  $J'$  such that for any distribution  $u \in J'$  from  $H^l(\mathbb{R}^N)$  following integral must be finite

$$\int_{\mathbb{R}^N} |Fu(\xi)|^2 \cdot (1 + |\xi|^2)^l d\xi$$

In case of integer  $l$  and  $\Omega = \mathbb{R}^N$  this definition is equivalent to the definition of the space given above. The space of linear continuous functionals on  $H^l(\mathbb{R}^N)$  is isomorphic to the space  $H^{-l}(\mathbb{R}^N)$ . Define the space  $H^l(\Omega)$  for arbitrary domain  $\Omega$  and for arbitrary real number  $l$ .

**Definition 1:** A distribution  $u \in J'$  belongs to the space  $H^l(\Omega)$ , if there exists extension  $u^*$  of this distribution from  $H^l(\mathbb{R}^N)$  that coincides with  $u$  in domain  $\Omega$ .

The norm  $\|u\|_l$  of the distribution  $u$  in this space define as infimum of the norms  $\|u^*\|_l$  by all possible such extensions (as in definition)  $u^*$ . If we consider only those distributions that have compact supports, then the extensions which defines the norm of the distribution is the extension by zero outside of the domain  $\Omega$ .

Denote by  $H_0^l(\Omega)$  a closure of the space of smooth functions with compact support  $C_0^\infty(\Omega)$  by the norm  $\|\cdot\|_l$ . From the embedding

$$\Sigma' \subset \bigcup_{l=-\infty}^{l=\infty} H^l$$

It follows that for any  $f \in \Sigma'$  there exist  $l$ , such that  $f \in H^l$ . Thus we can classify singularity of the distribution using the Sobolev spaces. Let distribution  $f \in \Sigma'$  is  $f \in H^{-1}$   $\forall l > 0$ . Then due to compactness of the support of  $f$  in  $\Omega$ , there exist  $\Omega^0 = \Omega^0(f)$  compactly embedded in  $\Omega$  such that for all  $u(x)$  from  $C^\infty(\Omega)$  following inequality holds

$$|\langle f, u \rangle| \leq \|f\|_{-1} \cdot \|u\|_{l,0} \quad (1)$$

where  $\|\cdot\|_{l,0}$ -denotes a norm in  $H^l(\Omega^0)$ ,  $\|\cdot\|_{-1}$ -is a norm in  $H^{-1}(\Omega)$ .

**Eigenfunction expansions associated with the Hamiltonian operator:** In the paper we consider the Schrödinger operator  $L = -\Delta + q(x)$  with potential  $q = \frac{a(x)}{|x-x_0|}$ ,  $x_0 \in \Omega$ , where  $a(x) \in C^\infty(\Omega)$ . This operator will be considered as a formal operator with domain of definition  $C_0^\infty(\Omega)$  which consists of functions from  $C^\infty(\Omega)$  with compact support.

Let  $A$  denotes a self-adjoint extension of the operator  $L$  with discrete spectrum without thickened points. Denote by  $\{u_n(x)\}$  system of eigenfunctions of operator  $A$ . Let  $f$  be a distribution with compact support and suppose that it vanishing in the neighborhood of  $x_0 \in \Omega$ . Then Fourier coefficients  $f_n$  of the distribution  $f$  by system  $\{u_n(x)\}$  can be defined as value of this functional on  $u_n(x)$ .

The Riesz means of order  $s \geq 0$  of the partial sums of eigenfunction expansions of the distribution  $f$  denoted by:

$$E_\lambda^s f(x) = \sum_{\lambda_n \leq \lambda} \left(1 - \frac{\lambda_n}{\lambda}\right)^s f_n u_n(x)$$

We prove following theorems on convergence of the spectral expansions of distributions in the domains where a distribution equal zero and also in the domains where it coincides with continuous function. Even in such a domain where it is very smooth high singularity of the distributions not allows convergence of its eigenfunction expansions. As an example we can consider the Dirac delta function. Below we use the Sobolev spaces with negative sign in order to classify singularities of the distributions. And also we study distributions with compact support and suppose that support of the distribution not contains a singular point of the operator.

**Theorem 1:** Let a distribution  $f$  has compact support in the domain  $\Omega$  and it is also vanishing at some neighborhood of the point  $x_0 \in \Omega$ . Let  $f \in H^1(\Omega)$  for some  $l > 0$  and let  $s \geq \frac{N-1}{2} + 1$ . Then uniformly on any compact set  $K$  from  $\Omega \setminus \{x_0 \cup \text{Supp} f\}$  following equality is valid

$$\lim_{\lambda \rightarrow \infty} E_\lambda^s f(x) = 0 \quad (2)$$

Theorem 1 states that for the equality (2) regularization index  $s$  must be greater or equal  $\frac{N-1}{2} + 1$ . Next theorem proves that this statement is sharp.

**Theorem 2:** Let  $l > 0$  and  $x_1, x_1 \neq x_0$ , an arbitrary point of domain  $\Omega$ . If  $s < \frac{N-1}{2} + 1$ , then there exists a distribution  $f$  from the Sobolev space  $H^1(\Omega)$  that has a compact support in the domain  $\Omega$  and vanishing at some neighborhoods of the points  $x_0, x_1 \in \Omega$  such that,

$$\lim_{\lambda \rightarrow \infty} E_\lambda^s f(x_1) = +\infty$$

Sharpness of the condition  $s \geq \frac{N-1}{2} + 1$  immediately follows from theorem 2 if we choose a compact  $K$  from the neighborhoods of the point  $x_1$  (where the distribution  $f$  vanishing) that does not contain a point  $x_0 \in \Omega$ .

**Theorem 3:** Let a distribution  $f$  has compact support in the domain  $\Omega$  and it is also vanishing at some neighborhood of the point  $x_0 \in \Omega$ . Let  $f \in H^1(\Omega)$  for some  $l > 0$  and let it coincides with continuous function  $g(x)$  in subdomain  $\Omega_0 \subset \Omega$  such that  $x_0 \notin \Omega_0$ .

If  $s \geq \frac{N-1}{2} + 1$ , then the Riesz means of the partial sums of eigenfunction expansions of distribution  $f$  uniformly convergence to the function  $g(x)$  on any compact set  $K$  from  $\Omega_0$ .

Note, statement of the theorem 3 is more general statement than statement of the theorem 1. However, theorem 3 is a consequence of theorem 1. This fact follows from (2) with application of the L. Hormander theorem in [12]. Thus in this paper we prove theorem 1 and 2.

**Some properties of the Bessel functions and eigenfunctions:** In many cases spectral expansions connected with differential operators has asymptotic relation with the Bessel functions. For example, spectral function of the free Hamiltonian operator can be expressed as

$$K(x) = (2\pi)^{-N} |x|^{-\frac{N}{2}} J_{\frac{N}{2}}(|x|)$$

where  $J_a(t)$  is the Bessel function of first kind and order  $a$ .

For  $t > 0$  we have an estimation of this function

$$|J_a(t)| \leq \frac{\text{const}}{\sqrt{t}}$$

And this estimation is used for the large values of  $t > 0$  and allows obtaining estimation of the spectral function above at the points far from the origin.

The spectral function for the Schrodinger operator can be obtained by using corresponding estimation for the spectral function of free Hamiltonian operator. In this, due to existence of the potential, we have more than one expression that involved the Bessel functions. For example, following estimation of the improper integrals of the following form is important

$$I_{a,b}(\lambda, \mu, \varphi) = \int_0^\infty J_a(r\lambda) J_b(r\mu) \varphi(r) r^{a-b+1} dr \quad (3)$$

where  $\varphi(r)$  is an arbitrary bounded in  $t \geq 0$  function,  $a \geq 0$  and  $b > a+1$  are given numbers.

In some cases there are exact formulas for these type integrals. For example,

$$\int_0^\infty J_{a+s}(\sqrt{\lambda} t) J_{a-1}(\sqrt{\mu} t) t^{-s} dt = \begin{cases} \frac{\left(1 - \frac{\mu}{\lambda}\right)^s \lambda^s \mu^{\frac{a-1}{2}}}{2^s \Gamma(s+1) \lambda^{\frac{a+s}{2}}}, & \mu \leq \lambda \\ 0, & \mu > \lambda \end{cases} \quad (4)$$

Following estimation for the function defined by formula (3) is true [2]:

**Lemma 1:** Let  $a \geq 0$ ,  $b > a+1$  and let  $|\varphi^{(k)}(t)| \leq \text{const} \cdot |t|^{-k}$  when  $t \geq \delta$  and  $\varphi(t) = 0$  when  $0 \leq t \leq \delta$ .

Then for  $I_{a,b}(\lambda, \mu, \varphi)$ , at  $\lambda > 0$  and  $\mu \geq 1$  following estimation is valid

$$|I_{a,b}(\lambda, \mu, \varphi)| \leq \frac{\text{const}}{\sqrt{\lambda\mu}} \cdot \frac{1}{1 + |\lambda - \mu|} \cdot \left(\frac{\lambda}{\mu}\right)^t \quad (5)$$

Using these formulas we obtain representation for the regularized partial sums of eigenfunction expansions. For this we use mean value formula for eigenfunctions. Mean value formula for eigenfunctions introduced in [4] and [11]. We use following mean value formula for eigenfunctions  $u_n(x)$ , which is valid in a ball  $\{r \leq R\}$  with the center at a point  $x \in \Omega_R$  [11]:

$$S_t(u_n) = (2\pi)^{N/2} J_\beta(r\sqrt{\lambda_n})(r\sqrt{\lambda_n})^{-\beta} u_n(x) + \frac{\pi}{2} r^{-\beta} \int_0^r \{ J_\beta(t\sqrt{\lambda_n}) Y_\beta(r\sqrt{\lambda_n}) - Y_\beta(t\sqrt{\lambda_n}) J_\beta(r\sqrt{\lambda_n}) \} t^{\beta+1} S_t(qu_n) dt \quad (6)$$

where

$$\beta = \frac{N-2}{2}$$

$$S_t(g)(x) = \int_{\theta} g(x + t\theta) d\theta$$

and  $J_v(t)$ ,  $Y_v(t)$  are the Bessel functions of order  $v$  first and second kind accordingly.

Let's fix number  $\delta > 0$  as  $R > \delta$ . Denote by  $\eta(t)$  infinite differentiable functions equal zero when  $t \geq R$  and equal 1 when  $t \leq \delta$ . Let  $x \in \Omega_h$  and  $y \in \Omega$ . Consider following function, which depends on distance  $r = |x-y|$ :

$$V(r) = \Gamma(s+1) 2^s (2\pi)^{\frac{-N}{2}} \lambda^{\frac{N-s}{4}} \frac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{r^{\frac{n}{2}+s}}$$

if  $r \leq R$  and otherwise it equal zero (7),

where a constant  $R$  less than  $h/4$ .

Denote  $\lambda_v |x-y| = V(r)\eta(r)$ . Then using (7) and using mean value formula (6) obtain following representation

for the Fourier coefficients of the function  $\lambda_v |x-y|$ :

$$\begin{aligned} \lambda_v(x) &= 2^s \Gamma(s+1) \lambda_n^{\frac{2-N}{4}} \lambda^{\frac{N-2s}{4}} u_n(x) * \int_0^R J_{\frac{N}{2}+s}(\sqrt{\lambda}r) J_{\frac{N}{2}-1}(\sqrt{\lambda_n}r) r^{-s} \eta(r) dr + \frac{2^s \Gamma(s+1) \pi}{(2\pi)^{N/2}} \lambda^{\frac{N}{2}} \\ &* \int_0^R (r\sqrt{\lambda})^{\frac{-N-s}{2}} J_{\frac{N}{2}+s}(r\sqrt{\lambda}) \eta(r) \cdot r^{N-1-\beta} * \int_0^r W_\beta(t, r, \sqrt{\lambda_n}) \cdot t^{\beta+1} \cdot S_t(q \cdot u_n) dt \end{aligned} \quad (8)$$

where

$$W_\beta(t, r, \sqrt{\lambda_n}) = J_\beta(t\sqrt{\lambda_n}) Y_\beta(r\sqrt{\lambda_n}) - Y_\beta(t\sqrt{\lambda_n}) J_\beta(r\sqrt{\lambda_n})$$

Integral in first term of (8), defined in the interval  $(0, R)$ , we split as difference of two integrals by intervals  $(0, \infty)$  and  $(R, \infty)$ . Then denote

$$I(\lambda, \lambda_n) = \lambda \cdot \lambda_n^{1/4} * \int_R^\infty J_{\frac{N}{2}+s}(r\sqrt{\lambda}) J_{\frac{N}{2}-1}(r\sqrt{\lambda_n}) r^{-s} dr \quad (9)$$

and by applying formula (4) for the integral by interval  $(0, \infty)$ , where

$$\operatorname{Re}(a-1) > \operatorname{Re}(a+s) > 0$$

obtain following expression for  $\lambda_v(x)$ :

$$\lambda_v(x) = \delta_n^\lambda u_n(x) (1 - \frac{\lambda_n}{\lambda})^s - 2^s \Gamma(s+1) \lambda_n^{\frac{1-N}{4}} \lambda^{\frac{N-1-s}{4}} u_n(x) I_1(\lambda, \lambda_n) + \frac{2^s \Gamma(s+1) \pi}{(2\pi)^{N/2}} \lambda^{\frac{N}{2}} I_2(\lambda, u_n) \quad (10)$$

where

$$\delta_n^\lambda = \begin{cases} 1, & \lambda_n < \lambda \\ 0, & \lambda_n \geq \lambda \end{cases} \quad \text{and} \quad I_2(\lambda, u_n) = \int_0^R (r\sqrt{\lambda})^{-\frac{N-1}{2}-s} J_{\frac{N}{2}+s}(r\sqrt{\lambda}) \eta(r) \cdot r^{N-1-\beta} * \int_0^r W_\beta(t, r, \sqrt{\lambda_n}) \cdot t^{\beta+1} \cdot S_t(q \cdot u_n) dt.$$

Let a distribution  $f$  has compact support in the domain  $\Omega$  and it is also vanishing at some neighborhood of the point  $x_0 \in \Omega$ . Then for arbitrary function  $w \in C_0^\infty(\Omega \setminus x_0)$  a sequence of numbers  $f(E_\lambda w(x))$  converges to  $f(w)$ , i.e. we have equality

$$f(w) = \sum_{n=1}^{\infty} f_n w_n \quad (11)$$

where  $w_n$  Fourier coefficients of the function  $w$  by system  $\{u_n(x)\}$ . Note that the function  $\frac{\lambda}{v} \langle x-y \rangle$  is smooth (infinite time differentiable) and has compact support in the domain  $\Omega$ . Then we replace  $w$  by  $\frac{\lambda}{v}(r)$  and taking into account (10) and (11) obtain

$$f(\frac{\lambda}{v}) = \delta_n^\lambda f_n u_n(x) (1 - \frac{\lambda_n}{\lambda})^s - 2^s \Gamma(s+1) \lambda^{\frac{N-1}{4} - \frac{s}{2}} \sum_{n=1}^{\infty} \lambda_n^{\frac{1-N}{2}} u_n(x) I_1(\lambda, \lambda_n) + \frac{2^s \Gamma(s+1) \pi}{(2\pi)^{\frac{N}{2}}} \frac{\pi}{2} \lambda^{\frac{N}{2}} \cdot \sum_{n=1}^{\infty} f_n I_2(\lambda, u_n) \quad (12)$$

From (12) by the definition we obtain following representation for the Riesz means of eigenfunction expansions of distribution  $f$ :

$$E_\lambda^s f(x) = f(\frac{\lambda}{v}(r)) + 2^s \Gamma(s+1) \lambda^{\frac{N-1}{4} - \frac{s}{2}} * \sum_{n=1}^{\infty} \lambda_n^{\frac{1-N}{2}} u_n(x) I_1(\lambda, \lambda_n) - \frac{2^s \Gamma(s+1) \pi}{(2\pi)^{\frac{N}{2}}} \frac{\pi}{2} \lambda^{\frac{N}{2}} \cdot \sum_{n=1}^{\infty} f_n I_2(\lambda, u_n) \quad (13)$$

Fix an arbitrary compact  $K$  from  $\Omega \setminus \{x_0 \cup \text{Supp} f\}$  and choose a number  $R > 0$  from the conditions

$$R < \text{dis} \left( K, \partial \Omega \cup \text{supp} f \cup \{x_0\} \right)$$

Then, if  $x \in K$ , then considering  $r$  as a function of variable  $y$  obtain

$$\text{supp} \frac{\lambda}{v}(r) \cap \text{supp} f = \emptyset$$

Therefore first term in (13) is equal zero and hence obtain following representation for the Riesz means of eigenfunction expansions of distribution  $f$

$$E_\lambda^s f(x) = 2^s \Gamma(s+1) \lambda^{\frac{N-1}{4} - \frac{s}{2}} * \sum_{n=1}^{\infty} \lambda_n^{\frac{1-N}{2}} u_n(x) I_1(\lambda, \lambda_n) - \frac{2^s \Gamma(s+1) \pi}{(2\pi)^{\frac{N}{2}}} \frac{\pi}{2} \lambda^{\frac{N}{2}} \cdot \sum_{n=1}^{\infty} f_n I_2(\lambda, u_n) \quad (14)$$

Representation (14) of the eigenfunction expansions we use to prove theorem 1 and 2 above. Infinite series in the right side of (14) needed estimation and evaluation for convergence. For this consider some preliminary statements.

**Lemma 2:** For an arbitrary compact set  $K$  from  $\Omega \setminus \{x_0\}$  there exists a constant  $c = c(K)$ , such that for all numbers  $\lambda > 0$  uniformly by  $x$  following inequality is valid

$$\sum_{n=1}^{\infty} \frac{|u_n(x)|^2}{1 + (\sqrt{\lambda_n} - \sqrt{\lambda})^2} \cdot \sqrt{\lambda_n}^{1-N} \leq c(K) \quad (15)$$

**Proof:** For the Hamiltonian described above uniformly on any compact set  $K$  from  $\Omega \setminus \{x_0\}$  it is valid (see for example in [4])

$$\sum_{\mu \leq \sqrt{\lambda_n} \leq \mu+1} |u_n(x)|^2 = O(\mu^{N-1}) \quad (16)$$

Estimation (16) we rewrite as

$$\sum_{\mu \leq \sqrt{\lambda_n} \leq \mu+1} |u_n(x)|^2 \sqrt{\lambda_n}^{1-N} = O(1) \quad (17)$$

Then by rearranging infinite sum in the right side of (15) obtain

$$\sum_{n=1}^{\infty} \frac{|u_n(x)|^2}{1 + (\sqrt{\lambda_n} - \sqrt{\lambda})^2} \cdot \sqrt{\lambda_n}^{1-N} = \sum_{k=0}^{\infty} \sum_{k \leq \sqrt{\lambda_n} - \sqrt{\lambda} \leq k+1} \frac{|u_n(x)|^2}{1 + (\sqrt{\lambda_n} - \sqrt{\lambda})^2} \sqrt{\lambda_n}^{1-N} \leq \sum_{k=0}^{\infty} \sum_{k \leq \sqrt{\lambda_n} - \sqrt{\lambda} \leq k+1} |u_n(x)|^2 \sqrt{\lambda_n}^{1-N} \leq c(K)$$

**Lemma 2:** Is proved.

**Lemma 3:** Let a distribution  $f$  has compact support in the domain  $\Omega$  and it is also vanishing at some neighborhood of the point  $x_0 \in \Omega$  and let  $f \in H^1(\Omega)$  for some  $l > 0$ . Then for any compact  $K \subset \Omega$ , that contains support of the distribution  $f$  and does not contain  $x_0 \in \Omega$ , following estimation is valid

$$\sum_{n=1}^{\infty} |f_n|^2 \cdot (1 + \lambda_n)^{-1} \leq c \cdot \|f\|_{-1}^2 \quad (18)$$

with constant  $c$  depends only from compact  $K$ .

**Proof:** Denote by  $A$  a formal operator defined by equality

$$Au(x) = \sum_{\lambda_n < \lambda} \lambda_n u_n(x) \quad (19)$$

Operator (19) is well defined in dense subspace of the Hilbert space  $L_2(\Omega)$ . Let  $B = A + I$ . Consider the following finite leaner combination of eigenfunctions

$$u(x) = \sum_{\lambda_n < \lambda} c_n \cdot u_n(x)$$

where  $\{c_k\}$  an arbitrary set of numbers.

Then for arbitrary positive number  $\tau$  following equality is valid

$$B^{-\tau} u(x) = \sum_{\lambda_n < \lambda} c_n \cdot (1 + \lambda_n)^{-\tau} u_n(x)$$

Thus

$$\langle f, B^{-\tau} u(x) \rangle = \sum_{\lambda_n < \lambda} c_n \cdot (1 + \lambda_n)^{-\tau} f_n \quad (20)$$

Since  $f$  is a linear continuous functional taking into account inequality (1) we obtain

$$|\langle f, B^{-\tau} u(x) \rangle| \leq \|f\|_{-1} \cdot \|B^{-\tau} u\|_{l,0} \quad (21)$$

where symbol  $\|\cdot\|_{-1,0}$  denotes a norm in the space  $H^{-1}(\Omega_0)$ ,  $K \subset \Omega_0 \subset \subset \Omega$ . Note that from the results of [4] it follows

$$\|B^{-\tau} u\|_{l,0} \leq \text{const} \cdot \|u\|_{L_2(\Omega)}$$

Then taking into account (20), (21) and using the Parsevval equality obtain

$$\sum_{\lambda_n < \lambda} c_n \cdot (1 + \lambda_n)^{-\tau} f_n \leq \text{const} \cdot \|f\|_{-1} \left( \sum c_n^2 \right)^{1/2}$$

As since this estimation is valid for an arbitrary set of numbers  $c_k$ , then we obtain required inequality. Lemma 3. is proved.

**Proof of the theorem 1:** Use the representation (14) for the Riesz means and apply Cauchy-Bunyakovski inequality. Then from (15) and (18) obtain

$$|E_{\lambda}^s f(x)| \leq \text{const} \cdot \|f\|_{-1} \cdot \sqrt{\lambda}^{1-s+(N-1)/2}$$

uniformly by  $x \in K \subset \Omega \setminus \{x_0 \cup \text{Supp} f\}$ . Theorem 1. is proved.

Note, that theorem 2. immediately follows from case  $a(x) = 0$  that was proved in [2, 6].

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