Variational Iteration Method for System of Higher-Order BVPs

Muhammad Shakeel, Qazi Mahmood ul Hassan and Syed Tauseef Mohyud-Din

Department of Mathematics, HITEC University, Taxila Cantt Pakistan

Abstract: In this paper, we apply Variational Iteration Method (VIM) to solve eighth-order Boundary Value Problems (BVPs) which arise frequently in astronomy and other physical problems. Numerical results completely reveal the efficiency and reliability of the proposed VIM.

Key words: Variational iteration method . astronomy . higher-order BVPs

INTRODUCTION

In this paper, we consider the general eighth-order boundary value problem of the type

\[ u^{(8)}(x) + \phi(x)u(x) = \psi(x), \quad a \leq x \leq b \]  

(1)

with boundary conditions

\[ u(a) = A_0, \quad u^{(2)}(a) = A_2, \quad u^{(4)}(a) = A_4, \quad u^{(6)}(a) = A_6 \]

\[ u(b) = B_0, \quad u^{(2)}(b) = B_2, \quad u^{(4)}(b) = B_4, \quad u^{(6)}(b) = B_6 \]  

(2)

where \( u(x), \phi(x) \) and \( \psi(x) \) are continuous functions defined on \([a, b]\) and the constants \( A_i \) and \( B_i \) are finite real numbers. These problems arise frequently in hydrodynamic, hydromagnetic stability and astronomy [6, 10, 11] and references therein. The basic motivation of this paper is the extension of Variational Iteration Method (VIM) to solve re-formulated eight-order boundary value problems. The eighth-order BVPs are converted into an equivalent system of integral equations by using a suitable transformation. The VIM [1-10] is applied to the re-formulated system of integral equations. Numerical results are very encouraging. Several examples are given to illustrate the reliability and performance of the proposed algorithm (MVIM).

VARIATIONAL ITERATION METHOD

To illustrate the basic concept of the technique, we consider the following general differential equation:

\[ Lu + Nu = g(x) \]  

(3)

where \( L \) is a linear operator, \( N \) a nonlinear operator and \( g(x) \) is the forcing term. According to variational iteration method [1-10], we can construct a correction functional as follows:

\[ u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s) \left( L u_n(s) + N u_n(s) - g(s) \right) ds \]  

(4)

where \( \lambda \) is a Lagrange multiplier [1-10], which can be identified optimally via variational iteration method. The subscripts \( n \) denote the \( n \)th approximation, \( u_n \) is considered as a restricted variation. That is, \( \delta u_n = 0 \); (4) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given [16-19]. For the sake of simplicity and to convey the idea of the technique, we consider the following system of differential equations:

\[ \dot{x}_i(t) = f_i(t, x_i), \quad i = 1, 2, 3, \ldots, n \]  

(5)

subject to the boundary conditions

\[ x_i(0) = c_i, \quad i = 1, 2, 3, \ldots, n \]  

(6)

To solve the system by means of the variational iteration method, we rewrite the system (5) in the following form:

\[ \dot{x}_i(t) = f_i(x_i) + \xi_i(t), \quad i = 1, 2, 3, \ldots, n \]  

(7)

subject to the boundary conditions \( x_i(0) = c_i \), \( i = 1, 2, 3, \ldots, n \) and \( \xi_i \) is defined in (5).

The correction functional for the nonlinear system (5) can be approximated as
\[ x^{(k+1)}_1(t) = x^{(k)}_1(t) + \int_0^t \lambda_1 \left( x_1^{(k)}(t), f_1 \left( x_1^{(k)}(t), x_2^{(k)}(t), \ldots, x_n^{(k)}(t) \right) - g_1(t) \right) dt \]
\[ x^{(k+1)}_2(t) = x^{(k)}_2(t) + \int_0^t \lambda_2 \left( x_2^{(k)}(t), f_2 \left( x_1^{(k)}(t), x_2^{(k)}(t), \ldots, x_n^{(k)}(t) \right) - g_2(t) \right) dt \]
\[ \vdots \]
\[ x^{(k+1)}_n(t) = x^{(k)}_n(t) + \int_0^t \lambda_n \left( x_n^{(k)}(t), f_n \left( x_1^{(k)}(t), x_2^{(k)}(t), \ldots, x_n^{(k)}(t) \right) - g_n(t) \right) dt \]

where \( \lambda_i = \pm 1, i = 1, 2, 3, \ldots, n \) are Lagrange multipliers, \( \delta_t, \delta_t, \ldots, \delta_t \) denote the restricted variations. For \( \lambda_i = -1, i = 1, 2, 3, \ldots, n \), we have the following iterative scheme:

\[ x^{(k+1)}_1(t) = x^{(k)}_1(t) - \int_0^t \lambda_1 \left( x_1^{(k)}(t), f_1 \left( x_1^{(k)}(t), x_2^{(k)}(t), \ldots, x_n^{(k)}(t) \right) - g_1(t) \right) dt \]
\[ x^{(k+1)}_2(t) = x^{(k)}_2(t) - \int_0^t \lambda_2 \left( x_2^{(k)}(t), f_2 \left( x_1^{(k)}(t), x_2^{(k)}(t), \ldots, x_n^{(k)}(t) \right) - g_2(t) \right) dt \]
\[ \vdots \]
\[ x^{(k+1)}_n(t) = x^{(k)}_n(t) - \int_0^t \lambda_n \left( x_n^{(k)}(t), f_n \left( x_1^{(k)}(t), x_2^{(k)}(t), \ldots, x_n^{(k)}(t) \right) - g_n(t) \right) dt \]

If we start with the initial approximations \( x_1^{(0)} = c_1, i = 1, 2, 3, \ldots, n \), then the approximations can be completely determined; finally we approximate the solution

\[ x_i(t) = \lim_{k \to \infty} x^{(k)}_i \]

by the nth term \( x^{(n)}_i(t) \) for \( i = 1, 2, 3, \ldots, n \) (8)

**NUMERICAL APPLICATIONS**

In this section, we first rewrite the eighth-order boundary value problem in an equivalent system of integral equations by using a suitable transformation. The VIM is applied to the re-formulated system of integral equations. Numerical results are very encouraging.

**Example 3.1:** Consider the Eq. (1) with

\[ [a, b] = [0, 1], \phi(x) = x \]

and

\[ \psi(x) = -(48 + 15x + x^3)e^x \]

and the boundary conditions

\[ A_0 = 0, A_2 = 0, A_4 = -8, A_6 = -24 \]

B_0 = 0, B_2 = -4e, B_4 = -16e, B_6 = -36e (9)

The exact solution is given by

\[ x(1-x)e^x \]

(10)

Using the transformation

\[ \frac{du}{dx} = a(x), \frac{da}{dx} = b(x), \frac{db}{dx} = c(x), \frac{dc}{dx} = f(x) \]

\[ \frac{df}{dx} = g(x), \frac{dg}{dx} = h(x), \frac{dh}{dx} = z(x) \]

we obtain the following system of differential equations:

\[ \frac{du}{dx} = a(x), \frac{da}{dx} = b(x), \frac{db}{dx} = c(x), \frac{dc}{dx} = f(x) \]

\[ \frac{df}{dx} = g(x), \frac{dg}{dx} = h(x), \frac{dh}{dx} = z(x), \frac{dz}{dx} = -xu(x) - \left( 48 + 15x + x^3 \right)e^x \]

(12)

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers \( \lambda_i = 1, i = 1, 2, 3, \ldots, 8 \):

\[ u^{(k+1)}(x) = 0 + \int_0^x a^{(k)}(s) \, ds, \quad a^{(k+1)} = A + \int_0^x b^{(k)}(s) \, ds \]

\[ b^{(k+1)} = 0 + \int_0^x e^{(k)}(s) \, ds, \quad e^{(k+1)} = B + \int_0^x f^{(k)}(s) \, ds \]

\[ f^{(k+1)} = -8 + \int_0^x g^{(k)}(s) \, ds, \quad g^{(k+1)} = C + \int_0^x h^{(k)}(s) \, ds \]

\[ h^{(k+1)} = -24 + \int_0^x z^{(k)}(s) \, ds \]

\[ z^{(k+1)} = L + \int_0^x \left(-su(s) - \left(48 + 15s + s^3\right) e^s\right) \, ds \quad (13) \]

Consequently, we obtain the following approximants:

\[ u^{(0)}(x) = 0, \quad a^{(0)}(x) = A, \quad b^{(0)}(x) = 0, \quad e^{(0)}(x) = B \]

\[ f^{(0)}(x) = -8, \quad g^{(0)}(x) = C, \quad h^{(0)}(x) = -24, \quad z^{(0)}(x) = L \]

\[ u^{(1)}(x) = Ax, \quad a^{(1)}(x) = A \]

\[ b^{(1)}(x) = Bx, \quad e^{(1)}(x) = B - 8x \]

\[ f^{(1)}(x) = -8 + Cx, \quad g^{(1)}(x) = C - 24x \]

\[ h^{(1)}(x) = -24 + Lx \]

\[ z^{(1)}(x) = L + 27 - 27e^x - 21e^x + 3e^x - e^x \]

\[ u^{(2)}(x) = Ax, \quad a^{(2)}(x) = A + \frac{1}{2} Bx^2 \]

\[ b^{(2)}(x) = Bx - 4x^2, \quad e^{(2)}(x) = B - 8x + \frac{1}{2} Cx^2 \]

The series solution is given by

\[ u(x) = Ax + \frac{1}{6} Bx^3 - \frac{1}{3} Cx^5 - \frac{1}{120} Lx^7 - \frac{1}{5040} x^9 + \left(\frac{1}{13} \cdot \frac{1}{604800} + \frac{1}{1814400} \right) A x^{10} - \frac{1}{403200} x^{11} \]

\[ -\left(\frac{1}{3628800} + \frac{1}{119750400} \right) B x^{12} - \frac{1}{43545600} x^{13} + \left(\frac{43}{14529715200} + \frac{1}{14529715200} \right) C x^{14} - \frac{1}{6706022400} x^{15} + O(x^{16}) \quad (15) \]

Imposing the boundary conditions at \( x = 1 \), we have

\[ A = 1.000000131, \quad B = -3.000001255, \quad C = -14.99998905, \quad L = -35.00006877 \quad (16) \]

Consequently, the series solution is given as

\[ u(x) = 1.000000131x - \frac{1}{3} x^4 - \frac{1}{120} Cx^5 - \frac{1}{5040} x^9 + \left(\frac{1}{30} \cdot \frac{1}{13} \cdot \frac{1}{604800} + \frac{1}{1814400} \right) A x^{10} - \frac{1}{403200} x^{11} \]

\[ -\left(\frac{1}{3628800} + \frac{1}{119750400} \right) B x^{12} - \frac{1}{43545600} x^{13} + \left(\frac{43}{14529715200} + \frac{1}{14529715200} \right) C x^{14} - \frac{1}{6706022400} x^{15} + O(x^{16}) \quad (17) \]

Table 3.1 exhibits the exact solution and the series solution along with the errors obtained by using the variational iteration method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of \( u(x) \).
Table 3.1: Error estimates

<table>
<thead>
<tr>
<th>x</th>
<th>Exact solution</th>
<th>Series solution</th>
<th>*Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000000000000</td>
<td>0.000000000000</td>
<td>0.000000000000+00</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0994653955</td>
<td>0.0994653826</td>
<td>1.2910000000E-08</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1954244659</td>
<td>0.1954244413</td>
<td>2.4600000000E-08</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2834703834</td>
<td>0.2834703497</td>
<td>3.3700000000E-08</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3580379674</td>
<td>0.3580379275</td>
<td>3.9900000000E-08</td>
</tr>
<tr>
<td>0.5</td>
<td>0.4121803599</td>
<td>0.4121803178</td>
<td>4.2100000000E-08</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4373085523</td>
<td>0.4373085120</td>
<td>4.0300000000E-08</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4228881028</td>
<td>0.4228880685</td>
<td>3.4300000000E-08</td>
</tr>
<tr>
<td>0.8</td>
<td>0.3560865733</td>
<td>0.3560865485</td>
<td>2.4800000000E-08</td>
</tr>
<tr>
<td>0.9</td>
<td>0.2213642927</td>
<td>0.2213642800</td>
<td>1.2700000000E-08</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.0000000006</td>
<td>0.0000000000</td>
<td>5.9599169680E-10</td>
</tr>
</tbody>
</table>

*Error = Exact solution-Series solution

Fig.1: 

Example 3.2: Consider the Eq. (1) with

\[a, b = 0, 1, x = x_\phi^-\]

and

\[\psi(x) = -(55 + 17x + x^2 - x^3)e^x\]

and the boundary conditions

\[A_0 = 0, A_2 = 2/e, A_4 = -4/e, A_6 = -18/e, B_0 = 0, B_2 = -6/e, B_4 = -20/e, B_6 = -42/e\]

(18)

The exact solution is given by

\[(1 - x^2)e^x\]

(19)

Using the transformation

\[
\begin{align*}
\frac{du}{dx} &= a(x), \quad \frac{da}{dx} = b(x), \quad \frac{db}{dx} = c(x), \quad \frac{dc}{dx} = f(x) \\
\frac{df}{dx} &= g(x), \quad \frac{dg}{dx} = h(x), \quad \frac{dh}{dx} = z(x)
\end{align*}
\]

(20)

we obtain the following system of differential equations:

\[
\begin{align*}
\frac{du}{dx} &= a(x), \quad \frac{da}{dx} = b(x), \quad \frac{db}{dx} = c(x), \quad \frac{dc}{dx} = f(x) \\
\frac{df}{dx} &= g(x), \quad \frac{dg}{dx} = h(x), \quad \frac{dh}{dx} = z(x), \\
\frac{dz}{dx} &= (su(x) - (55 + 17x + x^2 - x^3)e^x)
\end{align*}
\]

(21)

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers \(\lambda_i = 1, i = 1, 2, \ldots, 8\):

\[
\begin{align*}
\int_0^x a^{(k)}(s)ds, & \quad \int_0^x b^{(k)}(s)ds \\
\int_0^x c^{(k)}(s)ds, & \quad \int_0^x e^{(k)}(s)ds \\
\int_0^x g^{(k)}(s)ds, & \quad \int_0^x h^{(k)}(s)ds \\
\int_0^x z^{(k)}(s)ds & \quad \int_0^x L
\end{align*}
\]

(22)

Consequently, we obtain the following approximants:

\[
\begin{align*}
u^{(0)}(x) &= A, \quad a^{(0)}(x) = B, \quad b^{(0)}(x) = C, \quad c^{(0)}(x) = E \\
r^{(0)}(x) &= F, \quad g^{(0)}(x) = G, \quad h^{(0)}(x) = L, \quad z^{(0)}(x) = L \\
u^{(1)}(x) &= A + Bx, \quad a^{(1)}(x) = B + Cx \\
r^{(1)}(x) &= F + Gx, \quad g^{(1)}(x) = G + Hx \\
h^{(1)}(x) &= H + Lx, \quad z^{(1)}(x) = H + Lx
\end{align*}
\]

(23)

Consequently, we obtain the following approximants:

\[
\begin{align*}
u^{(2)}(x) &= A + Bx + \frac{1}{2}Cx^2, \quad a^{(2)}(x) = B + Cx + \frac{1}{2}Ex^2 \\
h^{(2)}(x) &= C + Ex + \frac{1}{2}Fx^2, \quad c^{(2)}(x) = E + Fx + \frac{1}{2}Gx^2 \\
h^{(2)}(x) &= F + Gx + \frac{1}{2}Hx^2, \quad g^{(2)}(x) = G + Hx + \frac{1}{2}Lx^2 \\
h^{(2)}(x) &= H + Lx + \frac{5}{6}Ax + \frac{1}{2}x^3
\end{align*}
\]

(24)
The series solution is given by

\[
\begin{align*}
 u(x) &= A + B x + \frac{1}{2} C x^2 + \frac{1}{6} E x^3 + \frac{1}{24} F x^4 + \frac{1}{120} G x^5 + \frac{1}{720} H x^6 + \frac{1}{5040} L x^7 - \frac{1}{8064} x^8 + \\
 &+ \left( \frac{1}{362880} - \frac{1}{5040} \right) x^9 + \left( -\frac{13}{518400} + \frac{1}{1814400} \right) x^{10} + \left( -\frac{53}{19958400} + \frac{1}{13305600} \right) x^{11} + \\
 &+ \left( -\frac{67}{87178291200} + \frac{1}{14529715200} \right) x^{12} + \left( -\frac{62270208}{1245404160} + \frac{1}{6} \right) x^{13} + \\
 &+ \left( -\frac{5}{182} \right) x^{14} + \left( -\frac{1}{87178291200} + \frac{1}{217945728000} + \frac{1}{186810624000} \right) x^{15} + O(x^{16})
\end{align*}
\]

Consequently, the series solution is given as

\[
\begin{align*}
 u(x) &= 1.0000005531 + 1.00000014 x - 0.5000068030 x^2 - 0.8333333555 x^3 - 0.4583319688 x^4 - 0.158333237 x^5 \\
 &- 0.04027787014 x^6 - 0.008134922115 x^7 - 0.0013640873 x^8 - 0.000195659512 x^9 - 0.00002452601410 x^{10} \\
 &- 0.000002730680837 x^{11} - 2.734855177 \times 10^{-7} x^{12} - 2.489149165 \times 10^{-8} x^{13} - 2.076204873 \times 10^{-9} x^{14} \\
 &- 1.598260779 \times 10^{-10} x^{15} + O(x^{16})
\end{align*}
\]

Table 3.2 exhibits the exact solution and the series solution along with the errors obtained by using the variational iteration method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of \( u(x) \).

**Example 3.3:** Consider the Eq. (1) with

\[ [a, b] = [-1, 1], \; \Phi(x) = -1 \; \text{and} \; \Psi(x) = -8 \left( 2 \cos(x) + 7 \sin(x) \right) \]

and the boundary conditions
Table 3.2: Error estimates

<table>
<thead>
<tr>
<th>x</th>
<th>Exact solution</th>
<th>Series solution</th>
<th>*Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>0.000000000020</td>
<td>0.000000000000</td>
<td>1.8053215490E-10</td>
</tr>
<tr>
<td>0.80</td>
<td>0.1617601361</td>
<td>0.1617584271</td>
<td>1.7090000000E-06</td>
</tr>
<tr>
<td>0.60</td>
<td>0.3512426973</td>
<td>0.3512394471</td>
<td>3.2502000000E-06</td>
</tr>
<tr>
<td>0.40</td>
<td>0.5630733113</td>
<td>0.5630688386</td>
<td>4.4727000000E-06</td>
</tr>
<tr>
<td>0.20</td>
<td>0.7859867813</td>
<td>0.7859815230</td>
<td>5.2583000000E-06</td>
</tr>
<tr>
<td>0.00</td>
<td>1.0000055310</td>
<td>1.0000000000</td>
<td>5.5310000000E-06</td>
</tr>
<tr>
<td>0.20</td>
<td>1.1725519120</td>
<td>1.1725466480</td>
<td>5.2640000000E-06</td>
</tr>
<tr>
<td>0.40</td>
<td>1.2531372290</td>
<td>1.2531327460</td>
<td>4.4830000000E-06</td>
</tr>
<tr>
<td>0.60</td>
<td>1.1661592910</td>
<td>1.1661560320</td>
<td>3.2590000000E-06</td>
</tr>
<tr>
<td>0.80</td>
<td>1.2531372290</td>
<td>1.2531327460</td>
<td>4.4830000000E-06</td>
</tr>
<tr>
<td>1.00</td>
<td>0.000000000020</td>
<td>0.000000000000</td>
<td>1.8382269910E-10</td>
</tr>
</tbody>
</table>

*Error = Exact solution -- Series solution

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = 1, i = 1, 2, 3, \ldots, 8$:

$$
\begin{align*}
    u^{(k+1)} &= A + \int_0^x a^{(k)}(s)ds, \quad a^{(k+1)} = B + \int_0^x b^{(k)}(s)ds \\
    b^{(k+1)} &= C + \int_0^x e^{(k)}(s)ds, \quad e^{(k+1)} = E + \int_0^x f^{(k)}(s)ds \\
    f^{(k+1)} &= F + \int_0^x g^{(k)}(s)ds, \quad g^{(k+1)} = G + \int_0^x h^{(k)}(s)ds \\
    h^{(k+1)} &= H + \int_0^x z^{(k)}(s)ds \\
    z^{(k+1)} &= L + \int_0^x \left[u(s) - 8\left[2\cos(x) + 7\sin(x)\right]\right]ds
\end{align*}
$$

Consequently, we obtain the following approximants:

$$
\begin{align*}
    u^{(0)}(x) &= A, \quad a^{(0)}(x) = B \\
    b^{(0)}(x) &= C, \quad e^{(0)}(x) = E \\
    f^{(0)}(x) &= F, \quad g^{(0)}(x) = G \\
    h^{(0)}(x) &= H, \quad z^{(0)}(x) = L \\
    u^{(1)}(x) &= A + Bx, \quad a^{(1)}(x) = B + Cx \\
    b^{(1)}(x) &= C + Ex, \quad e^{(1)}(x) = E + Fx \\
    f^{(1)}(x) &= F + Gx, \quad g^{(1)}(x) = G + Hx \\
    h^{(1)}(x) &= H + Lx, \quad z^{(1)}(x) = L - 40 + Ax + 40 \cos x - 16 \sin x \\
    u^{(2)}(x) &= A + Bx + \frac{1}{2}Cx^2, \quad a^{(2)}(x) = B + Cx + \frac{1}{2}Ex^2 \\
    b^{(2)}(x) &= C + Ex + \frac{1}{2}Fx^2, \quad e^{(2)}(x) = E + Fx + \frac{1}{2}Gx^2 \\
    f^{(2)}(x) &= F + Gx + \frac{1}{2}Hx^2, \quad g^{(2)}(x) = G + Hx + \frac{1}{2}Lx^2 \\
    h^{(2)}(x) &= H + Lx - 40x + \frac{1}{2}Ax^2 + 24 \sin x + 16 \cos x \\
    z^{(2)}(x) &= L - 40 + Ax + 40 \cos x - 16 \sin x + \frac{1}{2}Rx^2 \\
    u^{(3)}(x) &= A + Bx + \frac{1}{2}Cx^2 + \frac{1}{6}Ex^3 \\
    a^{(3)}(x) &= B + Cx + \frac{1}{2}Ex^2 + \frac{1}{6}Fx^3 \\
    b^{(3)}(x) &= C + Ex + \frac{1}{2}Fx^2 + \frac{1}{6}Gx^3
\end{align*}
$$

The exact solution is given by

$$
\left(x^2 - 1\right) \sin(x)
$$

Using the transformation

$$
\begin{align*}
    \frac{du}{dx} &= a(x), \quad \frac{da}{dx} = b(x), \quad \frac{db}{dx} = c(x), \quad \frac{dc}{dx} = f(x) \\
    \frac{df}{dx} &= g(x), \quad \frac{dg}{dx} = h(x), \quad \frac{dh}{dx} = z(x)
\end{align*}
$$

we obtain the following system of differential equations:
The series solution is given by

\[ u(x) = A + Bx + \frac{1}{2}Cx^2 + \frac{1}{6}Ex^3 + \frac{1}{24}Fx^4 + \frac{1}{120}Gx^5 + \frac{1}{720}Hx^6 + \frac{1}{5040}Lx^7 + \frac{1}{40320}Ax^8 + \left( \frac{1}{5040} \right) \left( \frac{1}{362880} \right) B x^9 + \frac{1}{3628800} C x^{10} + \left( \frac{13}{40989600} \right) \left( \frac{1}{9916800} \right) E x^{11} + \frac{1}{479001600} F x^{12} + \left( \frac{17}{778377600} \right) \left( \frac{1}{6227020800} \right) G x^{13} + \frac{1}{87178291200} H x^{14} + \left( \frac{1}{778377600} \right) \left( \frac{1}{130767436800} \right) L x^{15} + O(x^{16}) \]

Imposing the boundary conditions at \( x = -1 \) and \( x = 1 \), we have

\[
A = 0, \quad B = -0.9999999860, \quad C = 0, \quad E = 6.999999872, \quad F = 0, \quad G = -20.99999890, \quad H = 0, \quad L = -41.00000746
\]

Consequently, the series solution is given as

\[ u(x) = 0.9999999860x + 1.166666645x^3 - 0.17499999872x^5 + 0.008531744685x^7 - 0.0002011684303x^9 + 0.000002780784028x^{11} - 2.52126986510^{-8}x^{13} + 1.61355149610^{-10}x^{15} + O(x^{16}) \]

Table 3.3: Error estimates

<table>
<thead>
<tr>
<th>x</th>
<th>Exact solution</th>
<th>Series solution</th>
<th>*Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.00</td>
<td>-0.0000000002</td>
<td>0.0000000000</td>
<td>1.8738449960E-10</td>
</tr>
<tr>
<td>-0.80</td>
<td>0.2582481900</td>
<td>0.2582481927</td>
<td>2.7000000000E-09</td>
</tr>
<tr>
<td>-0.60</td>
<td>0.3613711786</td>
<td>0.3613711830</td>
<td>4.4000000000E-09</td>
</tr>
<tr>
<td>-0.40</td>
<td>0.3271114032</td>
<td>0.3271114075</td>
<td>4.3000000000E-09</td>
</tr>
<tr>
<td>-0.20</td>
<td>0.1907225549</td>
<td>0.1907225576</td>
<td>2.7000000000E-09</td>
</tr>
<tr>
<td>0.00</td>
<td>0.0000000000</td>
<td>0.0000000000</td>
<td>0.00000000000000</td>
</tr>
<tr>
<td>0.20</td>
<td>-0.1907225549</td>
<td>-0.1907225576</td>
<td>2.7000000000E-09</td>
</tr>
<tr>
<td>0.40</td>
<td>-0.3271114032</td>
<td>-0.3271114075</td>
<td>4.3000000000E-09</td>
</tr>
<tr>
<td>0.60</td>
<td>-0.3613711786</td>
<td>-0.3613711830</td>
<td>4.4000000000E-09</td>
</tr>
<tr>
<td>0.80</td>
<td>-0.2582481900</td>
<td>-0.2582481927</td>
<td>2.7000000000E-09</td>
</tr>
<tr>
<td>1.00</td>
<td>0.0000000000</td>
<td>0.0000000000</td>
<td>1.8738449960E-10</td>
</tr>
</tbody>
</table>

*Error = Exact solution-Series solution

Table 3.3 exhibits the exact solution and the series solution along with the errors obtained by using the variational iteration method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of \( u(x) \).

**Example 3.4:** Consider the Eq. (1) with

\[ \psi(x) = 8 \left[ 2\sin(x) - 7\cos(x) \right] \]

Fig. 3: 

\[ [a, b] = [-1, 1], \quad \phi(x) = -1 \]
and the boundary conditions

\[ A_0 = 0, A_2 = -4\sin(1) + 2\cos(1) \]
\[ A_4 = 8\sin(1) - 12\cos(1), A_6 = -12\sin(1) + 30\cos(1) \]
\[ B_0 = 0, B_2 = -4\sin(1) + 2\cos(1) \]
\[ B_4 = 8\sin(1) - 12\cos(1), B_6 = -12\sin(1) + 30\cos(1) \]  

(35)

The exact solution is given by

\[ (x^2 - 1)\cos(x) \]  

(36)

Using the transformation

\[ \frac{du}{dx} = a(x), \frac{da}{dx} = b(x), \frac{db}{dx} = c(x), \frac{dc}{dx} = f(x), \]
\[ \frac{df}{dx} = g(x), \frac{dg}{dx} = h(x), \frac{dh}{dx} = z(x) \]  

(37)

we obtain the following system of differential equations:

\[ \frac{du}{dx} = a(x), \frac{da}{dx} = b(x), \frac{db}{dx} = c(x), \frac{dc}{dx} = f(x), \]
\[ \frac{df}{dx} = g(x), \frac{dg}{dx} = h(x), \frac{dh}{dx} = z(x) \]  

(38)

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers \( \lambda_i = 1, i = 1,2,3,...,8 \):

\[ u^{(k+1)} = A + \int_0^x a^{(k)}(s) ds, a^{(k+1)} = B + \int_0^x b^{(k)}(s) ds \]
\[ b^{(k+1)} = C + \int_0^x c^{(k)}(s) ds, c^{(k+1)} = E + \int_0^x e^{(k)}(s) ds \]
\[ f^{(k+1)} = F + \int_0^x g^{(k)}(s) ds, g^{(k+1)} = G + \int_0^x h^{(k)}(s) ds \]
\[ h^{(k+1)} = H + \int_0^x z^{(k)}(s) ds \]
\[ z^{(k+1)} = L + \int_0^x \left[ u(s) + 8\left[ 2\sin(x) - 7\cos(x) \right] \right] ds \]  

(39)

The series solution is given by

\[ u(x) = A + Bx + \frac{1}{2}C x^2 + \frac{1}{6}Dx^3 + \frac{1}{24}Ex^4 + \frac{1}{120}Fx^5 + \frac{1}{720}Gx^6 + \frac{1}{5040}Hx^7 + \left( \frac{1}{720} + \frac{1}{40320} \right) Ax^8 + \frac{1}{362880}Bx^9 \]
\[ + \left( \frac{11}{453600} + \frac{1}{3628800} \right) C x^{10} + \frac{1}{39916800}Dx^{11} + \left( \frac{11}{479001600} + \frac{1}{1307674368000} \right) Ex^{12} \]
\[ + \frac{1}{6227020800}Fx^{13} + \left( \frac{19}{108972864000} + \frac{1}{87178291200} \right) Gx^{14} + \frac{1}{130767436800}Hx^{15} + O(x^{16}) \]  

(41)

Consequently, we obtain the following approximants:

\[ u^{(0)}(x) = A, a^{(0)}(x) = B \]
\[ h^{(0)}(x) = C, c^{(0)}(x) = E \]
\[ f^{(0)}(x) = F, g^{(0)}(x) = G \]
\[ h^{(0)}(x) = H, z^{(0)}(x) = L \]
\[ u^{(1)}(x) = A + Bx, a^{(1)}(x) = B + Cx \]
\[ b^{(1)}(x) = C + Ex, c^{(1)}(x) = E + Fx \]
\[ f^{(1)}(x) = F + Gx, g^{(1)}(x) = G + Hx \]
\[ h^{(1)}(x) = H + Lx, z^{(1)}(x) = L + Ax + 40\sin x - 16\cos x \]
\[ u^{(2)}(x) = A + Bx + \frac{1}{2}C x^2 + \frac{1}{6}E x^3 \]
\[ a^{(2)}(x) = B + Cx + \frac{1}{2}E x^2 + \frac{1}{6}F x^3 \]
\[ b^{(2)}(x) = C + Ex + \frac{1}{2}Fx^2 + \frac{1}{6}G x^3 \]
\[ c^{(2)}(x) = E + Fx + \frac{1}{2}G x^2 + \frac{1}{6}H x^3 \]
\[ f^{(2)}(x) = F + Gx + \frac{1}{2}H x^2 + \frac{1}{6}L x^3 \]
\[ g^{(2)}(x) = G + Hx + \frac{1}{2}L x^2 - 24x + \frac{1}{6}Ax^3 + 8\sin x + 16\cos x \]
\[ h^{(2)}(x) = H + Lx - 24 + \frac{1}{2}Ax^2 + 24\cos(x) - 16\sin(x) + \frac{1}{6}Bx^3 \]
\[ z^{(2)}(x) = L + Ax - 40\sin(x) - 16\cos(x) + \frac{1}{2}Bx^3 + \frac{1}{2}C x^2 \]  

(40)
Imposing the boundary conditions at \( x = -1 \) and \( x = 1 \), we have
\[
A = -0.9999943757, \quad B = 0, \quad C = 2.999986164, \quad E = 0
\]
\[
F = -12.99996668, \quad G = 0, \quad H = 30.99993139, \quad L = 0
\]  
(42)

Consequently, the series solution is given as
\[
u(x) = 0.9999943757 + 1.499993082x - 0.5416652783x^2 + 0.043055461x^3 - 0.001413690337x^4 + 0.00002507715668x^5 - 2.776607984 \times 10^{-7} x^6 + 2.099145664 \times 10^{-9} x^7 + O(x^{16})
\]  
(43)

Table 3.4: Error estimates

<table>
<thead>
<tr>
<th>x</th>
<th>Exact solution</th>
<th>Series solution</th>
<th>*Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.00</td>
<td>0.00000000003</td>
<td>0.0000000000</td>
<td>2.6802726400E-10</td>
</tr>
<tr>
<td>-0.80</td>
<td>0.02508126747</td>
<td>0.2508144153</td>
<td>1.7406000000E-06</td>
</tr>
<tr>
<td>-0.60</td>
<td>0.02582114843</td>
<td>0.528217935</td>
<td>3.3092000000E-06</td>
</tr>
<tr>
<td>-0.40</td>
<td>0.7736866824</td>
<td>0.7736912350</td>
<td>4.5526000000E-06</td>
</tr>
<tr>
<td>-0.20</td>
<td>0.9408585649</td>
<td>0.9408639147</td>
<td>5.3498000000E-06</td>
</tr>
<tr>
<td>0.00</td>
<td>0.9999943757</td>
<td>1.0000000000</td>
<td>5.6243000000E-06</td>
</tr>
<tr>
<td>0.20</td>
<td>0.9408585649</td>
<td>0.9408639147</td>
<td>5.3498000000E-06</td>
</tr>
<tr>
<td>0.40</td>
<td>0.7736866824</td>
<td>0.7736912350</td>
<td>4.5526000000E-06</td>
</tr>
<tr>
<td>0.60</td>
<td>0.02582114843</td>
<td>0.528217935</td>
<td>3.3092000000E-06</td>
</tr>
<tr>
<td>0.80</td>
<td>0.02508126747</td>
<td>0.2508144153</td>
<td>1.7406000000E-06</td>
</tr>
<tr>
<td>1.00</td>
<td>0.00000000003</td>
<td>0.0000000000</td>
<td>2.6802726400E-10</td>
</tr>
</tbody>
</table>

*Error = Exact solution--Series solution

Table 3.4 exhibits the exact solution and the series solution along with the errors obtained by using the variational iteration method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of \( u(x) \).

CONCLUSION

In this paper, we applied Variational Iteration Method (VIM) for finding the analytical solutions of a class of eighth-order BVPs. Numerical results explicitly reveal the complete reliability of the proposed VIM.

REFERENCES