# New Complex Solutions for Nonlinear Wave Equation with the Fifth Order Nonlinear Term and Foam Drainage Equation 

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#### Abstract

In this present work we applied new applications of direct algebraic method to foam drainage equation and to Nonlinear Wave equation with the fifth order nonlinear term, the balance numbers of which are not positive integers. Then new types of complex solutions are obtained to the foam drainage equation and Nonlinear Wave equation with the fifth order nonlinear term.


Key words:Direct algebraic method • Foam drainage equation • Nonlinear Wave equation with the fifth order nonlinear term

## INTRODUCTION

Seeking the complex and exact solutions of nonlinear partial differential equations plays an important role in nonlinear problems. When we want to understand the physical mechanism of phenomena in nature, described by nonlinear evolution equations, exact solutions for the nonlinear evolution equations have to be explored. Recently many new approaches to obtain the exact solutions of nonlinear differential equations have been proposed. When we want to understand the physical mechanism of phenomena in nature, described by nonlinear evolution equations, exact solutions for the nonlinear evolution equations have to be explored. Thus, the methods for deriving exact solutions for the governing equations have to be developed. Recently, many powerful methods have been established and improved. Among these methods, we cite the homogeneous balance method [1, 2], the tanh-function method [3], the extended tanh-function method [4], the Jacobi elliptic function expansion method [5, 6], the auxiliary equation method [7] and so on.

Recently, the direct algebraic method and symbolic computation have been suggested to obtain the exact complex solutions of nonlinear partial differential equations [8, 9].

Description of Direct Algebraic Method: For a given partial differential equation

$$
\begin{equation*}
G\left(u, u_{x}, u_{t}, u_{x x}, u_{t t}, \ldots .\right), \tag{1}
\end{equation*}
$$

Our method mainly consists of four steps:

Step 1: We seek complex solutions of Eq. (1) as the following form: c

Where k and c are real constants. Under the transformation (2), Eq. (1) becomes an ordinary differential equation

$$
\begin{equation*}
N\left(u, i k u^{\prime},-i k c u^{\prime},-k^{2} u^{\prime \prime}, \ldots . . .\right), \tag{2}
\end{equation*}
$$

Where $u^{\prime}=\frac{d u}{d \xi}$.

Step 2: We assume that the solution of Eq. (3) is of the form

$$
\begin{equation*}
u(\xi)=\sum_{i=0}^{n} a_{i} F^{i}(\xi) \tag{3}
\end{equation*}
$$

Where $a_{i}(i=1,2, \ldots, n)$ are real constants to be determined later. $F(\xi)$ expresses the solution of the auxiliary ordinary differential equation

$$
\begin{equation*}
F^{\prime}(\xi)=b+F^{2}(\xi) \tag{4}
\end{equation*}
$$

Eq. (5) admits the following solutions:
$F(\xi)= \begin{cases}-\sqrt{-b} \tanh (\sqrt{-b} \xi), & b \prec 0 \\ -\sqrt{-b} \operatorname{coth}(\sqrt{-b} \xi), & b \prec 0\end{cases}$
$F(\xi)=\left\{\begin{array}{lc}\sqrt{b} \tan (\sqrt{b} \xi), & b \succ 0 \\ -\sqrt{b} \cot (\sqrt{b} \xi), & b \succ 0\end{array}\right.$
$F(\xi)=-\frac{1}{\xi}, \quad b=0$
Integer $n$ in (4) can be determined by considering homogeneous balance [3] between the nonlinear terms and the highest derivatives of $u(\xi)$ in Eq. (3).

Step 3: Substituting (4) into (3) with (5), then the left hand side of Eq. (3) is converted into a polynomial in $F(\xi)$, equating each coefficient of the polynomial to zero yields a set of algebraic equations for $a_{i}, k, c$.

Step 4: Solving the algebraic equations obtained in step 3 and substituting the results into (4), then we obtain the exact traveling wave solutions for Eq. (1).

Application to Foam Drainage Equation: Consider the foam drainage equation

$$
\begin{equation*}
u_{t}+\left(u^{2}-\frac{\sqrt{u}}{2} u_{x}\right)_{x}=0 \tag{6}
\end{equation*}
$$

Where x and t are scaled position and time coordinates, respectively. We may choose the following complex travelling wave transformation:

$$
\begin{equation*}
u=u(\xi), \quad \xi=i k(x+c t) \tag{7}
\end{equation*}
$$

Where $c, k$ are constants to be determined later. Using the complex traveling wave solutions (7) we have the nonlinear ordinary differential equation

$$
\begin{equation*}
c i k u^{\prime}+i k\left(u^{2}-i k \frac{\sqrt{u}}{2} u^{\prime}\right)^{\prime}=0, \tag{8}
\end{equation*}
$$

Integrating (9) with respect to $\xi$ and considering the zero constants for integration we obtain

$$
\begin{equation*}
c u+\left(u^{2}-i k \frac{\sqrt{u}}{2} u^{\prime}\right)=0, \tag{9}
\end{equation*}
$$

then we use the transformation

$$
u(\xi)=v^{2}(\xi)
$$

that will convert Eq. (10) to

$$
\begin{equation*}
c+v^{2}-k v^{\prime}=0 \tag{10}
\end{equation*}
$$

Where the prime denotes differentiation with respect to $\xi$. Balancing $v^{\prime}$ with $v^{2}$ in (11) gives

$$
2 m=m+1 \quad \text { so } \quad m=1
$$

So

$$
\begin{equation*}
v=A_{1} F+A_{0}, \tag{11}
\end{equation*}
$$

Substituting (12) into Eq. (11) yields a set of algebraic equations for $A_{1}, A_{0}, k, b$ and by solving found equation we have

$$
\begin{gather*}
A_{1}=k, \quad A_{0}=0  \tag{13}\\
c=k b \tag{14}
\end{gather*}
$$

From (6),(12,13) and (14), we obtain the complex travelling wave solutions of (6) as follows

$$
\begin{aligned}
& v_{1}=k[-\sqrt{-b} \tanh (\sqrt{-b} i k(x+k b t))], \\
& u_{1}=-k^{2} b \tanh ^{2}(\sqrt{-b} i k(x+k b t))
\end{aligned}
$$

Where $b<0$ and $k$ is an arbitrary real constant. And

$$
\begin{aligned}
& v_{2}=k[-\sqrt{-b} \operatorname{coth}(\sqrt{-b} i k(x+k b t))], \\
& u_{2}=k^{2}[-\sqrt{-b} \operatorname{coth}(\sqrt{-b} i k(x+k b t))]^{2},
\end{aligned}
$$

Where $b \prec 0$ and $k$ is an arbitrary real constant.

$$
\begin{aligned}
& v_{3}=k[\sqrt{b} \tan (\sqrt{b} i k(x+k b t))], \\
& u_{3}=k^{2}[\sqrt{b} \tan (\sqrt{b} i k(x+k b t))]^{2},
\end{aligned}
$$

Where $b \succ 0$ and $k$ is an arbitrary real constant.

$$
v_{4}=k[-\sqrt{b} \cot (\sqrt{b} i k(x+k b t))],
$$

$$
u_{4}=k^{2}[-\sqrt{b} \cot (\sqrt{b} i k(x+k b t))]^{2},
$$

Where $b \succ 0$ and $k$ is an arbitrary real constant.

$$
\begin{aligned}
& v_{5}=-\frac{1}{i k x}, \\
& u_{5}=-\frac{1}{k^{2} x^{2}},
\end{aligned}
$$

Where $b=0$ and $k$ is an arbitrary real constant.
Application to Nonlinear Wave Equation with the Fifth Order Nonlinear Term: The Nonlinear Wave equation with the fifth order nonlinear term reads:

$$
\begin{equation*}
u_{t t}-a_{1} u_{x x}+a_{2} u+a_{3} u^{3}+a_{4} u^{5}=0 \tag{15}
\end{equation*}
$$

We may choose the following complex travelling wave transformation:

$$
\begin{equation*}
u=u(\xi), \quad \xi=i k(x-c t) \tag{16}
\end{equation*}
$$

Where $\alpha, k$ are constants to be determined later. Using the complex traveling wave solutions (16) we have the nonlinear ordinary differential equation

$$
\begin{equation*}
k^{2}\left(a_{1}-c^{2}\right) u^{\prime \prime}+a_{2} u+a_{3} u^{3}+a_{4} u^{5}=0 \tag{17}
\end{equation*}
$$

Considering the homogeneous balance between $u^{5}$ and $u^{\prime \prime}$ in (17), we required that $5 m=m+2 \Rightarrow m=\frac{1}{2}$. So

$$
\begin{equation*}
u=A F^{\frac{1}{2}} \tag{18}
\end{equation*}
$$

Hence

$$
\begin{equation*}
u^{\prime \prime}=-\frac{1}{4} A b^{2} F^{-\frac{3}{2}}+\frac{1}{2} A b F^{\frac{1}{2}}+\frac{3}{4} A F^{\frac{5}{2}} \tag{19}
\end{equation*}
$$

Substituting (18-19) into Eq. (17) yields a set of algebraic equations for $A, k, \alpha$ and b . These equations are found as

$$
\begin{aligned}
& \frac{1}{2} k^{2} A b\left(a_{1}-c^{2}\right)+a_{2} A=0 \\
& \frac{3}{4} k^{2} A\left(a_{1}-c^{2}\right)+a_{4} A^{5}=0
\end{aligned}
$$

By solving algebraic relation above we obtain

$$
\begin{equation*}
A= \pm \sqrt{\frac{3 a_{2}}{2 a_{4} b}}, \quad c= \pm \sqrt{a_{1}+\frac{2 a_{2}}{b k^{2}}} \tag{20}
\end{equation*}
$$

From (6),(18) and (20), we obtain the complex travelling wave solutions of (15) as follows

$$
u_{1}= \pm \sqrt{\frac{3 a_{2}}{2 a_{4} b}}\left[-\sqrt{-b} \tanh \left(\sqrt{-b} i k\left(x \mp \sqrt{\left.a_{1}+\frac{2 a_{2}}{b k^{2}} t\right)}\right)\right]^{\frac{1}{2}}\right.
$$

Where $b \prec 0$ and $k$ is an arbitrary real constant. And

$$
u_{2}= \pm \sqrt{\frac{3 a_{2}}{2 a_{4} b}}\left[-\sqrt{-b} \operatorname{coth}\left(\sqrt{-b} i k\left(x \mp \sqrt{a_{1}+\frac{2 a_{2}}{b k^{2}}} t\right)\right)\right]^{\frac{1}{2}}
$$

Where $b \prec 0$ and $k$ is an arbitrary real constant.

$$
u_{3}= \pm \sqrt{\frac{3 a_{2}}{2 a_{4} b}}\left[\sqrt{b} \tan \left(\sqrt{b} i k\left(x \mp \sqrt{a_{1}+\frac{2 a_{2}}{b k^{2}}} t\right)\right)\right]^{\frac{1}{2}}
$$

Where $b \succ 0$ and $k$ is an arbitrary real constant.

$$
u_{4}= \pm \sqrt{\frac{3 a_{2}}{2 a_{4} b}}\left[-\sqrt{b} \cot \left(\sqrt{b} i k\left(x \mp \sqrt{a_{1}+\frac{2 a_{2}}{b k^{2}}} t\right)\right)\right]^{\frac{1}{2}}
$$

Where $b \succ 0$ and $k$ is an arbitrary real constant. If $b=0$ we don't have new solution.

## CONCLUSION

The application of direct algebraic method was still limited to those equations the balance numbers of which are positive integers. In this paper, we explore a new application of the direct algebraic method and obtain new types of complex wave solutions to the foam drainage equation and Nonlinear Wave equation with the fifth order nonlinear term.

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